

# About the author

*The simpler and sharper, the more beautiful.*

Vasile Cîrtoaje

**Vasile Cîrtoaje** is a Professor at the Department of Automatic Control and Computers from Petroleum-Gas University of Ploiesti, Romania, where he teaches university courses such as Control System Theory and Digital Control Systems.

Since 1970, he published many mathematical problems, solutions and articles in the Romanian journals *Gazeta Matematica-B*, *Gazeta Matematica-A* and *Mathematical Review of Timisoara*. In addition, from 2000 to present, Vasile Cîrtoaje has published many interesting problems and articles in *Art of Problem Solving* website, *Mathematical Reflections*, *Crux Mathematicorum*, *Journal of Inequalities and Applications*, *Journal of Inequalities in Pure and Applied Mathematics*, *Mathematical Inequalities and Applications*, *Banach Journal of Mathematical Analysis*, *Journal of Nonlinear Science and Applications*, *Journal of Nonlinear Analysis and Application*, *Australian Journal of Mathematical Analysis and Applications*, *British Journal of Mathematical and Computer Science*, *International Journal of Pure and Applied Mathematics*, *Journal of Inequalities and Special Functions*, *American Mathematical Monthly*, *School Science and Mathematics*, *Creative Mathematics and Informatics*.

He collaborated with Titu Andreescu, Gabriel Dospinescu and Mircea Lascu in writing the book *Old and New Inequalities*, with Vo Quoc Ba Can and Tran Quoc Anh in writing the book *Inequalities with Beautiful Solution*, with Xiong Bin, Iurie Boreico and Vo Quoc Ba Can in writing the book *An Introduction to Inequalities*, and he wrote his own books *Algebraic Inequalities - Old and New Methods* and *Mathematical Inequalities* (Volume 1,2,3,4,5).

Notice that Vasile Cîrtoaje is the author of some well-known results and strong methods for proving and creating inequalities, such as:

- Half convex function method (HCF method) for Jensen type inequalities;
- Partial convex function method (PCF method) for Jensen type inequalities;
- Jensen type inequalities with ordered variables;
- Equal variables method (EV method) for real or nonnegative variables;
- Arithmetic compensation method (AC method);
- Best lower and upper bounds for Jensen's inequality;
- Necessary and sufficient conditions for symmetric homogeneous polynomial inequalities of degree six in real variables;
- Necessary and sufficient conditions for symmetric homogeneous polynomial inequalities of degree six in nonnegative variables;

- Highest coefficient cancellation method (HCC method) for symmetric homogeneous polynomial inequalities of degree six and eight in real variables;
- Highest coefficient cancellation method for symmetric homogeneous polynomial inequalities of degree six, seven and eight in nonnegative variables;
- Necessary and sufficient conditions for cyclic homogeneous polynomial inequalities of degree four in real variables;
- Necessary and sufficient conditions for cyclic homogeneous polynomial inequalities of degree four in nonnegative variables;
- Strong sufficient conditions for cyclic homogeneous polynomial inequalities of degree four in real or nonnegative variables;
- Inequalities with power-exponential functions.

# Foreword

The author, Vasile Cîrtoaje, professor at University of Ploiesti-Romania, has become well-known for his excellent creations in the mathematical inequality field, ever since the time when he was student in high school (in Breaza city, Prahova Valley). As a student (quite some time ago, oh yes!), I was already familiar with the name of Vasile Cîrtoaje. For me, and many others of my age, it is the name of someone who helped me to grow in mathematics, even though I never met him face to face. It is a name synonymous to hard and beautiful problems involving inequalities. When you say Vasile Cîrtoaje (*Vasc* username on the site Art of Problem Solving), you say inequalities. I remember how happy I was when I could manage to solve one of the problems proposed by professor Cîrtoaje in *Gazeta Matematica* or *Revista Matematica Timisoara*.

The first three volumes of this book are a great opportunity to see and know many old and new elementary methods for solving mathematical inequalities: Volume 1 - *Symmetric polynomial inequalities* (in real variables and nonnegative real variables), Volume 2 - *Symmetric rational and nonrational inequalities*, Volume 3 - *Cyclic and noncyclic inequalities*. As a rule, the inequalities from each section of these volumes are increasingly ordered by the number of variables: two, three, four, five, six and n-variables.

The last two volumes (Volume 4 - *Extensions and refinements of Jensen's inequality*, Volume 5 - *Other recent methods for creating and solving inequalities*) contain beautiful and efficient original methods for creating and solving inequalities, such as *half or partially convex function method* - for Jensen's type inequalities, *Popoviciu's method* for convex function, *equal variables method* and *arithmetic compensation method* - for symmetric inequalities, *the highest coefficient cancellation method* - for symmetric homogeneous polynomial inequalities of degree six, seven and eight, *pqr method* - for cyclic homogeneous polynomial inequalities of degree four, et al.

Many problems, the majority I would say, are made up by the author himself. The chapters and volumes are relatively independent, and you can open the book somewhere to solve an inequality or only read its solution. If you carefully make a thorough study of the book, then you will find that your skills in solving inequalities are considerably improved.

The book contains more than 3000 beautiful inequalities, hints, solutions and methods. Most inequalities and methods are old and recent own creations of the author. Among these, I would like to point out the following inequalities:

$$(a^2 + b^2 + c^2)^2 \geq 3(a^3b + b^3c + c^3a), \quad a, b, c \in \mathbb{R};$$
$$\left(\frac{a}{a+b}\right)^2 + \left(\frac{b}{b+c}\right)^2 + \left(\frac{c}{c+d}\right)^2 + \left(\frac{d}{d+a}\right)^2 \geq 1, \quad a, b, c, d \geq 0;$$

$$\begin{aligned}
& \sum \frac{a}{a + (3 - k)b + kc} \geq 1, \quad 0 \leq k \leq 1, \quad a, b, c, d \geq 0; \\
& \sum (a - kb)(a - kc)(a - b)(a - c) \geq \frac{(k + 2)^2(a - b)^2(b - c)^2(c - a)^2}{2(a^2 + b^2 + c^2)}, \quad a, b, c, k \in \mathbb{R};^1 \\
& a^{5/4} + b^{5/4} + c^{5/4} + d^{5/4} + e^{5/4} \geq 5, \quad a \geq b \geq \cdots \geq e \geq 0, \quad ab + bc + \cdots + ea = 5; \\
& a^{7/6} + b^{7/6} + c^{7/6} + d^{7/6} + e^{7/6} \geq 5, \quad 0 \leq a \leq 1, \quad b, c, d, e \geq 1, \quad ab + bc + \cdots + ea = 5; \\
& \frac{1}{a + 3} + \frac{1}{b + 3} + \cdots + \frac{1}{f + 3} \geq 2, \quad a \geq b \geq \cdots \geq f \geq 0, \quad ab + bc + \cdots + fa = 6; \\
& (a_1 + a_2 + \cdots + a_n)^2 \geq n(a_1 a_{k+1} + a_2 a_{k+2} + \cdots + a_n a_{n+k}), \quad \frac{n}{4} \leq k \leq \frac{n}{2}, \quad a_1 \leq \cdots \leq a_n; \\
& \sum_{i=1}^n \frac{1}{1 + a_i + \cdots + a_i^{n-1}} \geq 1, \quad a_1, a_2, \dots, a_n > 0, \quad a_1 a_2 \cdots a_n = 1; \\
& \frac{1}{1 + a_1^2} + \frac{1}{1 + a_2^2} + \cdots + \frac{1}{1 + a_n^2} \geq \frac{n}{2}, \quad a_1, a_2, \dots, a_n \geq 0, \quad \sum_{1 \leq i < j \leq n} a_i a_j = n(n - 1)/2; \\
& a^{ea} + b^{eb} \geq a^{eb} + b^{ea}, \quad a, b > 0, \quad e \approx 2.7182818; \\
& a^{3b} + b^{3a} \leq 2, \quad a, b \geq 0, \quad a + b = 2.
\end{aligned}$$

The book represents a rich source of beautiful, serious and profound mathematics, dealing with classical and new approaches and techniques which help the reader to develop his inequality-solving skills, intuition and creativity. As a result, it is suitable for a wide audience: advanced middle school students, high school students, college and university students, and teachers. Each problem has a hint, and many problems have multiple solutions, almost all of which are, not surprisingly, quite ingenious. Almost all inequalities require careful thought and analysis, making the book a rewarding source for anyone interested in Olympiad-type problems and in the development of the inequality field. Many problems and methods can be used as group projects for advanced high school students.

What makes this book so attractive? The answer is simple: the great number of new and beautiful inequalities, their quality and freshness, as well as the new approaches and methods for solving mathematical inequalities. Nevertheless, you will find this book to be delightful, inspired, original and enjoyable. Any interested reader will remark the tenacity, enthusiasm and ability of the author in creating and solving nice and difficult inequalities. This book is neither more, nor less than a Work of a Master. I highly recommend it.

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<sup>1</sup>Throughout the book, the math symbol  $\sum$  means  $\sum_{cyc}$ ;

$$\sum_{cyc} f(a_1, a_2, \dots, a_n) = f(a_1, a_2, \dots, a_n) + f(a_2, a_3, \dots, a_1) + \cdots + f(a_n, a_1, \dots, a_{n-1}).$$

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# Chapter 1

## Some Classic and New Inequalities and Methods

### 1. AM-GM (ARITHMETIC MEAN-GEOMETRIC MEAN) INEQUALITY

If  $a_1, a_2, \dots, a_n$  are nonnegative real numbers, then

$$a_1 + a_2 + \dots + a_n \geq n \sqrt[n]{a_1 a_2 \dots a_n},$$

with equality if and only if  $a_1 = a_2 = \dots = a_n$ .

### 2. WEIGHTED AM-GM INEQUALITY

Let  $p_1, p_2, \dots, p_n$  be positive real numbers satisfying

$$p_1 + p_2 + \dots + p_n = 1.$$

If  $a_1, a_2, \dots, a_n$  are nonnegative real numbers, then

$$p_1 a_1 + p_2 a_2 + \dots + p_n a_n \geq a_1^{p_1} a_2^{p_2} \dots a_n^{p_n},$$

with equality if and only if  $a_1 = a_2 = \dots = a_n$ .

### 3. AM-HM (ARITHMETIC MEAN-HARMONIC MEAN) INEQUALITY

If  $a_1, a_2, \dots, a_n$  are positive real numbers, then

$$(a_1 + a_2 + \dots + a_n) \left( \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \geq n^2,$$

with equality if and only if  $a_1 = a_2 = \dots = a_n$ .

#### 4. POWER MEAN INEQUALITY

The power mean of order  $k$  of positive real numbers  $a_1, a_2, \dots, a_n$ , that is

$$M_k = \begin{cases} \left( \frac{a_1^k + a_2^k + \dots + a_n^k}{n} \right)^{\frac{1}{k}}, & k \neq 0 \\ \sqrt[n]{a_1 a_2 \cdots a_n}, & k = 0 \end{cases},$$

is an increasing function with respect to  $k \in \mathbb{R}$ . For instant,  $M_2 \geq M_1 \geq M_0 \geq M_{-1}$  is equivalent to

$$\sqrt{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}}.$$

#### 5. BERNOULLI'S INEQUALITY

For any real number  $x \geq -1$ , we have

- a)  $(1+x)^r \geq 1+rx$  for  $r \geq 1$  and  $r \leq 0$ ;
- b)  $(1+x)^r \leq 1+rx$  for  $0 \leq r \leq 1$ .

If  $a_1, a_2, \dots, a_n$  are real numbers such that either  $a_1, a_2, \dots, a_n \geq 0$  or

$$-1 \leq a_1, a_2, \dots, a_n \leq 0,$$

then

$$(1+a_1)(1+a_2) \cdots (1+a_n) \geq 1+a_1+a_2+\dots+a_n.$$

#### 6. SCHUR'S INEQUALITY

For any nonnegative real numbers  $a, b, c$  and any positive number  $k$ , the inequality holds

$$a^k(a-b)(a-c) + b^k(b-c)(b-a) + c^k(c-a)(c-b) \geq 0,$$

with equality for  $a = b = c$ , and for  $a = 0$  and  $b = c$  (or any cyclic permutation).

For  $k = 1$ , we get the third degree Schur's inequality, which can be rewritten as follows

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a),$$

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

$$a^2 + b^2 + c^2 + \frac{9abc}{a+b+c} \geq 2(ab+bc+ca),$$

$$(b-c)^2(b+c-a) + (c-a)^2(c+a-b) + (a-b)^2(a+b-c) \geq 0.$$



For  $k = 2$ , we get the fourth degree Schur's inequality, which holds for any real numbers  $a, b, c$ , and can be rewritten as follows

$$\begin{aligned} a^4 + b^4 + c^4 + abc(a + b + c) &\geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2), \\ a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 &\geq (ab + bc + ca)(a^2 + b^2 + c^2 - ab - bc - ca), \\ (b - c)^2(b + c - a)^2 + (c - a)^2(c + a - b)^2 + (a - b)^2(a + b - c)^2 &\geq 0, \\ 6abcp &\geq (p^2 - q)(4q - p^2), \quad p = a + b + c, \quad q = ab + bc + ca. \end{aligned}$$

A generalization of the fourth degree Schur's inequality for any real numbers  $a, b, c$  and any real number  $m$ , is the following (V. Cîrtoaje, 2004):

$$\sum (a - mb)(a - mc)(a - b)(a - c) \geq 0,$$

where the equality holds for  $a = b = c$ , and for  $a/m = b = c$  (or any cyclic permutation). This inequality is equivalent to

$$\begin{aligned} \sum a^4 + m(m + 2) \sum a^2b^2 + (1 - m^2)abc \sum a &\geq (m + 1) \sum ab(a^2 + b^2), \\ \sum (b - c)^2(b + c - a - ma)^2 &\geq 0. \end{aligned}$$

A more general result is given by the following theorem (V. Cîrtoaje, 2008).

**Theorem.** *Let*

$$f_4(a, b, c) = \sum a^4 + \alpha \sum a^2b^2 + \beta abc \sum a - \gamma \sum ab(a^2 + b^2),$$

where  $\alpha, \beta, \gamma$  are real constants such that  $1 + \alpha + \beta = 2\gamma$ . Then,

(a)  $f_4(a, b, c) \geq 0$  for all  $a, b, c \in \mathbb{R}$  if and only if

$$1 + \alpha \geq \gamma^2;$$

(b)  $f_4(a, b, c) \geq 0$  for all  $a, b, c \geq 0$  if and only if

$$\alpha \geq (\gamma - 1) \max\{2, \gamma + 1\}.$$

## 7. CAUCHY-SCHWARZ INEQUALITY

If  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  are real numbers, then

$$(a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2) \geq (a_1b_1 + a_2b_2 + \dots + a_nb_n)^2,$$

with equality for

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots = \frac{a_n}{b_n}.$$

Notice that the equality conditions are also valid for  $a_i = b_i = 0$ , where  $1 \leq i \leq n$ .

## 8. HÖLDER'S INEQUALITY

If  $x_{ij}$  ( $i = 1, 2, \dots, m; j = 1, 2, \dots, n$ ) are nonnegative real numbers, then

$$\prod_{i=1}^m \left( \sum_{j=1}^n x_{ij} \right) \geq \left( \sum_{j=1}^n \sqrt[m]{\prod_{i=1}^m x_{ij}} \right)^m.$$

## 9. REARRANGEMENT INEQUALITY

(1) If  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  are two increasing (or decreasing) real sequences, and  $(i_1, i_2, \dots, i_n)$  is an arbitrary permutation of  $(1, 2, \dots, n)$ , then

$$a_1 b_1 + a_2 b_2 + \dots + a_n b_n \geq a_1 b_{i_1} + a_2 b_{i_2} + \dots + a_n b_{i_n}.$$

(2) If  $a_1, a_2, \dots, a_n$  is decreasing and  $b_1, b_2, \dots, b_n$  is increasing, then

$$a_1 b_1 + a_2 b_2 + \dots + a_n b_n \leq a_1 b_{i_1} + a_2 b_{i_2} + \dots + a_n b_{i_n}.$$

(3) Let  $b_1, b_2, \dots, b_n$  and  $c_1, c_2, \dots, c_n$  be two real sequences such that

$$b_1 + \dots + b_k \geq c_1 + \dots + c_k, \quad k = 1, 2, \dots, n.$$

If  $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$ , then

$$a_1 b_1 + a_2 b_2 + \dots + a_n b_n \geq a_1 c_1 + a_2 c_2 + \dots + a_n c_n.$$

## 10. CHEBYSHEV'S INEQUALITY

Let  $a_1 \geq a_2 \geq \dots \geq a_n$  be real numbers.

a) If  $b_1 \geq b_2 \geq \dots \geq b_n$ , then

$$n \sum_{i=1}^n a_i b_i \geq \left( \sum_{i=1}^n a_i \right) \left( \sum_{i=1}^n b_i \right);$$

b) If  $b_1 \leq b_2 \leq \dots \leq b_n$ , then

$$n \sum_{i=1}^n a_i b_i \leq \left( \sum_{i=1}^n a_i \right) \left( \sum_{i=1}^n b_i \right).$$

## 11. MINKOWSKI'S INEQUALITY

For any real number  $k \geq 1$  and any positive real numbers  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$ , the following inequalities hold:

$$\sum_{i=1}^n (a_i^k + b_i^k)^{\frac{1}{k}} \geq \left[ \left( \sum_{i=1}^n a_i \right)^k + \left( \sum_{i=1}^n b_i \right)^k \right]^{\frac{1}{k}} ;$$

$$\sum_{i=1}^n (a_i^k + b_i^k + c_i^k)^{\frac{1}{k}} \geq \left[ \left( \sum_{i=1}^n a_i \right)^k + \left( \sum_{i=1}^n b_i \right)^k + \left( \sum_{i=1}^n c_i \right)^k \right]^{\frac{1}{k}} .$$

## 12. MACLAURIN'S INEQUALITY and NEWTON'S INEQUALITY

If  $a_1, a_2, \dots, a_n$  are nonnegative real numbers, then

$$S_1 \geq S_2 \geq \dots \geq S_n \quad (\text{Maclaurin})$$

and

$$S_k^2 \geq S_{k-1} S_{k+1}, \quad (\text{Newton})$$

where

$$S_k = \sqrt[k]{\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} a_{i_1} a_{i_2} \dots a_{i_k}}{\binom{n}{k}}}.$$

## 13. CONVEX FUNCTIONS

A function  $f$  defined on a real interval  $\mathbb{I}$  is said to be *convex* if

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$$

for all  $x, y \in \mathbb{I}$  and any  $\alpha, \beta \geq 0$  with  $\alpha + \beta = 1$ . If the inequality is reversed, then  $f$  is said to be concave.

If  $f$  is differentiable on  $\mathbb{I}$ , then  $f$  is (strictly) convex if and only if the derivative  $f'$  is (strictly) increasing. If  $f'' \geq 0$  on  $\mathbb{I}$ , then  $f$  is convex on  $\mathbb{I}$ .

Let  $\mathbb{I}$  be a real interval,  $s$  an interior point of  $\mathbb{I}$  and

$$\mathbb{I}_{\geq s} = \{u | u \in \mathbb{I}, u \geq s\}, \quad \mathbb{I}_{\leq s} = \{u | u \in \mathbb{I}, u \leq s\}.$$

A function  $f : \mathbb{I} \rightarrow \mathbb{R}$  is *half convex* if there exists an interior point  $s \in \mathbb{I}$  such that  $f$  is convex on  $\mathbb{I}_{\leq s}$  or  $\mathbb{I}_{\geq s}$ .

A function  $f : \mathbb{I} \rightarrow \mathbb{R}$  is *right partially convex* related to an interior point  $s \in \mathbb{I}$  if there exists an interior point  $s_0 \in \mathbb{I}$ ,  $s_0 > s$ , such that  $f$  is convex on  $[s, s_0]$ . Also, a function  $f : \mathbb{I} \rightarrow \mathbb{R}$  is *left partially convex* related to an interior point  $s \in \mathbb{I}$  if there exists an interior point  $s_0 \in \mathbb{I}$ ,  $s_0 < s$ , such that  $f$  is convex on  $[s_0, s]$ .

**Jensen's inequality.** *Let  $p_1, p_2, \dots, p_n$  be positive real numbers. If  $f$  is a convex function on a real interval  $\mathbb{I}$ , then for any  $a_1, a_2, \dots, a_n \in \mathbb{I}$ , the inequality holds*

$$\frac{p_1 f(a_1) + p_2 f(a_2) + \dots + p_n f(a_n)}{p_1 + p_2 + \dots + p_n} \geq f\left(\frac{p_1 a_1 + p_2 a_2 + \dots + p_n a_n}{p_1 + p_2 + \dots + p_n}\right).$$

For  $p_1 = p_2 = \dots = p_n$ , Jensen's inequality becomes

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq n f\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right).$$

We can extend Jensen's inequality for convex functions to half or partially convex functions (V. Cîrtoaje, 2004-2013).

**Half Convex Function-Theorem.** *Let  $f$  be a real function defined on an interval  $\mathbb{I}$  and convex on  $\mathbb{I}_{\geq s}$  or  $\mathbb{I}_{\leq s}$ , where  $s \in \text{int}(\mathbb{I})$ . The inequality*

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq n f\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)$$

*holds for all  $a_1, a_2, \dots, a_n \in \mathbb{I}$  satisfying  $a_1 + a_2 + \dots + a_n = ns$  if and only if*

$$f(x) + (n-1)f(y) \geq nf(s)$$

*for all  $x, y \in \mathbb{I}$  such that  $x + (n-1)y = ns$ .*

**Right Half Convex Function Theorem for Ordered Variables.** *Let  $f$  be a real function defined on an interval  $\mathbb{I}$  and convex on  $\mathbb{I}_{\geq s}$ , where  $s \in \text{int}(\mathbb{I})$ . The inequality*

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq n f\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)$$

*holds for all  $a_1, a_2, \dots, a_n \in \mathbb{I}$  satisfying  $a_1 + a_2 + \dots + a_n = ns$  and*

$$a_1 \leq a_2 \leq \dots \leq a_m \leq s, \quad m \in \{1, 2, \dots, n-1\},$$

*if and only if*

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s)$$

*for all  $x, y \in \mathbb{I}$  such that*

$$x \leq s \leq y, \quad x + (n-m)y = (1+n-m)s.$$

**Left Half Convex Function Theorem for Ordered Variables.** Let  $f$  be a real function defined on an interval  $\mathbb{I}$  and convex on  $\mathbb{I}_{\leq s}$ , where  $s \in \text{int}(\mathbb{I})$ . The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right)$$

holds for all  $a_1, a_2, \dots, a_n \in \mathbb{I}$  satisfying  $a_1 + a_2 + \cdots + a_n = ns$  and

$$a_1 \geq a_2 \geq \cdots \geq a_m \geq s, \quad m \in \{1, 2, \dots, n-1\},$$

if and only if

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s)$$

for all  $x, y \in \mathbb{I}$  such that

$$x \geq s \geq y, \quad x + (n-m)y = (1+n-m)s.$$

**Right Partially Convex Function-Theorem.** Let  $f$  be a real function defined on an interval  $\mathbb{I}$  and convex on  $[s, s_0]$ , where  $s, s_0 \in \text{int}(\mathbb{I})$ ,  $s < s_0$ . In addition,  $f$  is decreasing on  $\mathbb{I}_{\leq s_0}$  and  $f(u) \geq f(s_0)$  for  $u \in \mathbb{I}_{\geq s_0}$ . The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right)$$

holds for all  $a_1, a_2, \dots, a_n \in \mathbb{I}$  satisfying  $a_1 + a_2 + \cdots + a_n = ns$  if and only if

$$f(x) + (n-1)f(y) \geq nf(s)$$

for all  $x, y \in \mathbb{I}$  such that  $x \leq s \leq y$  and  $x + (n-1)y = ns$ .

**Left Partially Convex Function-Theorem.** Let  $f$  be a real function defined on an interval  $\mathbb{I}$  and convex on  $[s_0, s]$ , where  $s_0, s \in \mathbb{I}$ ,  $s_0 < s$ . In addition,  $f$  is increasing on  $\mathbb{I}_{\geq s_0}$  and  $f(u) \geq f(s_0)$  for  $u \in \mathbb{I}_{\leq s_0}$ . The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right)$$

holds for all  $a_1, a_2, \dots, a_n \in \mathbb{I}$  satisfying  $a_1 + a_2 + \cdots + a_n = ns$  if and only if

$$f(x) + (n-1)f(y) \geq nf(s)$$

for all  $x, y \in \mathbb{I}$  such that  $x \geq s \geq y$  and  $x + (n-1)y = ns$ .

**Right Partially Convex Function Theorem for Ordered Variables.** Let  $f$  be a real function defined on an interval  $\mathbb{I}$  and convex on  $[s, s_0]$ , where  $s, s_0 \in \mathbb{I}$ ,  $s < s_0$ . In addition,  $f$  is decreasing on  $\mathbb{I}_{\leq s_0}$  and  $f(u) \geq f(s_0)$  for  $u \in \mathbb{I}_{\geq s_0}$ . The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right)$$

holds for all  $a_1, a_2, \dots, a_n \in \mathbb{I}$  satisfying  $a_1 + a_2 + \dots + a_n = ns$  and

$$a_1 \leq a_2 \leq \dots \leq a_m \leq s, \quad m \in \{1, 2, \dots, n-1\},$$

if and only if

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s)$$

for all  $x, y \in \mathbb{I}$  such that  $x \leq s \leq y$  and  $x + (n-m)y = (1+n-m)s$ .

**Left Partially Convex Function Theorem for Ordered Variables.** *Let  $f$  be a real function defined on an interval  $\mathbb{I}$  and convex on  $[s_0, s]$ , where  $s_0, s \in \mathbb{I}$ ,  $s_0 < s$ . In addition,  $f$  is increasing on  $\mathbb{I}_{\geq s_0}$  and  $f(u) \geq f(s_0)$  for  $u \in \mathbb{I}_{\leq s_0}$ . The inequality*

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)$$

holds for all  $a_1, a_2, \dots, a_n \in \mathbb{I}$  satisfying  $a_1 + a_2 + \dots + a_n = ns$  and

$$a_1 \geq a_2 \geq \dots \geq a_m \geq s, \quad m \in \{1, 2, \dots, n-1\},$$

if and only if

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s)$$

for all  $x, y \in \mathbb{I}$  such that  $x \geq s \geq y$  and  $x + (n-m)y = (1+n-m)s$ .

In all these theorems, we may replace the hypothesis condition

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s),$$

by the equivalent condition

$$h(x, y) \geq 0 \quad \text{for all } x, y \in \mathbb{I} \text{ such that } x + (n-m)y = (1+n-m)s,$$

where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(s)}{u - s}.$$

The following theorem is also useful to prove some symmetric inequalities.

**Left Convex-Right Concave Function Theorem** (V. Cîrtoaje, 2004). *Let  $a < c$  be real numbers, let  $f$  be a continuous function on  $\mathbb{I} = [a, \infty)$ , strictly convex on  $[a, c]$  and strictly concave on  $[c, \infty)$ , and let*

$$E(a_1, a_2, \dots, a_n) = f(a_1) + f(a_2) + \dots + f(a_n).$$

If  $a_1, a_2, \dots, a_n \in \mathbb{I}$  such that

$$a_1 + a_2 + \dots + a_n = S = \text{constant},$$

then

- (a)  $E$  is minimum for  $a_1 = a_2 = \dots = a_{n-1} \leq a_n$ ;
- (b)  $E$  is maximum for either  $a_1 = a$  or  $a < a_1 \leq a_2 = \dots = a_n$ .

On the other hand, it is known the following result concerning the best upper bound of Jensen's difference.

**Best Upper Bound of Jensen's Difference-Theorem** (V. Cîrtoaje, 1990). *Let  $p_1, p_2, \dots, p_n$  be fixed positive real numbers, and let  $f$  be a convex function on a closed interval  $\mathbb{I} = [a, b]$ . If  $a_1, a_2, \dots, a_n \in \mathbb{I}$ , then Jensen's difference*

$$D = \frac{p_1 f(a_1) + p_2 f(a_2) + \dots + p_n f(a_n)}{p_1 + p_2 + \dots + p_n} - f\left(\frac{p_1 a_1 + p_2 a_2 + \dots + p_n a_n}{p_1 + p_2 + \dots + p_n}\right)$$

*is maximum when some of  $a_i$  are equal to  $a$  and the others  $a_i$  are equal to  $b$ ; that is, when all  $a_i \in \{a, b\}$ .*

#### 14. KARAMATA'S MAJORIZATION INEQUALITY

Karamata's inequality is also called the H-L-P inequality (Hardy-Littlewood-Polya inequality).

Let  $f$  be a convex function on a real interval  $\mathbb{I}$ . If a decreasingly ordered sequence

$$A = (a_1, a_2, \dots, a_n), \quad a_i \in \mathbb{I},$$

majorizes a decreasingly ordered sequence

$$B = (b_1, b_2, \dots, b_n), \quad b_i \in \mathbb{I},$$

then

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq f(b_1) + f(b_2) + \dots + f(b_n).$$

We say that a sequence  $A = (a_1, a_2, \dots, a_n)$  with  $a_1 \geq a_2 \geq \dots \geq a_n$  majorizes a sequence  $B = (b_1, b_2, \dots, b_n)$  with  $b_1 \geq b_2 \geq \dots \geq b_n$ , and write it as

$$A \succ B,$$

if

$$\begin{aligned} a_1 &\geq b_1, \\ a_1 + a_2 &\geq b_1 + b_2, \\ &\dots\dots\dots \\ a_1 + a_2 + \dots + a_{n-1} &\geq b_1 + b_2 + \dots + b_{n-1}, \\ a_1 + a_2 + \dots + a_n &= b_1 + b_2 + \dots + b_n. \end{aligned}$$

#### 15. POPOVICIU'S INEQUALITY

**Theorem.** *If  $f$  is a convex function on a real interval  $\mathbb{I}$  and  $a_1, a_2, \dots, a_n \in \mathbb{I}$ , then*

$$\begin{aligned} f(a_1) + f(a_2) + \dots + f(a_n) + n(n-2)f\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right) &\geq \\ &\geq (n-1)[f(b_1) + f(b_2) + \dots + f(b_n)], \end{aligned}$$

where

$$b_i = \frac{1}{n-1} \sum_{j \neq i} a_j, \quad i = 1, 2, \dots, n.$$

In the same conditions, the following similar inequality holds:

$$\begin{aligned} f(a_1) + f(a_2) + \dots + f(a_n) + \frac{n}{n-2} f\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right) &\geq \\ &\geq \frac{2}{n-2} \sum_{1 \leq i < j \leq n} f\left(\frac{a_i + a_j}{2}\right). \end{aligned}$$

## 16. SQUARE PRODUCT INEQUALITY

Let  $a, b, c$  be real numbers, and let

$$\begin{aligned} p &= a + b + c, \quad q = ab + bc + ca, \quad r = abc, \\ s &= \sqrt{p^2 - 3q} = \sqrt{a^2 + b^2 + c^2 - ab - bc - ca}. \end{aligned}$$

From the identity

$$\begin{aligned} (a-b)^2(b-c)^2(c-a)^2 &= -27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3 \\ &= \frac{4(p^2 - 3q)^3 - (2p^3 - 9pq + 27r)^2}{27}, \end{aligned}$$

it follows that

$$\frac{-2p^3 + 9pq - 2(p^2 - 3q)\sqrt{p^2 - 3q}}{27} \leq r \leq \frac{-2p^3 + 9pq + 2(p^2 - 3q)\sqrt{p^2 - 3q}}{27},$$

which is equivalent to

$$\frac{p^3 - 3ps^2 - 2s^3}{27} \leq r \leq \frac{p^3 - 3ps^2 + 2s^3}{27}.$$

Therefore, for constant  $p$  and  $q$ , the product  $r$  is minimum and maximum when two of  $a, b, c$  are equal.

## 17. SYMMETRIC INEQUALITIES OF DEGREE THREE, FOUR OR FIVE

**Theorem** (V. Cîrtoaje, 2012). *Let  $f_n(a, b, c)$  be a symmetric homogeneous polynomial of degree  $n$ .*

(a) *The inequality  $f_4(a, b, c) \geq 0$  holds for all real numbers  $a, b, c$  if and only if  $f_4(a, 1, 1) \geq 0$  for all real  $a$ ;*

(b) *For  $n \in \{3, 4, 5\}$ , the inequality  $f_n(a, b, c) \geq 0$  holds for all  $a, b, c \geq 0$  if and only if  $f_n(a, 1, 1) \geq 0$  and  $f_n(0, b, c) \geq 0$  for all  $a, b, c \geq 0$ .*



## 18. SYMMETRIC HOMOGENEOUS INEQUALITIES OF DEGREE SIX

Any sixth degree symmetric homogeneous polynomial  $f_6(a, b, c)$  can be written in the form

$$f_6(a, b, c) = Ar^2 + B(p, q)r + C(p, q),$$

where  $A$  is called *the highest coefficient* of  $f_6$ , and

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

**Theorem** (V. Cîrtoaje, 2008). *Let  $f_6(a, b, c)$  be a sixth degree symmetric homogeneous polynomial having the highest coefficient  $A \leq 0$ .*

(a) *The inequality  $f_6(a, b, c) \geq 0$  holds for all real numbers  $a, b, c$  if and only if  $f_6(a, 1, 1) \geq 0$  for all real  $a$ ;*

(b) *The inequality  $f_6(a, b, c) \geq 0$  holds for all  $a, b, c \geq 0$  if and only if  $f_6(a, 1, 1) \geq 0$  and  $f_6(0, b, c) \geq 0$  for all  $a, b, c \geq 0$ .*

This theorem is also valid for the case where  $B(p, q)$  and  $C(p, q)$  are homogeneous rational functions.

For  $A > 0$ , we can use the *highest coefficient cancellation method* (V. Cîrtoaje, 2008). This method consists in finding some suitable real numbers  $B$ ,  $C$  and  $D$  such that the following sharper inequality holds

$$f_6(a, b, c) \geq A \left( r + Bp^3 + Cpq + D\frac{q^2}{p} \right)^2.$$

Because the function  $g_6$  defined by

$$g_6(a, b, c) = f_6(a, b, c) - A \left( r + Bp^3 + Cpq + D\frac{q^2}{p} \right)^2$$

has the highest coefficient equal to zero, we can prove the inequality  $g_6(a, b, c) \geq 0$  using Theorem above.

Notice that sometimes it is useful to break the problem into two parts,  $p^2 \leq \xi q$  and  $p^2 > \xi q$ , where  $\xi$  is a suitable real number.

A symmetric homogeneous polynomial of degree six in three variables has the form

$$\begin{aligned} f_6(a, b, c) = & A_1 \sum a^6 + A_2 \sum ab(a^4 + b^4) + A_3 \sum a^2b^2(a^2 + b^2) \\ & + A_4 \sum a^3b^3 + A_5abc \sum a^3 + A_6abc \sum ab(a + b) + 3A_7a^2b^2c^2, \end{aligned}$$

where  $A_1, \dots, A_7$  are real constants. In order to write this polynomial as a function of  $p$ ,  $q$  and  $r$ , the following relations are useful:

$$\begin{aligned}
\sum a^3 &= 3r + p^3 - 3pq, \\
\sum ab(a+b) &= -3r + pq, \\
\sum a^3b^3 &= 3r^2 - 3pqr + q^3, \\
\sum a^2b^2(a^2+b^2) &= -3r^2 - 2(p^3 - 2pq)r + p^2q^2 - 2q^3, \\
\sum ab(a^4+b^4) &= -3r^2 - 2(p^3 - 7pq)r + p^4q - 4p^2q^2 + 2q^3, \\
\sum a^6 &= 3r^2 + 6(p^3 - 2pq)r + p^6 - 6p^4q + 9p^2q^2 - 2q^3, \\
(a-b)^2(b-c)^2(c-a)^2 &= -27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3.
\end{aligned}$$

According to these relations, the highest coefficient  $A$  of the polynomial  $f_6(a, b, c)$  is

$$A = 3(A_1 - A_2 - A_3 + A_4 + A_5 - A_6 + A_7).$$

The polynomials

$$\begin{aligned}
P_1(a, b, c) &= \sum (A_1a^2 + A_2bc)(B_1a^2 + B_2bc)(C_1a^2 + C_2bc), \\
P_2(a, b, c) &= \sum (A_1a^2 + A_2bc)(B_1b^2 + B_2ca)(C_1c^2 + C_2ab)
\end{aligned}$$

and

$$P_3(a, b, c) = (A_1a^2 + A_2bc)(A_1b^2 + A_2ca)(A_1c^2 + A_2ab)$$

has the highest coefficients

$$P_1(1, 1, 1), \quad P_2(1, 1, 1), \quad P_3(1, 1, 1),$$

respectively. The polynomial

$$P_4(a, b, c) = (a^2 + mab + b^2)(b^2 + mbc + c^2)(c^2 + mca + a^2)$$

has the highest coefficient

$$A = (m - 1)^3.$$

## 19. EQUAL VARIABLES METHOD

The Equal Variables Theorem (EV-Theorem) for nonnegative real variables has the following statement (V. Cîrtoaje, 2005).

**EV-Theorem** (for nonnegative variables). Let  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) be fixed nonnegative real numbers, and let  $x_1 \leq x_2 \leq \dots \leq x_n$  be nonnegative real variables such that

$$x_1 + x_2 + \dots + x_n = a_1 + a_2 + \dots + a_n,$$

$$x_1^k + x_2^k + \dots + x_n^k = a_1^k + a_2^k + \dots + a_n^k,$$

where  $k$  is a real number; for  $k = 0$ , assume that  $x_1 x_2 \dots x_n = a_1 a_2 \dots a_n > 0$ . Let  $f : \mathbb{I} \rightarrow \mathbb{R}$ , where  $\mathbb{I} = [0, \infty)$  when  $f$  is continuous at  $x = 0$ , and  $\mathbb{I} = (0, \infty)$  when  $f(0_+) = \pm\infty$ . In addition,  $f$  is differentiable on  $(0, \infty)$  and the associated function  $g : (0, \infty) \rightarrow \mathbb{R}$  defined by

$$g(x) = f' \left( x^{\frac{1}{k-1}} \right)$$

is strictly convex on  $(0, \infty)$ . Let

$$S_n = f(x_1) + f(x_2) + \dots + f(x_n).$$

(1) If  $k \leq 0$ , then  $S_n$  is maximum for

$$x_1 = x_2 = \dots = x_{n-1} \leq x_n,$$

and is minimum for

$$0 < x_1 \leq x_2 = x_3 = \dots = x_n;$$

(2) If  $k > 0$  and either  $f$  is continuous at  $x = 0$  or  $f(0_+) = -\infty$ , then  $S_n$  is maximum for

$$x_1 = x_2 = \dots = x_{n-1} \leq x_n;$$

(3) If  $k > 0$  and either  $f$  is continuous at  $x = 0$  or  $f(0_+) = \infty$ , then  $S_n$  is minimum for

$$x_1 = \dots = x_{j-1} = 0, \quad x_{j+1} = \dots = x_n, \quad j \in \{1, 2, \dots, n\}.$$

For  $f(x) = x^m$ , we get the following corollary.

**EV-Corollary** (for nonnegative variables). Let  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) be fixed nonnegative real numbers, let  $x_1 \leq x_2 \leq \dots \leq x_n$  be nonnegative real variables such that

$$x_1 + x_2 + \dots + x_n = a_1 + a_2 + \dots + a_n,$$

$$x_1^k + x_2^k + \dots + x_n^k = a_1^k + a_2^k + \dots + a_n^k,$$

and let

$$S_n = x_1^m + x_2^m + \dots + x_n^m.$$

**Case 1:**  $k \leq 0$  (for  $k = 0$ , assume that  $x_1 x_2 \dots x_n = a_1 a_2 \dots a_n > 0$ ).

(a) If  $m \in (k, 0) \cup (1, \infty)$ , then  $S_n$  is maximum for

$$x_1 = x_2 = \dots = x_{n-1} \leq x_n,$$

and is minimum for

$$x_1 \leq x_2 = x_3 = \cdots = x_n;$$

(b) If  $m \in (-\infty, k) \cup (0, 1)$ , then  $S_n$  is minimum for

$$x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and is maximum for

$$x_1 \leq x_2 = x_3 = \cdots = x_n.$$

**Case 2:**  $0 < k < 1$ .

(a) If  $m \in (0, k) \cup (1, \infty)$ , then  $S_n$  is maximum for

$$x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and is minimum for

$$x_1 = \cdots = x_{j-1} = 0, \quad x_{j+1} = \cdots = x_n, \quad j \in \{1, 2, \dots, n\};$$

(b) If  $m \in (-\infty, 0) \cup (k, 1)$ , then  $S_n$  is minimum for

$$x_1 = x_2 = \cdots = x_{n-1} \leq x_n;$$

(c) If  $m \in (k, 1)$ , then  $S_n$  is maximum for

$$x_1 = \cdots = x_{j-1} = 0, \quad x_{j+1} = \cdots = x_n, \quad j \in \{1, 2, \dots, n\}.$$

**Case 3:**  $k > 1$ .

(a) If  $m \in (0, 1) \cup (k, \infty)$ , then  $S_n$  is maximum for

$$x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and is minimum for

$$x_1 = \cdots = x_{j-1} = 0, \quad x_{j+1} = \cdots = x_n, \quad j \in \{1, 2, \dots, n\};$$

(b) If  $m \in (-\infty, 0) \cup (1, k)$ , then  $S_n$  is minimum for

$$0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n;$$

(c) If  $m \in (1, k)$ , then  $S_n$  is maximum for

$$x_1 = \cdots = x_{j-1} = 0, \quad x_{j+1} = \cdots = x_n, \quad j \in \{1, 2, \dots, n\}.$$

The Equal Variables Theorem (EV-Theorem) for real variables has the following statement (V. Cîrtoaje, 2014).

**EV-Theorem** (for real variables). Let  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) be fixed real numbers, let  $x_1 \leq x_2 \leq \dots \leq x_n$  be real variables such that

$$x_1 + x_2 + \dots + x_n = a_1 + a_2 + \dots + a_n,$$

$$x_1^k + x_2^k + \dots + x_n^k = a_1^k + a_2^k + \dots + a_n^k,$$

where  $k$  is an even positive integer, and let  $f$  be a differentiable function on  $\mathbb{R}$  such that the associated function  $g : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$g(x) = f' \left( {}^{k-1}\sqrt{x} \right)$$

is strictly convex on  $\mathbb{R}$ . Then, the sum

$$S_n = f(x_1) + f(x_2) + \dots + f(x_n)$$

is minimum for

$$x_2 = x_3 = \dots = x_n,$$

and is maximum for

$$x_1 = x_2 = \dots = x_{n-1}.$$

For  $n = 3$ , the following results are valid.

**Theorem 1.** Let  $a \geq b \geq c$  be real numbers such that

$$a + b + c = p, \quad ab + bc + ca = q,$$

where  $p$  and  $q$  are fixed real numbers satisfying  $p^2 \geq 3q$ . The product

$$r = abc$$

is minimum when  $a = b$ , and is maximum when  $b = c$ .

**Theorem 2.** Let  $a \geq b \geq c$  such that

$$a + b + c = p, \quad ab + bc + ca = q,$$

where  $p$  and  $q$  are fixed real numbers satisfying  $p^2 \geq 3q$ .

(a) If  $a, b, c$  are nonnegative real numbers, then the product  $r = abc$  is maximum when  $b = c$ , and is minimum when  $a = b$  or  $c = 0$ ;

(b) If  $a, b, c$  are the lengths of the sides of a triangle (non-degenerate or degenerate), then the product  $r = abc$  is maximum when  $b = c \geq \frac{a}{2}$  or  $b + c = a$ , and is minimum when  $a = b \geq c$ .

**Theorem 3.** Let  $a \geq b \geq c$  be positive real numbers such that

$$a + b + c = p, \quad abc = r,$$

where  $p$  and  $r$  are fixed positive numbers satisfying  $p^3 \geq 27r$ . Then,

$$q = ab + bc + ca$$

is minimum when  $b = c$ , and is maximum when  $a = b$ .

**Theorem 4.** Let  $a \geq 1 \geq b \geq c \geq 0$  such that

$$a + b + c = 3, \quad ab + bc + ca = q,$$

where  $q \in [0, 3]$  is a fixed number. The product  $r = abc$  is minimum when  $b = 1$  or  $c = 0$ , and maximum when  $b = c$ .

**Theorem 5.** Let  $a \geq b \geq 1 \geq c \geq 0$  such that

$$a + b + c = 3, \quad ab + bc + ca = q,$$

where  $q \in [0, 3]$  is a fixed number. The product  $r = abc$  is minimum when  $a = b$  or  $c = 0$ , and maximum when  $b = 1$ .

## 20. ARITHMETIC COMPENSATION METHOD

The Arithmetic Compensation Theorem (AC-Theorem) has the following statement (V. Cîrtoaje, 2005).

**AC-Theorem.** Let  $s > 0$  and let  $F$  be a symmetric continuous function on the compact set in  $\mathbb{R}^n$

$$S = \{(x_1, x_2, \dots, x_n) : x_1 + x_2 + \dots + x_n = s, x_i \geq 0, i = 1, 2, \dots, n\}.$$

If

$$\begin{aligned} & F(x_1, x_2, x_3, \dots, x_n) \geq \\ & \geq \min \left\{ F\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, \dots, x_n\right), F(0, x_1 + x_2, x_3, \dots, x_n) \right\} \end{aligned}$$

for all  $(x_1, x_2, \dots, x_n) \in S$ , then  $F(x_1, x_2, x_3, \dots, x_n)$  is minimum when

$$x_1 = x_2 = \dots = x_k = \frac{s}{k}, \quad x_{k+1} = \dots = x_n = 0;$$

that is,

$$F(x_1, x_2, x_3, \dots, x_n) \geq \min_{1 \leq k \leq n} F\left(\frac{s}{k}, \dots, \frac{s}{k}, 0, \dots, 0\right)$$

for all  $(x_1, x_2, \dots, x_n) \in S$ .

Notice that if

$$F(x_1, x_2, x_3, \dots, x_n) < F\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, \dots, x_n\right)$$

involves

$$F(x_1, x_2, x_3, \dots, x_n) \geq F(0, x_1 + x_2, x_3, \dots, x_n),$$

then the hypothesis

$$\begin{aligned} & F(x_1, x_2, x_3, \dots, x_n) \geq \\ & \geq \min \left\{ F\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, \dots, x_n\right), F(0, x_1 + x_2, x_3, \dots, x_n) \right\} \end{aligned}$$

is satisfied.

## 21. VASC'S CYCLIC INEQUALITY

The following theorem gives Vasc's cyclic inequality (*Vasile Cîrtoaje*, 1991).

**Theorem 1.** *If  $a, b, c$  are real numbers, then*

$$(a^2 + b^2 + c^2)^2 \geq 3(a^3b + b^3c + c^3a),$$

*with equality for  $a = b = c$ , and also for*

$$\frac{a}{\sin^2 \frac{4\pi}{7}} = \frac{b}{\sin^2 \frac{2\pi}{7}} = \frac{c}{\sin^2 \frac{\pi}{7}}$$

*(or any cyclic permutation).*

A generalization of Vasc's inequality is the following (*V. Cîrtoaje*, 2008).

**Theorem 2.** *Let*

$$f_4(a, b, c) = \sum a^4 + A \sum a^2b^2 + Babc \sum a + C \sum a^3b + D \sum ab^3,$$

*where  $A, B, C, D$  are real constants such that*

$$1 + A + B + C + D = 0.$$

*The inequality  $f_4(a, b, c) \geq 0$  holds for all real numbers  $a, b, c$  if and only if*

$$3(1 + A) \geq C^2 + CD + D^2.$$

Notice that

$$\begin{aligned} \frac{4}{S} f_4(a, b, c) &= (U + V + C + D)^2 + 3 \left( U - V + \frac{C - D}{3} \right)^2 \\ &+ \frac{4}{3} (3 + 3A - C^2 - CD - D^2), \end{aligned}$$

where

$$S = \sum a^2b^2 - \sum a^2bc,$$

$$U = \frac{\sum a^3b - \sum a^2bc}{S},$$

$$V = \frac{\sum ab^3 - \sum a^2bc}{S}.$$

For  $A = B = 0$ ,  $C = -2$  and  $D = 1$ , we get the following inequality

$$a^4 + b^4 + c^4 + ab^3 + bc^3 + ca^3 \geq 2(a^3b + b^3c + c^3a),$$

with equality for  $a = b = c$ , and also for

$$\frac{a}{\sin \frac{\pi}{9}} = \frac{b}{\sin \frac{7\pi}{9}} = \frac{c}{\sin \frac{13\pi}{9}}$$

(or any cyclic permutation).

## 22. CYCLIC INEQUALITIES OF DEGREE THREE AND FOUR

Consider the third degree cyclic homogeneous polynomial

$$f_3(a, b, c) = \sum a^3 + Babc + C \sum a^2b + D \sum ab^2,$$

where  $B, C, D$  are real constants. The following theorem holds.

**Theorem 1** (Pham Kim Hung, 2007). *The cyclic inequality  $f_3(a, b, c) \geq 0$  holds for all nonnegative numbers  $a, b, c$  if and only if*

$$f_3(1, 1, 1) \geq 0$$

and

$$f_3(a, 1, 0) \geq 0$$

for all  $a \geq 0$ .

Consider now the fourth degree cyclic homogeneous polynomial

$$f_4(a, b, c) = \sum a^4 + A \sum a^2b^2 + Babc \sum a + C \sum a^3b + D \sum ab^3,$$

where  $A, B, C, D$  are real constants.

The following theorem states the necessary and sufficient conditions such that  $f_4(a, b, c) \geq 0$  for all real numbers  $a, b, c$ .

**Theorem 2** (V. Cîrtoaje and Y. Zhou, 2011). *The inequality  $f_4(a, b, c) \geq 0$  holds for all real numbers  $a, b, c$  if and only if  $g_4(t) \geq 0$  for all  $t \geq 0$ , where*

$$g_4(t) = 3(2 + A - C - D)t^4 - Ft^3 + 3(4 - B + C + D)t^2 + 1 + A + B + C + D,$$

$$F = \sqrt{27(C - D)^2 + E^2},$$

$$E = 8 - 4A + 2B - C - D.$$



The following theorem states some strong sufficient conditions such that  $f_4(a, b, c) \geq 0$  for all real numbers  $a, b, c$ .

**Theorem 3** (V. Cîrtoaje and Y. Zhou, 2011). *The inequality  $f_4(a, b, c) \geq 0$  holds for all real numbers  $a, b, c$  if the following two conditions are satisfied:*

- (a)  $1 + A + B + C + D \geq 0$ ;
- (b) *there exists a real number  $t \in (-\sqrt{3}, \sqrt{3})$  such that  $f(t) \geq 0$ , where*

$$f(t) = 2Gt^3 - (6 + 2A + B + 3C + 3D)t^2 + 2(1 + C + D)Gt + H,$$

$$G = \sqrt{1 + A + B + C + D}, \quad H = 2 + 2A - B - C - D - C^2 - CD - D^2.$$

The following theorem states the necessary and sufficient conditions that  $f_4(a, b, c) \geq 0$  for all  $a, b, c \geq 0$ .

**Theorem 4** (V. Cîrtoaje, 2012). *Let*

$$E = 8 - 4A + 2B - C - D,$$

$$F = \sqrt{27(C - D)^2 + E^2},$$

$$g_4(t) = 3(2 + A - C - D)t^4 - Ft^3 + 3(4 - B + C + D)t^2 + 1 + A + B + C + D,$$

$$g_3(t) = \frac{2E}{F}t^3 + 3t^2 - 1.$$

*For  $F = 0$ , the inequality  $f_4(a, b, c) \geq 0$  holds for all  $a, b, c \geq 0$  if and only if  $g_4(t) \geq 0$  for all  $t \in [0, 1]$ .*

*For  $F \neq 0$ , the inequality  $f_4(a, b, c) \geq 0$  holds for all  $a, b, c \geq 0$  if and only if the following two conditions are satisfied:*

- (a)  $g_4(t) \geq 0$  for all  $t \in [0, t_1]$ , where  $t_1 \in [1/2, 1]$  such that  $g_3(t_1) = 0$ ;
- (b)  $f_4(a, 1, 0) \geq 0$  for all  $a \geq 0$ .

The following theorem states some strong sufficient conditions that  $f_4(a, b, c) \geq 0$  for all  $a, b, c \geq 0$ .

**Theorem 5** (V. Cîrtoaje and Y. Zhou, 2012). *The inequality  $f_4(a, b, c) \geq 0$  holds for all  $a, b, c \geq 0$  if*

$$1 + A + B + C + D \geq 0$$

*and one of the following two conditions is satisfied:*

- (a)  $3(1 + A) \geq C^2 + CD + D^2$ ;
- (b)  $3(1 + A) < C^2 + CD + D^2$ , and there exists  $t \geq 0$  such that

$$(C + 2D)t^2 + 6t + 2C + D \geq 2\sqrt{(t^4 + t^2 + 1)(C^2 + CD + D^2 - 3 - 3A)}.$$

### 23. VASC'S POWER EXPONENTIAL INEQUALITY

**Theorem.** *Let  $0 < k \leq e$ .*

(a) *If  $a, b > 0$ , then (Vasile Cîrtoaje, 2006)*

$$a^{ka} + b^{kb} \geq a^{kb} + b^{ka};$$

(b) *If  $a, b \in (0, 1]$ , then (Vasile Cîrtoaje, 2010)*

$$2\sqrt{a^{ka}b^{kb}} \geq a^{kb} + b^{ka}.$$

# Chapter 2

## Symmetric Polynomial Inequalities in Real Variables

### 2.1 Application

**2.1.** Let  $a, b, c, d$  be real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 9.$$

Prove that

$$a^3 + b^3 + c^3 + d^3 \leq 27.$$

**2.2.** If  $a, b, c$  are real numbers such that

$$a + b + c = 0,$$

then

$$(2a^2 + bc)(2b^2 + ca)(2c^2 + ab) \leq 0.$$

**2.3.** Let  $a, b, c$  be real numbers such that

$$a + b \geq 0, \quad b + c \geq 0, \quad c + a \geq 0.$$

Prove that

$$9(a + b)(b + c)(c + a) \geq 8(a + b + c)(ab + bc + ca).$$

**2.4.** Let  $a, b, c$  be real numbers such that

$$ab + bc + ca = 3.$$

Prove that

$$(3a^2 + 1)(3b^2 + 1)(3c^2 + 1) \geq 64.$$

When does equality hold?

**2.5.** If  $a$  and  $b$  are real numbers, then

$$3(1 - a + a^2)(1 - b + b^2) \geq 2(1 - ab + a^2b^2).$$

**2.6.** If  $a, b, c$  are real numbers, then

$$3(1 - a + a^2)(1 - b + b^2)(1 - c + c^2) \geq 1 + abc + a^2b^2c^2.$$

**2.7.** If  $a, b, c$  are real numbers, then

$$(a^2 + b^2 + c^2)^3 \geq (a + b + c)(ab + bc + ca)(a^3 + b^3 + c^3).$$

**2.8.** If  $a, b, c$  are real numbers, then

$$2(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \geq [ab(a + b) + bc(b + c) + ca(c + a) - 2abc]^2.$$

**2.9.** If  $a, b, c$  are real numbers, then

$$2(a^2 + 1)(b^2 + 1)(c^2 + 1) \geq (ab + bc + ca + 1)^2.$$

**2.10.** If  $a, b, c$  are real numbers, then

$$(a^2 + 1)(b^2 + 1)(c^2 + 1) \geq \frac{5}{16}(a + b + c + 1)^2.$$

**2.11.** If  $a, b, c$  are real numbers, then

$$(a) \quad a^6 + b^6 + c^6 - 3a^2b^2c^2 + 2(a^2 + bc)(b^2 + ca)(c^2 + ab) \geq 0;$$

$$(b) \quad a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq (a^2 - 2bc)(b^2 - 2ca)(c^2 - 2ab).$$

**2.12.** If  $a, b, c$  are real numbers, then

$$\frac{2}{3}(a^6 + b^6 + c^6) + a^3b^3 + b^3c^3 + c^3a^3 + abc(a^3 + b^3 + c^3) \geq 0.$$

**2.13.** If  $a, b, c$  are real numbers, then

$$4(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq (a - b)^2(b - c)^2(c - a)^2.$$

**2.14.** If  $a, b, c$  are real numbers, then

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq 3(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2).$$

**2.15.** If  $a, b, c$  are real numbers such that  $abc > 0$ , then

$$4\left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right)\left(c + \frac{1}{c}\right) \geq 9(a + b + c).$$

**2.16.** If  $a, b, c$  are real numbers, then

- (a)  $(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) \leq (a^2 + b^2 + c^2)(ab + bc + ca)^2;$
- (b)  $(2a^2 + bc)(2b^2 + ca)(2c^2 + ab) \leq (a + b + c)^2(a^2b^2 + b^2c^2 + c^2a^2).$

**2.17.** If  $a, b, c$  are real numbers such that

$$ab + bc + ca \geq 0,$$

then

$$27(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) \leq (a + b + c)^6.$$

**2.18.** If  $a, b, c$  are real numbers such that

$$a^2 + b^2 + c^2 = 2,$$

then

$$(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) + 2 \geq 0.$$

**2.19.** If  $a, b, c$  are real numbers such that

$$a + b + c = 3,$$

then

$$3(a^4 + b^4 + c^4) + a^2 + b^2 + c^2 + 6 \geq 6(a^3 + b^3 + c^3).$$

**2.20.** If  $a, b, c$  are real numbers such that

$$abc = 1,$$

then

$$3(a^2 + b^2 + c^2) + 2(a + b + c) \geq 5(ab + bc + ca).$$

**2.21.** If  $a, b, c$  are real numbers such that

$$abc = 1,$$

then

$$a^2 + b^2 + c^2 + 6 \geq \frac{3}{2} \left( a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right).$$

**2.22.** If  $a, b, c$  are real numbers, then

$$(1 + a^2)(1 + b^2)(1 + c^2) + 8abc \geq \frac{1}{4}(1 + a)^2(1 + b)^2(1 + c)^2.$$

**2.23.** Let  $a, b, c$  be real numbers such that

$$a + b + c = 0.$$

Prove that

$$a^{12} + b^{12} + c^{12} \geq \frac{2049}{8} a^4 b^4 c^4.$$

**2.24.** If  $a, b, c$  are real numbers such that  $abc \geq 0$ , then

$$a^2 + b^2 + c^2 + 2abc + 4 \geq 2(a + b + c) + ab + bc + ca.$$

**2.25.** Let  $a, b, c$  be real numbers such that

$$a + b + c = 3.$$

(a) If  $a, b, c \geq -3$ , then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

(b) If  $a, b, c \geq -7$ , then

$$\frac{1-a}{(1+a)^2} + \frac{1-b}{(1+b)^2} + \frac{1-c}{(1+c)^2} \geq 0.$$

**2.26.** If  $a, b, c$  are real numbers, then

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$

**2.27.** If  $a, b, c$  are real numbers, then

$$\left(\frac{a^2 + b^2 + c^2}{3}\right)^3 \geq a^2b^2c^2 + \frac{1}{16}(a-b)^2(b-c)^2(c-a)^2.$$

**2.28.** If  $a, b, c$  are real numbers, then

$$(a^2 + b^2 + c^2)^3 \geq \frac{108}{5}a^2b^2c^2 + 2(a-b)^2(b-c)^2(c-a)^2.$$

**2.29.** If  $a, b, c$  are real numbers, then

$$2(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \geq (a-b)^2(b-c)^2(c-a)^2.$$

**2.30.** If  $a, b, c$  are real numbers, then

$$32(a^2 + bc)(b^2 + ca)(c^2 + ab) + 9(a-b)^2(b-c)^2(c-a)^2 \geq 0.$$

**2.31.** If  $a, b, c$  are real numbers, then

$$a^4(b-c)^2 + b^4(c-a)^2 + c^4(a-b)^2 \geq \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$

**2.32.** If  $a, b, c$  are real numbers, then

$$a^2(b-c)^4 + b^2(c-a)^4 + c^2(a-b)^4 \geq \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$

**2.33.** If  $a, b, c$  are real numbers, then

$$a^2(b^2 - c^2)^2 + b^2(c^2 - a^2)^2 + c^2(a^2 - b^2)^2 \geq \frac{3}{8}(a-b)^2(b-c)^2(c-a)^2.$$

**2.34.** If  $a, b, c$  are real numbers such that

$$ab + bc + ca = 3,$$

then

$$(a) \quad (a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq 3(a + b + c)^2;$$

$$(b) \quad (a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq \frac{3}{2}(a^2 + b^2 + c^2).$$

**2.35.** If  $a, b, c$  are real numbers, then

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq 3(ab + bc + ca)(a^2b^2 + b^2c^2 + c^2a^2).$$

**2.36.** If  $a, b, c$  are real numbers, not all of the same sign, then

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq 3(ab + bc + ca)^3.$$

**2.37.** If  $a, b, c$  are real numbers, then

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq \frac{3}{8}(a^2 + b^2)(b^2 + c^2)(c^2 + a^2).$$

**2.38.** If  $a, b, c$  are real numbers, then

$$2(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \geq (a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2).$$



**2.39.** If  $a, b, c$  are real numbers, then

$$9(1 + a^4)(1 + b^4)(1 + c^4) \geq 8(1 + abc + a^2b^2c^2)^2.$$

**2.40.** If  $a, b, c$  are real numbers, then

$$2(1 + a^2)(1 + b^2)(1 + c^2) \geq (1 + a)(1 + b)(1 + c)(1 + abc).$$

**2.41.** Let  $a, b, c$  be real numbers. Prove that

$$(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) \geq 1$$

for

$$(a) \quad a^3b^3 + b^3c^3 + c^3a^3 = 3;$$

$$(b) \quad a^4b^4 + b^4c^4 + c^4a^4 = 3.$$

**2.42.** If  $a, b, c$  are nonzero real numbers, then

$$\sum \frac{b^2 - bc + c^2}{a^2} + 2 \sum \frac{a^2}{bc} \geq \left( \sum a \right) \left( \sum \frac{1}{a} \right).$$

**2.43.** Let  $a, b, c$  be real numbers. Prove that

(a) if  $a, b, c \in [0, 1]$ , then

$$abc - (b + c - a)(c + a - b)(a + b - c) \leq 1;$$

(b) if  $a, b, c \in [-1, 1]$ , then

$$abc - (b + c - a)(c + a - b)(a + b - c) \leq 4.$$

**2.44.** Let  $a, b, c$  be real numbers. Prove that

(a) if  $a, b, c \in [0, 1]$ , then

$$\sum a^2(a - b)(a - c) \leq 1;$$

(b) if  $a, b, c \in [-1, 1]$ , then

$$\sum a^2(a - b)(a - c) \leq 4.$$

**2.45.** Let  $a, b, c$  be real numbers such that

$$ab + bc + ca = abc + 2.$$

Prove that

$$a^2 + b^2 + c^2 - 3 \geq (2 + \sqrt{3})(a + b + c - 3).$$

**2.46.** Let  $a, b, c$  be real numbers such that

$$(a + b)(b + c)(c + a) = 10.$$

Prove that

$$(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) + 12a^2b^2c^2 \geq 30.$$

**2.47.** Let  $a, b, c$  be real numbers such that

$$(a + b)(b + c)(c + a) = 5.$$

Prove that

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) + 12a^2b^2c^2 \geq 15.$$

**2.48.** Let  $a, b, c$  be real numbers such that

$$a + b + c = 1, \quad a^3 + b^3 + c^3 = k.$$

Prove that

- (a) if  $k = 25$ , then  $|a| \leq 1$  or  $|b| \leq 1$  or  $|c| \leq 1$ ;
- (b) if  $k = -11$ , then  $1 < a \leq 2$  or  $1 < b \leq 2$  or  $1 < c \leq 2$ .

**2.49.** Let  $a, b, c$  be real numbers such that

$$a + b + c = a^3 + b^3 + c^3 = 2.$$

Prove that  $a, b, c \notin \left[\frac{5}{4}, 2\right]$ .

**2.50.** If  $a, b, c$  and  $k$  are real numbers, then

$$\sum (a - b)(a - c)(a - kb)(a - kc) \geq 0.$$

**2.51.** If  $a, b, c$  are real numbers, then

$$\sum a^2(a-b)(a-c) \geq \frac{(a-b)^2(b-c)^2(c-a)^2}{a^2+b^2+c^2+ab+bc+ca}.$$

**2.52.** If  $a, b, c$  are real numbers, then

$$3(a^2+3)(b^2+3)(c^2+3) \geq 16(a+b+c)^2+48.$$

**2.53.** Let  $a \geq b \geq c$  be real numbers. Prove that for fixed

$$a+b+c=p, \quad ab+bc+ca=q,$$

the product

$$r=abc$$

has the minimum value only when  $a=b$ , and the maximum value only when  $b=c$ .

**2.54.** Let  $a, b, c$  be real numbers. Prove that for fixed

$$a+b+c=p, \quad abc=r,$$

the sum

$$q=ab+bc+ca$$

has the maximum value only when two of  $a, b, c$  are equal.

**2.55.** Let  $a, b, c$  be real numbers such that  $a+b+c=3$ . Prove that

- (a)  $(ab+bc+ca)^2 \geq 9abc;$
- (b)  $(ab+bc+ca)^2+9 \geq 18abc;$
- (c)  $(ab+bc+ca-3)^2 \geq 27(abc-1).$

**2.56.** Let  $a, b, c$  be real numbers such that

$$ab+bc+ca+abc=4.$$

Prove that

- (a) if  $abc > 0$ , then

$$2(a+b+c)+ab+bc+ca \leq \frac{9}{abc};$$

- (b) if  $abc < 0$ , then

$$2(a+b+c)+ab+bc+ca \geq \frac{9}{abc}.$$

**2.57.** If  $a, b, c$  are real numbers such that

$$a + b + c + abc = 4,$$

then

$$a^2 + b^2 + c^2 + 3 \geq 2(ab + bc + ca).$$

**2.58.** If  $a, b, c$  are real numbers such that

$$ab + bc + ca = 3abc,$$

then

$$4(a^2 + b^2 + c^2) + 9 \geq 7(ab + bc + ca).$$

**2.59.** Let  $a, b, c \leq \frac{6}{5}$  be real numbers such that  $a^2 + b^2 + c^2 = 4$ . If

$$k = \frac{16(2 + 15\sqrt{2})}{125} \approx 2.97,$$

then

$$ab + bc + ca + k \geq abc.$$

**2.60.** Let  $f_4(a, b, c)$  be a symmetric homogeneous polynomial of degree four. Prove that the inequality  $f_4(a, b, c) \geq 0$  holds for all real numbers  $a, b, c$  if and only if  $f_4(a, 1, 1) \geq 0$  for all real  $a$ .

**2.61.** If  $a, b, c$  are real numbers, then

$$10(a^4 + b^4 + c^4) + 64(a^2b^2 + b^2c^2 + c^2a^2) \geq 33 \sum ab(a^2 + b^2).$$

**2.62.** If  $a, b, c$  are real numbers such that

$$a + b + c = 3,$$

then

$$3(a^4 + b^4 + c^4) + 33 \geq 14(a^2 + b^2 + c^2).$$

**2.63.** If  $a, b, c$  are real numbers such that

$$a^2 + b^2 + c^2 = 3,$$

then

$$a^4 + b^4 + c^4 + 3(ab + bc + ca) \leq 12.$$

**2.64.** Let  $\alpha, \beta, \gamma$  be real numbers such that

$$1 + \alpha + \beta = 2\gamma.$$

The inequality

$$\sum a^4 + \alpha \sum a^2 b^2 + \beta abc \sum a \geq \gamma \sum ab(a^2 + b^2)$$

holds for any real numbers  $a, b, c$  if and only if

$$1 + \alpha \geq \gamma^2.$$

**2.65.** If  $a, b, c$  are real numbers such that

$$a^2 + b^2 + c^2 = 2,$$

then

$$ab(a^2 - ab + b^2 - c^2) + bc(b^2 - bc + c^2 - a^2) + ca(c^2 - ca + a^2 - b^2) \leq 1.$$

**2.66.** If  $a, b, c$  are real numbers, then

$$(a + b)^4 + (b + c)^4 + (c + a)^4 \geq \frac{4}{7}(a^4 + b^4 + c^4).$$

**2.67.** Let  $a, b, c$  be real numbers. If

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc,$$

then

$$(3 - p)r + \frac{p^2 + q^2 - pq}{3} \geq q.$$

**2.68.** If  $a, b, c$  are real numbers, then

$$\frac{ab(a + b) + bc(b + c) + ca(c + a)}{(a^2 + 1)(b^2 + 1)(c^2 + 1)} \leq \frac{3}{4}.$$

**2.69.** If  $a, b, c$  are real numbers such that  $abc > 0$ , then

$$\left(a + \frac{1}{a} - 1\right) \left(b + \frac{1}{b} - 1\right) \left(c + \frac{1}{c} - 1\right) + 2 \geq \frac{1}{3}(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

**2.70.** If  $a, b, c$  are real numbers, then

$$\left(a^2 + \frac{1}{2}\right) \left(b^2 + \frac{1}{2}\right) \left(c^2 + \frac{1}{2}\right) \geq \left(a + b - \frac{1}{2}\right) \left(b + c - \frac{1}{2}\right) \left(c + a - \frac{1}{2}\right).$$

**2.71.** If  $a, b, c$  are real numbers such that

$$a + b + c = 3,$$

then

$$\frac{a(a-1)}{8a^2+9} + \frac{b(b-1)}{8b^2+9} + \frac{c(c-1)}{8c^2+9} \geq 0.$$

**2.72.** If  $a, b, c$  are real numbers such that

$$a + b + c = 3,$$

then

$$\frac{(a-11)(a-1)}{2a^2+1} + \frac{(b-11)(b-1)}{2b^2+1} + \frac{(c-11)(c-1)}{2c^2+1} \geq 0.$$

**2.73.** If  $a, b, c$  are real numbers, then

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \geq 9(ab + bc + ca).$$

**2.74.** If  $a, b, c$  are real numbers such that

$$ab + bc + ca = 3,$$

then

$$4(a^4 + b^4 + c^4) + 11abc(a + b + c) \geq 45.$$

**2.75.** Any sixth degree symmetric homogeneous polynomial  $f_6(a, b, c)$  can be written in the form

$$f_6(a, b, c) = Ar^2 + B(p, q)r + C(p, q),$$

where  $A$  is called *the highest coefficient* of  $f_6$ , and

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

In the case  $A \leq 0$ , prove that the inequality  $f_6(a, b, c) \geq 0$  holds for all real numbers  $a, b, c$  if and only if  $f_6(a, 1, 1) \geq 0$  for all real  $a$ .

**2.76.** If  $a, b, c$  are real numbers such that

$$ab + bc + ca = -1,$$

then

$$(a) \quad 5(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \geq 8;$$

$$(b) \quad (a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq 1.$$

**2.77.** If  $a, b, c$  are real numbers, then

$$(a) \quad \sum a^2(a - b)(a - c)(a + 2b)(a + 2c) + (a - b)^2(b - c)^2(c - a)^2 \geq 0;$$

$$(b) \quad \sum a^2(a - b)(a - c)(a - 4b)(a - 4c) + 7(a - b)^2(b - c)^2(c - a)^2 \geq 0.$$

**2.78.** If  $a, b, c$  are real numbers, then

$$(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) + (a - b)^2(b - c)^2(c - a)^2 \geq 0.$$

**2.79.** If  $a, b, c$  are real numbers, then

$$(2a^2 + 5ab + 2b^2)(2b^2 + 5bc + 2c^2)(2c^2 + 5ca + 2a^2) + (a - b)^2(b - c)^2(c - a)^2 \geq 0.$$

**2.80.** If  $a, b, c$  are real numbers, then

$$\left(a^2 + \frac{2}{3}ab + b^2\right) \left(b^2 + \frac{2}{3}bc + c^2\right) \left(c^2 + \frac{2}{3}ca + a^2\right) \geq \frac{64}{27}(a^2 + bc)(b^2 + ca)(c^2 + ab).$$

**2.81.** If  $a, b, c$  are real numbers, then

$$\sum a^2(a-b)(a-c) \geq \frac{2(a-b)^2(b-c)^2(c-a)^2}{a^2+b^2+c^2}.$$

**2.82.** If  $a, b, c$  are real numbers, then

$$\sum (a-b)(a-c)(a-2b)(a-2c) \geq \frac{8(a-b)^2(b-c)^2(c-a)^2}{a^2+b^2+c^2}.$$

**2.83.** If  $a, b, c$  are real numbers, no two of which are zero, then

$$\frac{a^2+3bc}{b^2+c^2} + \frac{b^2+3ca}{c^2+a^2} + \frac{c^2+3ab}{a^2+b^2} \geq 0.$$

**2.84.** If  $a, b, c$  are real numbers, no two of which are zero, then

$$\frac{a^2+6bc}{b^2-bc+c^2} + \frac{b^2+6ca}{c^2-ca+a^2} + \frac{c^2+6ab}{a^2-ab+b^2} \geq 0.$$

**2.85.** If  $a, b, c$  are real numbers such that

$$ab+bc+ca \geq 0,$$

then

$$\frac{4a^2+23bc}{b^2+c^2} + \frac{4b^2+23ca}{c^2+a^2} + \frac{4c^2+23ab}{a^2+b^2} \geq 0.$$

**2.86.** If  $a, b, c$  are real numbers such that

$$ab+bc+ca = 3,$$

then

$$20(a^6+b^6+c^6) + 43abc(a^3+b^3+c^3) \geq 189.$$

**2.87.** If  $a, b, c$  are real numbers such that

$$ab+bc+ca \geq 0,$$

then

$$(a) \quad (a^2+b^2+c^2)(ab+bc+ca)^2 \geq abc(4a^3+4b^3+4c^3+15abc);$$

$$(b) \quad 4(a+b+c)^6 \geq 81abc(5a^3+5b^3+5c^3+21abc).$$



**2.88.** If  $a, b, c$  are real numbers, then

$$4 \sum (a^2 + bc)(a - b)(a - c)(a - 3b)(a - 3c) \geq 7(a - b)^2(b - c)^2(c - a)^2.$$

**2.89.** Let  $a, b, c$  be real numbers such that

$$ab + bc + ca \geq 0.$$

For any real  $k$ , prove that

$$\sum 4bc(a - b)(a - c)(a - kb)(a - kc) + (a - b)^2(b - c)^2(c - a)^2 \geq 0.$$

**2.90.** If  $a, b, c$  are real numbers, then

$$[(a^2b + b^2c + c^2a) + (ab^2 + bc^2 + ca^2)]^2 \geq 4(ab + bc + ca)(a^2b^2 + b^2c^2 + c^2a^2).$$

**2.91.** If  $a, b, c$  are real numbers such that

$$a + b + c = 3,$$

then

$$\frac{(a - 1)(a - 25)}{a^2 + 23} + \frac{(b - 1)(b - 25)}{b^2 + 23} + \frac{(c - 1)(c - 25)}{c^2 + 23} \geq 0.$$

**2.92.** If  $a, b, c$  are real numbers such that  $abc \neq 0$ , then

$$\left(\frac{b + c}{a}\right)^2 + \left(\frac{c + a}{b}\right)^2 + \left(\frac{a + b}{c}\right)^2 > 2.$$

**2.93.** If  $a, b, c$  are real numbers, then

- (a)  $(a^2 + 1)(b^2 + 1)(c^2 + 1) \geq \frac{8}{3\sqrt{3}} |(a - b)(b - c)(c - a)|;$
- (b)  $(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1) \geq |(a - b)(b - c)(c - a)|.$

**2.94.** If  $a, b, c$  are real numbers such that

$$a + b + c = 3,$$

then

$$(1 - a + a^2)(1 - b + b^2)(1 - c + c^2) \geq 1.$$

**2.95.** If  $a, b, c$  are real numbers such that

$$a + b + c = 0,$$

then

$$\frac{a(a-4)}{a^2+2} + \frac{b(b-4)}{b^2+2} + \frac{c(c-4)}{c^2+2} \geq 0.$$

**2.96.** If  $a, b, c$  are real numbers such that

$$a, b, c \leq 1 + \sqrt{2}, \quad a + b + c \geq 0,$$

then

$$2abc + a^2 + b^2 + c^2 + 1 \geq 2(ab + bc + ca).$$

**2.97.** If  $a, b, c$  are real numbers such that  $a + b + c = 2$  and  $ab + bc + ca > 0$ , then

$$(a^2 + bc)(b^2 + ca)(c^2 + ab) + abc \leq 1.$$

**2.98.** If  $a, b, c$  are real numbers such that

$$a^2 + b^2 + c^2 = 3, \quad a \geq \frac{4}{3},$$

then

$$3(abc + 1) \geq 2(ab + bc + ca).$$

**2.99.** If  $a, b, c$  are real numbers such that  $a \geq \frac{8}{7}$  and  $a^2 + b^2 + c^2 = 3$ , then

$$\frac{3 - a - b - c}{1 - abc} \geq \frac{49}{100}.$$

**2.100.** If  $a, b, c \in [-1, 1]$ , then

$$a^3 + b^3 + c^3 + abc \leq \frac{15}{16}(a + b + c) + \frac{19}{16}.$$

**2.101.** If  $a, b, c$  are real numbers, then

$$(a^3 + b^3 + c^3)^2 \geq (a^4 + b^4 + c^4)(ab + bc + ca).$$

**2.102.** Let  $a_1, a_2, \dots, a_n$  be real numbers such that

$$a_1^2 + a_2^2 + \dots + a_n^2 = n.$$

Prove that:

(a) for  $n = 3$ ,

$$\frac{a_1 + a_2 + a_3}{3} + \min_{i \neq j} (a_i - a_j)^2 \leq \frac{5}{3};$$

(b) for  $n = 5$ ,

$$\frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} + \min_{i \neq j} (a_i - a_j)^2 \leq 1.$$

**2.103.** Let  $a_1, a_2, \dots, a_7$  be real numbers such that

$$a_1^2 + a_2^2 + \dots + a_7^2 = n.$$

Prove that:

$$(a) \quad \sqrt{\frac{|a_1 + a_2 + \dots + a_7|}{7}} + \min_{i \neq j} (a_i - a_j)^2 \leq 1;$$

$$(b) \quad \sqrt{\frac{|a_1 + a_2 + \dots + a_7|}{7}} + 8 \min_{i \neq j} (a_i - a_j)^2 \leq \frac{19}{8}.$$

**2.104.** Let  $f$  be a differentiable convex function on a closed interval  $\mathbb{I} = [a, b]$ . If  $a_1, a_2, \dots, a_n \in \mathbb{I}$ , then Jensen's difference

$$D = f(a_1) + f(a_2) + \dots + f(a_n) - nf\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)$$

has the maximum value when all  $a_i \in \{a, b\}$ .

**2.105.** If  $a, b, c$  are real numbers, then

$$2(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1) \geq (abc - 1)^2.$$

**2.106.** If  $a, b, c$  are real numbers, then

$$(1 + \sqrt{2})(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1) \geq a^2b^2c^2 + 1.$$

**2.107.** If  $a, b, c, d$  are real numbers, then

$$\begin{aligned} \text{(a)} \quad & 4(1 - a + a^2)(1 - b + b^2)(1 - c + c^2) \geq 1 + a^2b^2 + b^2c^2 + c^2a^2; \\ \text{(b)} \quad & (1 - a + a^2)(1 - b + b^2)(1 - c + c^2)(1 - d + d^2) \geq \left( \frac{1 + abcd}{2} \right)^2. \end{aligned}$$

**2.108.** If  $a, b, c, d$  are real numbers, then

$$3(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1) \geq a^2b^2c^2d^2 - abcd + 1.$$

**2.109.** If  $a, b, c, d$  are real numbers, then

$$(a^2 - a + 2)(b^2 - b + 2)(c^2 - c + 2)(d^2 - d + 2) \geq (a + b + c + d)^2.$$

**2.110.** If  $a, b, c, d$  are real numbers, then

$$(a^2 + 3)(b^2 + 3)(c^2 + 3)(d^2 + 3) \geq 16(a + b + c + d)^2.$$

**2.111.** Let  $a, b, c, d$  be real numbers such that  $abcd > 0$ . Prove that

$$\left(a + \frac{1}{a}\right) \left(b + \frac{1}{b}\right) \left(c + \frac{1}{c}\right) \left(d + \frac{1}{d}\right) \geq (a + b + c + d) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right).$$

**2.112.** Let  $a, b, c, d$  be real numbers such that

$$a + b + c + d = 4, \quad a^2 + b^2 + c^2 + d^2 = 7.$$

Prove that

$$a^3 + b^3 + c^3 + d^3 \leq 16.$$

**2.113.** Let  $a, b, c, d$  be real numbers such that

$$a + b + c + d = 0.$$

Prove that

$$12(a^4 + b^4 + c^4 + d^4) \leq 7(a^2 + b^2 + c^2 + d^2)^2.$$

**2.114.** Let  $a, b, c, d$  be real numbers such that

$$a + b + c + d = 0.$$

Prove that

$$(a^2 + b^2 + c^2 + d^2)^3 \geq 3(a^3 + b^3 + c^3 + d^3)^2.$$

**2.115.** If  $a, b, c, d$  are real numbers such that

$$a + b + c + d = 0,$$

then

$$a^4 + b^4 + c^4 + d^4 + 28abcd \geq 0.$$

**2.116.** If  $a, b, c, d$  are real numbers such that

$$abcd = 1,$$

then

$$(1 + a^2)(1 + b^2)(1 + c^2)(1 + d^2) \geq (a + b + c + d)^2.$$

**2.117.** Let  $a, b, c, d$  be real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

Prove that

$$(abc)^3 + (bcd)^3 + (cda)^3 + (dab)^3 \leq 4.$$

**2.118.** Let  $a, b, c, d$  be real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 1.$$

Prove that

$$(1 - a)^4 + (1 - b)^4 + (1 - c)^4 + (1 - d)^4 \geq a^4 + b^4 + c^4 + d^4.$$

**2.119.** If  $a, b, c, d \geq \frac{-1}{2}$  such that

$$a + b + c + d = 4,$$

then

$$\frac{1-a}{1-a+a^2} + \frac{1-b}{1-b+b^2} + \frac{1-c}{1-c+c^2} + \frac{1-d}{1-d+d^2} \geq 0.$$

**2.120.** If  $a, b, c, d$  are real numbers such that  $a \geq b \geq c \geq d$  and

$$a^2 + b^2 + c^2 + d^2 = 4,$$

then

$$a^2c^2 + b^2d^2 \leq 2.$$

**2.121.** Let  $a, b, c, d$  be real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

If  $a \geq b \geq c \geq d$ , then

$$1 - abcd \leq (a - d)^2.$$

**2.122.** If  $a, b, c, d, e \geq -3$  such that

$$a + b + c + d + e = 5,$$

then

$$\frac{1-a}{1+a+a^2} + \frac{1-b}{1+b+b^2} + \frac{1-c}{1+c+c^2} + \frac{1-d}{1+d+d^2} + \frac{1-e}{1+e+e^2} \geq 0.$$

**2.123.** Let  $a, b, c, d, e$  be real numbers such that

$$a + b + c + d + e = 0.$$

Prove that

$$30(a^4 + b^4 + c^4 + d^4 + e^4) \geq 7(a^2 + b^2 + c^2 + d^2 + e^2)^2.$$

**2.124.** If  $a, b, c, d, e$  are real numbers such that  $a + b + c + d + e = 5$ , then

$$(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1)(e^2 - e + 1) \geq 1.$$

**2.125.** If  $a, b, c, d, e$  are real numbers, then

$$4(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1)(e^2 - e + 1) \geq (abcde - 1)^2.$$

**2.126.** If  $a_1, a_2, \dots, a_5$  are real numbers such that

$$a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3 = 0,$$

then

$$\sum_{i < j} a_i a_j \leq 0.$$

**2.127.** Let  $a_1, a_2, \dots, a_n \geq -1$  such that

$$a_1 + a_2 + \dots + a_n = 0.$$

Prove that

$$(n-2)(a_1^2 + a_2^2 + \dots + a_n^2) \geq a_1^3 + a_2^3 + \dots + a_n^3.$$

**2.128.** Let  $a_1, a_2, \dots, a_n \geq -1$  such that

$$a_1 + a_2 + \dots + a_n = 0.$$

Prove that

$$(n-2)(a_1^2 + a_2^2 + \dots + a_n^2) + (n-1)(a_1^3 + a_2^3 + \dots + a_n^3) \geq 0.$$

**2.129.** Let  $a_1, a_2, \dots, a_n \geq n-1 - \sqrt{n^2 - n + 1}$  be nonzero real numbers such that

$$a_1 + a_2 + \dots + a_n = n.$$

Prove that

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \dots + \frac{1}{a_n^2} \geq n.$$

**2.130.** Let  $a_1, a_2, \dots, a_n \leq \frac{n}{n-2}$  be real numbers such that

$$a_1 + a_2 + \dots + a_n = n.$$

If  $k$  is a positive integer,  $k \geq 2$ , then

$$a_1^k + a_2^k + \dots + a_n^k \geq n.$$

**2.131.** If  $a_1, a_2, \dots, a_n \geq \frac{-n}{n-2}$ ,  $n \geq 3$ , then

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \dots + \frac{1}{a_n^2} \geq \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}.$$

**2.132.** If  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) are real numbers such that

$$a_1, a_2, \dots, a_n \geq \frac{-(3n-2)}{n-2}, \quad a_1 + a_2 + \dots + a_n = n,$$

then

$$\frac{1-a_1}{(1+a_1)^2} + \frac{1-a_2}{(1+a_2)^2} + \dots + \frac{1-a_n}{(1+a_n)^2} \geq 0.$$

**2.133.** Let  $a_1, a_2, \dots, a_n$  be real numbers.

(a) If  $k \geq n$ , then

$$\frac{(a_1 + a_2 + \dots + a_n + k - n)^2}{(a_1^2 + k - 1)(a_2^2 + k - 1) \dots (a_n^2 + k - 1)} \leq \frac{1}{k^{n-2}};$$

(b) If  $k \geq \frac{n}{2}$ , then

$$\frac{a_1 + a_2 + \dots + a_n + k - n}{(a_1^2 + 2k - 1)(a_2^2 + 2k - 1) \dots (a_n^2 + 2k - 1)} \leq \frac{1}{2(2k)^{n-1}};$$

$$(c) \quad \frac{(a_1 + a_2 + \dots + a_n)^2}{(a_1^2 + n - 1)(a_2^2 + n - 1) \dots (a_n^2 + n - 1)} \leq \frac{1}{n^{n-2}};$$

$$(d) \quad \frac{a_1 + a_2 + \dots + a_n}{(a_1^2 + 2n - 1)(a_2^2 + 2n - 1) \dots (a_n^2 + 2n - 1)} \leq \frac{1}{2(2n)^{n-1}}.$$

**2.134.** Let  $a_1, a_2, \dots, a_n$  be real numbers. If  $k \geq \frac{n}{4}$ , then

$$(a) \quad \frac{(a_1 + a_2 + \dots + a_n + 2k - n)^2}{(a_1^2 - a_1 + k)(a_2^2 - a_2 + k) \dots (a_n^2 - a_n + k)} \leq \frac{4}{k^{n-2}};$$

$$(b) \quad \frac{(a_1 + a_2 + \dots + a_n)^2}{(a_1^2 - a_1 + \frac{n}{2})(a_2^2 - a_2 + \frac{n}{2}) \dots (a_n^2 - a_n + \frac{n}{2})} \leq \frac{2^n}{n^{n-2}}.$$



**2.135.** Let  $a_1, a_2, \dots, a_n$  be real numbers. If  $k \geq n$ , then

$$\begin{aligned} \text{(a)} \quad & \frac{(a_1 + a_2 + \dots + a_n)^2 + n(k - n)}{(a_1^2 + k - 1)(a_2^2 + k - 1) \cdots (a_n^2 + k - 1)} \leq \frac{n}{k^{n-1}}; \\ \text{(b)} \quad & \frac{(a_1 + a_2 + \dots + a_n)^2 + n^2}{(a_1^2 + 2n - 1)(a_2^2 + 2n - 1) \cdots (a_n^2 + 2n - 1)} \leq \frac{n}{(2n)^{n-1}}. \end{aligned}$$

**2.136.** If  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) are real numbers such that

$$a_1 \leq a_2 \leq \dots \leq a_n, \quad a_1 + a_2 + \dots + a_n = 0,$$

then

$$a_1^2 + a_2^2 + \dots + a_n^2 + na_1a_n \leq 0.$$



## 2.2 Solutions

**P 2.1.** Let  $a, b, c, d$  be real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 9.$$

Prove that

$$a^3 + b^3 + c^3 + d^3 \leq 27.$$

**Solution.** From  $a^2 + b^2 + c^2 + d^2 = 9$ , we get

$$a^2 \leq 9, \quad a \leq 3, \quad a^2(a - 3) \leq 0, \quad a^3 \leq 3a^2.$$

Similarly,

$$b^3 \leq 3b^2, \quad c^3 \leq 3c^2, \quad d^3 \leq 3d^2.$$

Therefore, we have

$$a^3 + b^3 + c^3 + d^3 \leq 3(a^2 + b^2 + c^2 + d^2) = 27.$$

The equality holds for  $a = 3$  and  $b = c = d = 0$  (or any cyclic permutation thereof). □

**P 2.2.** If  $a, b, c$  are real numbers such that

$$a + b + c = 0,$$

then

$$(2a^2 + bc)(2b^2 + ca)(2c^2 + ab) \leq 0.$$

**First Solution.** Among  $a, b, c$  there are two with the same sign; assume that  $bc \geq 0$ . We need to show that

$$(2b^2 + ca)(2c^2 + ab) \leq 0.$$

This is equivalent to

$$[2b^2 - c(b + c)][2c^2 - (b + c)b] \leq 0,$$

$$(b - c)^2(2b + c)(b + 2c) \geq 0.$$

Since

$$(2b + c)(b + 2c) = 2(b^2 + c^2) + 5bc \geq 0,$$

the conclusion follows. The equality holds for  $\frac{-a}{2} = b = c$  (or any cyclic permutation).

**Second Solution.** We have

$$2a^2 + bc = (a - b)(a - c) + a(a + b + c) = (a - b)(a - c),$$

$$\begin{aligned} 2b^2 + ca &= (b - c)(b - a) + b(a + b + c) = (b - c)(b - a), \\ 2c^2 + ab &= (c - a)(c - b) + c(a + b + c) = (c - a)(c - b). \end{aligned}$$

Therefore,

$$(2a^2 + bc)(2b^2 + ca)(2c^2 + ab) = -(a - b)^2(b - c)^2(c - a)^2 \leq 0.$$

□

**P 2.3.** Let  $a, b, c$  be real numbers such that

$$a + b \geq 0, \quad b + c \geq 0, \quad c + a \geq 0.$$

Prove that

$$9(a + b)(b + c)(c + a) \geq 8(a + b + c)(ab + bc + ca).$$

(Nguyen Van Huyen, 2014)

**Solution.** Write the desired inequality in the form

$$a(b - c)^2 + b(c - a)^2 + c(a - b)^2 \geq 0.$$

For  $a, b, c \geq 0$ , the inequality is clearly true. Otherwise, without loss of generality, assume that  $a \leq b \leq c$ ,  $a < 0$ . From  $a + b \geq 0$ , it follows that

$$a < 0 < b \leq c, \quad a + b \geq 0.$$

Replacing  $a$  by  $-a$ , we need to show that

$$0 < a \leq b \leq c$$

involves

$$-a(c - b)^2 + b(c + a)^2 + c(a + b)^2 \geq 0.$$

This is true since  $c(a + b)^2 > 0$  and

$$-a(c - b)^2 + b(c + a)^2 \geq -b(c - b)^2 + b(c + a)^2 = b(a + b)(a - b + 2c) > 0.$$

The equality holds for  $a = b = c \geq 0$ .

□

**P 2.4.** Let  $a, b, c$  be real numbers such that

$$ab + bc + ca = 3.$$

Prove that

$$(3a^2 + 1)(3b^2 + 1)(3c^2 + 1) \geq 64.$$

When does equality hold?

**Solution.** Using the substitution

$$a = \frac{x}{\sqrt{3}}, \quad b = \frac{y}{\sqrt{3}}, \quad c = \frac{z}{\sqrt{3}},$$

we need to show that

$$(x^2 + 1)(y^2 + 1)(z^2 + 1) \geq 64$$

for all real  $x, y, z$  such that  $xy + yz + zx = 9$ .

**First Solution.** Applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned} (x^2 + 1)(y^2 + 1)(z^2 + 1) &= (x^2 + 1)[(y + z)^2 + (yz - 1)^2] \\ &\geq [x(y + z) + (yz - 1)]^2 = 64. \end{aligned}$$

The equality holds for  $xy + yz + zx = 9$  and  $\frac{y + z}{x} = yz - 1$ ; that is, for

$$y + z = (yz - 1)x = \frac{(yz - 1)(9 - yz)}{y + z},$$

$$(y + z)^2 + (yz - 1)(yz - 9) = 0,$$

$$(y - z)^2 + (yz - 3)^2 = 0,$$

$$y = z = \pm\sqrt{3}.$$

In addition, from  $xy + yz + zx = 9$ , we get

$$x = y = z = \pm\sqrt{3}.$$

Therefore, the original inequality becomes an equality for

$$a = b = c = \pm 1.$$

**Second Solution.** We have

$$\begin{aligned} (x^2 + 1)(y^2 + 1)(z^2 + 1) - 64 &= x^2y^2z^2 + \sum x^2y^2 + \sum x^2 - 63 \\ &= x^2y^2z^2 + \left(\sum xy\right)^2 - 2xyz \sum x + \left(\sum x\right)^2 - 2 \sum xy - 63 \\ &= x^2y^2z^2 - 2xyz \sum x + \left(\sum x\right)^2 = \left(xyz - \sum x\right)^2 \geq 0. \end{aligned}$$

**Third Solution.** We have

$$\begin{aligned} (3a^2 + 1)(3b^2 + 1)(3c^2 + 1) - 64 &= 27a^2b^2c^2 + 9 \sum a^2b^2 + 3 \sum a^2 - 63 \\ &= 27a^2b^2c^2 + 9 \left(\sum ab\right)^2 - 18abc \sum a + 3 \left(\sum a\right)^2 - 6 \sum ab - 63 \\ &= 27a^2b^2c^2 - 18abc \sum a + 3 \left(\sum a\right)^2 = 3 \left(3abc - \sum a\right)^2 \geq 0. \end{aligned}$$

□

**P 2.5.** If  $a$  and  $b$  are real numbers, then

$$3(1 - a + a^2)(1 - b + b^2) \geq 2(1 - ab + a^2b^2).$$

(Titu Andreescu, 2006)

**Solution.** We write the inequality as follows:

$$(3 - 3b + b^2)a^2 - (3 - 5b + 3b^2)a + 1 - 3b + 3b^2 \geq 0.$$

Since

$$3 - 3b + b^2 > \frac{9}{4} - 3b + b^2 = \frac{(3 - 2b)^2}{4} \geq 0,$$

$$1 - 3b + 3b^2 > \frac{3}{4} - 3b + 3b^2 = \frac{3(1 - 2b)^2}{4} \geq 0,$$

$$3 - 5b + 3b^2 > \frac{25}{12} - 5b + 3b^2 = \frac{(5 - 6b)^2}{12} \geq 0,$$

it suffices to consider the case  $a > 0$ . By the AM-GM, we have

$$(3 - 3b + b^2)a^2 + 1 - 3b + 3b^2 \geq 2a\sqrt{(3 - 3b + b^2)(1 - 3b + 3b^2)}.$$

Thus, we only need to show that

$$2\sqrt{(3 - 3b + b^2)(1 - 3b + 3b^2)} \geq 3 - 5b + 3b^2.$$

This is true if

$$4(3 - 3b + b^2)(1 - 3b + 3b^2) \geq (3 - 5b + 3b^2)^2,$$

which is equivalent to

$$3(b^2 - 3b + 1)^2 \geq 0.$$

The equality occurs for  $a > 0$ ,  $(3 - 3b + b^2)a^2 = 1 - 3b + 3b^2$  and  $b^2 - 3b + 1 = 0$ . Since

$$a^2 = \frac{1 - 3b + 3b^2}{3 - 3b + b^2} = \frac{(-b^2) + 3b^2}{3 - 3b + (3b - 1)} = b^2,$$

the equality occurs for

$$a = b = \frac{3 \pm \sqrt{5}}{2}.$$

□

**P 2.6.** If  $a, b, c$  are real numbers, then

$$3(1 - a + a^2)(1 - b + b^2)(1 - c + c^2) \geq 1 + abc + a^2b^2c^2.$$

(Vasile Cîrtoaje and Mircea Lascu, 1989)

**First Solution.** From the identity

$$2(1 - a + a^2)(1 - b + b^2) = 1 + a^2b^2 + (a - b)^2 + (1 - a)^2(1 - b)^2,$$

it follows that

$$2(1 - a + a^2)(1 - b + b^2) \geq 1 + a^2b^2.$$

Thus, it is enough to prove that

$$3(1 + a^2b^2)(1 - c + c^2) \geq 2(1 + abc + a^2b^2c^2).$$

This inequality is equivalent to

$$(3 + a^2b^2)c^2 - (3 + 2ab + 3a^2b^2)c + 1 + 3a^2b^2 \geq 0,$$

$$[2(3 + a^2b^2)c - 3 - 2ab - 3a^2b^2]^2 + 3(1 - ab)^4 \geq 0.$$

The equality holds for  $a = b = c = 1$ .

**Second Solution.** Write the required inequality as

$$3(1 - a + a^2)(1 - b + b^2)(1 - c + c^2) - abc \geq 1 + a^2b^2c^2.$$

Replacing  $a, b, c$  with  $|a|, |b|, |c|$ , respectively, the left side of this inequality remains unchanged or decreases, while the right side remains unchanged. Therefore, it suffices to prove the inequality only for  $a, b, c \geq 0$ . For  $a = b = c$ , the inequality is true since

$$3(1 - a + a^2)^3 - (1 + a^3 + a^6) = (1 - a)^4(2 - a + 2a^2) \geq 0.$$

Multiplying the inequalities

$$\sqrt[3]{3}(1 - a + a^2) \geq \sqrt[3]{1 + a^3 + a^6},$$

$$\sqrt[3]{3}(1 - b + b^2) \geq \sqrt[3]{1 + b^3 + b^6},$$

$$\sqrt[3]{3}(1 - c + c^2) \geq \sqrt[3]{1 + c^3 + c^6},$$

we get

$$3(1 - a + a^2)(1 - b + b^2)(1 - c + c^2) \geq \sqrt[3]{(1 + a^3 + a^6)(1 + b^3 + b^6)(1 + c^3 + c^6)}.$$

Therefore, it suffices to prove that

$$\sqrt[3]{(1 + a^3 + a^6)(1 + b^3 + b^6)(1 + c^3 + c^6)} \geq 1 + abc + a^2b^2c^2,$$

which follows immediately from Hölder's inequality.

□

**P 2.7.** If  $a, b, c$  are real numbers, then

$$(a^2 + b^2 + c^2)^3 \geq (a + b + c)(ab + bc + ca)(a^3 + b^3 + c^3).$$

(Vasile Cîrtoaje, 2007)

**Solution.** Substituting  $a, b, c$  by  $|a|, |b|, |c|$ , respectively, the left side of the inequality remains unchanged, while the right side either remains unchanged or increases. Therefore, it suffices to prove the inequality only for  $a, b, c \geq 0$ . Let

$$p = a + b + c, \quad q = ab + bc + ca.$$

Since

$$q^2 - 3abcp = \frac{a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2}{2} \geq 0,$$

we have

$$(a + b + c)(a^3 + b^3 + c^3) = p(p^3 - 3pq + 3abc) \leq p^4 - 3p^2q + q^2.$$

Thus, it suffices to show that

$$(p^2 - 2q)^3 \geq q(p^4 - 3p^2q + q^2),$$

which is equivalent to the obvious inequality

$$(p^2 - 3q)^2(p^2 - q) \geq 0.$$

The equality holds for  $a = b = c$ .

□

**P 2.8.** If  $a, b, c$  are real numbers, then

$$2(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \geq [ab(a + b) + bc(b + c) + ca(c + a) - 2abc]^2.$$

(Vo Quoc Ba Can, 2009)

**Solution.** Since

$$(a^2 + b^2)(a^2 + c^2) = (a^2 + bc)^2 + (ab - ac)^2$$

and

$$2(b^2 + c^2) = (b + c)^2 + (b - c)^2,$$

the required inequality follows by applying the Cauchy-Schwarz inequality as follows

$$\begin{aligned} 2(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) &\geq [(a^2 + bc)(b + c) + (ab - ac)(b - c)]^2 \\ &= [ab(a + b) + bc(b + c) + ca(c + a) - 2abc]^2. \end{aligned}$$

The equality holds when two of  $a, b, c$  are equal.

□



**P 2.9.** If  $a, b, c$  are real numbers, then

$$2(a^2 + 1)(b^2 + 1)(c^2 + 1) \geq (ab + bc + ca + 1)^2.$$

(Vasile Cîrtoaje, 2007)

**Solution.** By the Cauchy-Schwarz inequality, we have

$$2(a^2 + 1)(b^2 + 1)(c^2 + 1) = (a^2 + a^2 + 1 + 1)(b^2 + c^2 + b^2c^2 + 1) \geq (ab + ac + bc + 1)^2.$$

The equality occurs for  $a = b = c = 1$  or  $a = b = c = -1$ .

□

**P 2.10.** If  $a, b, c$  are real numbers, then

$$(a^2 + 1)(b^2 + 1)(c^2 + 1) \geq \frac{5}{16}(a + b + c + 1)^2.$$

(Vasile Cîrtoaje, 2006)

**First Solution.** Since the equality holds for

$$a = b = c = \frac{1}{2},$$

we replace  $a, b, c$  respectively by  $a/2, b/2, c/2$ . Thus, the inequality becomes

$$(a^2 + 4)(b^2 + 4)(c^2 + 4) \geq 5(a + b + c + 2)^2,$$

with equality for  $a = b = c = 1$ . To prove this, we apply the Cauchy-Schwarz inequality in the form

$$(a^2 + 4) \left[ 1 + \left( \frac{b + c + 2}{2} \right)^2 \right] \geq (a + b + c + 2)^2.$$

Therefore, it suffices to prove that

$$(b^2 + 4)(c^2 + 4) \geq 5 \left[ 1 + \left( \frac{b + c + 2}{2} \right)^2 \right].$$

This inequality is equivalent to

$$11(b + c)^2 - 20(b + c) + 4b^2c^2 - 32bc + 24 \geq 0.$$

Since

$$4b^2c^2 - 8bc + 4 = 4(bc - 1)^2 \geq 0,$$

it suffices to show that

$$11(b+c)^2 - 20(b+c) - 24bc + 20 \geq 0.$$

Indeed,

$$\begin{aligned} 11(b+c)^2 - 20(b+c) - 24bc + 20 &\geq 11(b+c)^2 - 20(b+c) - 6(b+c)^2 + 20 \\ &= 5(b+c-2)^2 \geq 0. \end{aligned}$$

**Second Solution.** Among  $a^2, b^2, c^2$  there are two either less than or equal to  $\frac{1}{4}$ , or larger than or equal to  $\frac{1}{4}$ . Let  $b^2$  and  $c^2$  be these numbers; that is,

$$(4b^2 - 1)(4c^2 - 1) \geq 0.$$

Then, we have

$$\begin{aligned} \frac{16}{5}(b^2 + 1)(c^2 + 1) &= 5 \left( \frac{4b^2 - 1}{5} + 1 \right) \left( \frac{4c^2 - 1}{5} + 1 \right) \\ &\geq 5 \left( \frac{4b^2 - 1}{5} + \frac{4c^2 - 1}{5} + 1 \right) = 4b^2 + 4c^2 + 3. \end{aligned}$$

Therefore, it suffices to prove that

$$(a^2 + 1)(4b^2 + 4c^2 + 3) \geq (a + b + c + 1)^2.$$

Writing this inequality as

$$\left( a^2 + \frac{1}{4} + \frac{1}{4} + \frac{1}{2} \right) (1 + 4b^2 + 4c^2 + 2) \geq (a + b + c + 1)^2,$$

we recognize the Cauchy-Schwarz inequality.

□

**P 2.11.** If  $a, b, c$  are real numbers, then

$$(a) \quad a^6 + b^6 + c^6 - 3a^2b^2c^2 + 2(a^2 + bc)(b^2 + ca)(c^2 + ab) \geq 0;$$

$$(b) \quad a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq (a^2 - 2bc)(b^2 - 2ca)(c^2 - 2ab).$$

**Solution.** (a) Since

$$(a^2 + bc)(b^2 + ca)(c^2 + ab) = 2a^2b^2c^2 + \sum a^3b^3 + abc \sum a^3,$$

we can write the desired inequality as follows

$$\sum a^6 + 2 \sum a^3b^3 + 2abc \sum a^3 + a^2b^2c^2 \geq 0,$$

$$(\sum a^3)^2 + 2abc \sum a^3 + a^2b^2c^2 \geq 0,$$

$$(\sum a^3 + abc)^2 \geq 0.$$

The equality holds for

$$a^3 + b^3 + c^3 + abc = 0.$$

(b) Since

$$(a^2 - 2bc)(b^2 - 2ca)(c^2 - 2ab) = -7a^2b^2c^2 - 2 \sum a^3b^3 + 4abc \sum a^3,$$

we can write the desired inequality as follows

$$\sum a^6 + 2 \sum a^3b^3 - 4abc \sum a^3 + 4a^2b^2c^2 \geq 0,$$

$$(\sum a^3)^2 - 4abc \sum a^3 + 4a^2b^2c^2 \geq 0,$$

$$(\sum a^3 - 2abc)^2 \geq 0.$$

The equality holds for

$$a^3 + b^3 + c^3 - 2abc = 0.$$

□

**P 2.12.** If  $a, b, c$  are real numbers, then

$$\frac{2}{3}(a^6 + b^6 + c^6) + a^3b^3 + b^3c^3 + c^3a^3 + abc(a^3 + b^3 + c^3) \geq 0.$$

**Solution.** Write the inequality as follows

$$\frac{4}{3}(a^6 + b^6 + c^6) + 2(a^3b^3 + b^3c^3 + c^3a^3) + 2abc(a^3 + b^3 + c^3) \geq 0,$$

$$\frac{1}{3}(a^6 + b^6 + c^6) + (a^3 + b^3 + c^3)^2 + 2abc(a^3 + b^3 + c^3) \geq 0.$$

By virtue of the AM-GM inequality, we have

$$\frac{1}{3}(a^6 + b^6 + c^6) \geq a^2b^2c^2.$$

Therefore, it suffices to show that

$$a^2b^2c^2 + (a^3 + b^3 + c^3)^2 + 2abc(a^3 + b^3 + c^3) \geq 0,$$

which is equivalent to

$$(abc + a^3 + b^3 + c^3)^2 \geq 0.$$

The equality holds for  $-a = b = c$  (or any cyclic permutation).

□

**P 2.13.** If  $a, b, c$  are real numbers, then

$$4(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq (a - b)^2(b - c)^2(c - a)^2.$$

(Vasile Cîrtoaje, 2009)

**Solution.** Using the identity

$$4xy = (x + y)^2 - (x - y)^2,$$

we have

$$\begin{aligned} 4(a^2 + ab + b^2)(a^2 + ac + c^2) &= [(2a^2 + ab + ac + 2bc) + (b - c)^2]^2 \\ &\quad - (a + b + c)^2(b - c)^2 \\ &= (2a^2 + ab + ac + 2bc)^2 + 3a^2(b - c)^2. \end{aligned}$$

From this result and

$$4(b^2 + bc + c^2) = (b - c)^2 + 3(b + c)^2,$$

we get

$$16 \prod (a^2 + ab + b^2) = [(2a^2 + ab + ac + 2bc)^2 + 3a^2(b - c)^2] [(b - c)^2 + 3(b + c)^2].$$

Next, the Cauchy-Schwarz inequality gives

$$\begin{aligned} 16 \prod (a^2 + ab + b^2) &\geq [(2a^2 + ab + ac + 2bc)(b - c) + 3a(b - c)(b + c)]^2 \\ &= 4(b - c)^2(a - b)^2(a - c)^2. \end{aligned}$$

The equality holds for

$$ab(a + b) + bc(b + c) + ca(c + a) = 0.$$

**Remark.** The inequality is a consequence of the identity

$$4 \prod (a^2 + ab + b^2) = 3[ab(a + b) + bc(b + c) + ca(c + a)]^2 + (a - b)^2(b - c)^2(c - a)^2.$$

□

**P 2.14.** If  $a, b, c$  are real numbers, then

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq 3(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2).$$

(Gabriel Dospinescu, 2009)

**Solution** (by Vo Quoc Ba Can). As we have shown in the proof of the preceding P 2.13,

$$16 \prod (a^2 + ab + b^2) = [(2a^2 + ab + ac + 2bc)^2 + 3a^2(b - c)^2] [3(b + c)^2 + (b - c)^2].$$

Thus, by the Cauchy-Schwarz inequality, we get

$$\begin{aligned} 16 \prod (a^2 + ab + b^2) &\geq 3[(b + c)(2a^2 + ab + ac + 2bc) + a(b - c)^2]^2 \\ &= 12[(a^2b + b^2c + c^2a) + (ab^2 + bc^2 + ca^2)]^2. \end{aligned}$$

To prove the desired inequality, it suffices to show that

$$[(a^2b + b^2c + c^2a) + (ab^2 + bc^2 + ca^2)]^2 \geq 4(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2).$$

Indeed, this is equivalent to

$$\begin{aligned} [(a^2b + b^2c + c^2a) - (ab^2 + bc^2 + ca^2)]^2 &\geq 0, \\ (a - b)^2(b - c)^2(c - a)^2 &\geq 0. \end{aligned}$$

The equality holds when two of  $a, b, c$  are equal.

**Remark.** The inequality is a consequence of the identity

$$\prod (a^2 + ab + b^2) = 3(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) + (a - b)^2(b - c)^2(c - a)^2.$$

□

**P 2.15.** If  $a, b, c$  are real numbers such that  $abc > 0$ , then

$$4 \left( a + \frac{1}{a} \right) \left( b + \frac{1}{b} \right) \left( c + \frac{1}{c} \right) \geq 9(a + b + c).$$

**Solution.** It suffices to show that

$$4(a^2 + 1)(b^2 + 1)(c^2 + 1) \geq 9abc(a + b + c)$$

for any real  $a, b, c$ .

**First Solution.** It is easy to check that the equality occurs for  $a = b = c = \sqrt{2}$ . Therefore, using the notation

$$a = x\sqrt{2}, \quad b = y\sqrt{2}, \quad c = z\sqrt{2},$$

the inequality can be written as

$$(2x^2 + 1)(2y^2 + 1)(2z^2 + 1) \geq 9xyz(x + y + z),$$

with equality for  $x = y = z = 1$ . Since

$$(xy + yz + zx)^2 - 3xyz(x + y + z) = \frac{1}{2} \sum x^2(y - z)^2 \geq 0,$$

it suffices to prove the stronger inequality

$$(2x^2 + 1)(2y^2 + 1)(2z^2 + 1) \geq 3(xy + yz + zx)^2.$$

Let

$$A = (y^2 - 1)(z^2 - 1), \quad B = (z^2 - 1)(x^2 - 1), \quad C = (x^2 - 1)(y^2 - 1).$$

From

$$ABC = (x^2 - 1)^2(y^2 - 1)^2(z^2 - 1)^2 \geq 0,$$

it follows that at least one of  $A, B, C$  is nonnegative. Due to symmetry, assume that

$$A = (y^2 - 1)(z^2 - 1) \geq 0.$$

Applying the Cauchy-Schwarz inequality, we have

$$(xy + yz + zx)^2 \leq (x^2 + 1 + x^2)(y^2 + y^2z^2 + z^2).$$

Therefore, it suffices to show that

$$(2y^2 + 1)(2z^2 + 1) \geq 3(y^2 + y^2z^2 + z^2),$$

which reduces to the obvious inequality

$$(y^2 - 1)(z^2 - 1) \geq 0.$$

**Second Solution.** Since

$$4(b^2 + 1)(c^2 + 1) - 3[(b + c)^2 + b^2c^2] = (b - c)^2 + (bc - 2)^2 \geq 0,$$

it suffices to show that

$$(a^2 + 1)[(b + c)^2 + b^2c^2] \geq 3abc(a + b + c).$$

Applying the Cauchy-Schwarz inequality, we get

$$(a^2 + 1)[(b + c)^2 + b^2c^2] \geq [a(b + c) + bc]^2 \geq 3abc(a + b + c).$$

□

**P 2.16.** If  $a, b, c$  are real numbers, then

$$(a) \quad (a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) \leq (a^2 + b^2 + c^2)(ab + bc + ca)^2;$$

$$(b) \quad (2a^2 + bc)(2b^2 + ca)(2c^2 + ab) \leq (a + b + c)^2(a^2b^2 + b^2c^2 + c^2a^2).$$

(Vasile Cîrtoaje, 2005)

**Solution.** (a) Let  $q = ab + bc + ca$ . Since

$$a^2 + 2bc = q + (a - b)(a - c),$$

$$b^2 + 2ca = q + (b - c)(b - a),$$

$$c^2 + 2ab = q + (c - a)(c - b),$$

we can rewrite the required inequality as follows

$$[q + (a - b)(a - c)][q + (b - c)(b - a)][q + (c - a)(c - b)] \leq q^2(a^2 + b^2 + c^2),$$

$$q^3 + q^2 \sum (a - b)(a - c) - (a - b)^2(b - c)^2(c - a)^2 \leq q^2(a^2 + b^2 + c^2).$$

Since

$$\sum (a - b)(a - c) = a^2 + b^2 + c^2 - q,$$

the inequality reduces to the obvious form

$$(a - b)^2(b - c)^2(c - a)^2 \geq 0.$$

The equality holds for  $a = b$ , or  $b = c$ , or  $c = a$ .

(b) For  $a = 0$ , the required inequality reduces to

$$b^2c^2(b - c)^2 \geq 0.$$

Otherwise, for  $abc \neq 0$ , the inequality follows from the inequality in (a) by substituting  $a, b, c$  with  $1/a, 1/b, 1/c$ , respectively. The equality occurs for  $a = b$ , or  $b = c$ , or  $c = a$ .

□

**P 2.17.** If  $a, b, c$  are real numbers such that

$$ab + bc + ca \geq 0,$$

then

$$27(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) \leq (a + b + c)^6.$$

**Solution.** In virtue of the AM-GM inequality, we have

$$\begin{aligned}(a + b + c)^6 &= [a^2 + b^2 + c^2 + (ab + bc + ca) + (ab + bc + ca)]^3 \\ &\geq 27(a^2 + b^2 + c^2)(ab + bc + ca)^2.\end{aligned}$$

Thus, the required inequality follows immediately from the inequality (a) in P 2.16. The equality holds for  $a = b = c$ . □

**P 2.18.** If  $a, b, c$  are real numbers such that

$$a^2 + b^2 + c^2 = 2,$$

then

$$(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) + 2 \geq 0.$$

(Vasile Cîrtoaje, 2011)

**Solution** (by Vo Quoc Ba Can). If  $a, b, c$  have the same sign, then the inequality is trivial. Otherwise, since the inequality is symmetric and does not change by substituting  $-a, -b, -c$  for  $a, b, c$ , respectively, it suffices to consider the case where  $a \leq 0$  and  $b, c \geq 0$ . Replacing now  $-a$  with  $a$ , we need to prove the inequality

$$(a^2 + 2bc)(b^2 - 2ac)(c^2 - 2ab) + 2 \geq 0 \quad (*)$$

for all  $a, b, c \geq 0$  satisfying

$$a^2 + b^2 + c^2 = 2.$$

If  $b^2 - 2ac$  and  $c^2 - 2ab$  have the same sign, then the inequality is also trivial. Due to symmetry in  $b$  and  $c$ , we may assume that

$$b^2 - 2ac \geq 0 \geq c^2 - 2ab.$$

On the other hand, it is easy to check that (\*) becomes an equality for  $a = b = 1$  and  $c = 0$ , when

$$a^2 + 2bc = b^2 - 2ac = ab - \frac{c^2}{2}.$$

Then, we rewrite the inequality (\*) in the form

$$(a^2 + 2bc)(b^2 - 2ac) \left( ab - \frac{c^2}{2} \right) \leq 1.$$

Using the AM-GM inequality, we have

$$27(a^2 + 2bc)(b^2 - 2ac) \left( ab - \frac{c^2}{2} \right) \leq \left[ (a^2 + 2bc) + (b^2 - 2ac) + \left( ab - \frac{c^2}{2} \right) \right]^3.$$



Thus, it suffices to prove that

$$(a^2 + 2bc) + (b^2 - 2ac) + \left(ab - \frac{c^2}{2}\right) \leq 3.$$

This inequality can be written in the homogeneous form

$$2(a^2 + 2bc) + 2(b^2 - 2ac) + (2ab - c^2) \leq 3(a^2 + b^2 + c^2),$$

which is equivalent to

$$(a - b + 2c)^2 \geq 0.$$

The original inequality is an equality for  $a = -1$ ,  $b = 1$  and  $c = 0$  (or any permutation).

**Remark.** In the same manner, we can prove the following generalization.

- Let  $a, b, c$  be real numbers such that  $a^2 + b^2 + c^2 = 2$ . If  $0 < k \leq 2$ , then

$$(a^2 + kbc)(b^2 + kca)(c^2 + kab) + k \geq 0.$$

□

**P 2.19.** If  $a, b, c$  are real numbers such that

$$a + b + c = 3,$$

then

$$3(a^4 + b^4 + c^4) + a^2 + b^2 + c^2 + 6 \geq 6(a^3 + b^3 + c^3).$$

(Vasile Cîrtoaje, 2006)

**Solution.** Write the inequality as  $F(a, b, c) \geq 0$ , where

$$F(a, b, c) = 3(a^4 + b^4 + c^4) + (a^2 + b^2 + c^2) - 6(a^3 + b^3 + c^3) + 6.$$

Due to symmetry, we may assume that  $a \leq b \leq c$ . To prove the required inequality, we use the mixing variables method. More precisely, we show that

$$F(a, b, c) \geq F(a, x, x) \geq 0,$$

where  $x = (b + c)/2$ ,  $x \geq 1$ . We have

$$\begin{aligned} F(a, b, c) - F(a, x, x) &= 3(b^4 + b^4 - 2x^4) + (b^2 + c^2 - 2x^2) - 6(b^3 + c^3 - 2x^3) \\ &= 3[(b^2 + c^2)^2 - 4x^4] + 6(x^4 - b^2c^2) + (b^2 + c^2 - 2x^2) - 6(b^3 + c^3 - 2x^3) \\ &= (b^2 + c^2 - 2x^2)[3(b^2 + c^2 + 2x^2) + 1] + 6(x^2 - bc)(x^2 + bc) - 12x(b^2 + c^2 - bc - x^2). \end{aligned}$$

Since

$$b^2 + c^2 - 2x^2 = \frac{1}{2}(b - c)^2,$$

$$x^2 - bc = \frac{1}{4}(b - c)^2,$$

$$b^2 + c^2 - bc - x^2 = \frac{3}{4}(b - c)^2,$$

we get

$$\begin{aligned} F(a, b, c) - F(a, x, x) &= \frac{1}{2}(b - c)^2[3(b^2 + c^2 + 2x^2) + 1 + 3(x^2 + bc) - 18x] \\ &= \frac{1}{2}(b - c)^2[3(x^2 - bc) + 18x(x - 1) + 1] \geq 0. \end{aligned}$$

Also,

$$F(a, x, x) = F(3 - 2x, x, x) = 6(x - 1)^2(3x - 4)^2 \geq 0.$$

This completes the proof. The equality holds for  $a = b = c = 1$ , and for  $a = 1/3$  and  $b = c = 4/3$  (or any cyclic permutation). □

**P 2.20.** If  $a, b, c$  are real numbers such that

$$abc = 1,$$

then

$$3(a^2 + b^2 + c^2) + 2(a + b + c) \geq 5(ab + bc + ca).$$

**Solution.** Without loss of generality, assume that  $a \geq b \geq c$ . From  $abc = 1$ , it follows that either  $a, b, c > 0$  or  $a > 0$  and  $b, c < 0$ .

Case 1:  $a, b, c > 0$ . Let

$$p = a + b + c, \quad q = ab + bc + ca.$$

The AM-GM inequality

$$a + b + c \geq 3\sqrt[3]{abc}$$

gives  $p \geq 3$ , while Schur's inequality

$$p^3 + 9abc \geq 4pq$$

gives

$$q \leq \frac{p^3 + 9}{4p}.$$

Write the required inequality as

$$3(p^2 - 2q) + 2p \geq 5q,$$

$$3p^2 + 2p \geq 11q.$$

This is true since

$$3p^2 + 2p - 11q \geq 3p^2 + 2p - \frac{11(p^3 + 9)}{4p} = \frac{(p-3)(p^2 + 11p + 33)}{4p} \geq 0.$$

Case 2:  $a > 0$  and  $b, c < 0$ . Substituting  $-b$  for  $b$  and  $-c$  for  $c$ , we need to prove that

$$3(a^2 + b^2 + c^2) + 2a + 5a(b + c) \geq 2(b + c) + 5bc$$

for  $a, b, c > 0$  satisfying  $abc = 1$ . It suffices to show that

$$3(b^2 + c^2) - 5bc \geq (2 - 5a)(b + c).$$

Since

$$\frac{3(b^2 + c^2) - 5bc}{b + c} \geq \frac{b + c}{4} \geq \frac{\sqrt{bc}}{2} = \frac{1}{2\sqrt{a}},$$

we only need to prove that

$$\frac{1}{2\sqrt{a}} \geq 2 - 5a.$$

Indeed, by the AM-GM inequality, we get

$$5a + \frac{1}{2\sqrt{a}} = 5a + \frac{1}{4\sqrt{a}} + \frac{1}{4\sqrt{a}} \geq 3\sqrt[3]{5a \cdot \frac{1}{4\sqrt{a}} \cdot \frac{1}{4\sqrt{a}}} > 2.$$

This completes the proof. The equality holds for  $a = b = c = 1$ .

□

**P 2.21.** If  $a, b, c$  are real numbers such that

$$abc = 1,$$

then

$$a^2 + b^2 + c^2 + 6 \geq \frac{3}{2} \left( a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right).$$

**Solution.** Substituting  $a, b, c$  by  $|a|, |b|, |c|$ , respectively, the left side of the inequality remains unchanged, while the right side either remains unchanged or increases. Therefore, it suffices to prove the inequality only for  $a, b, c > 0$ .

**First Solution.** Write the inequality in the form

$$3(6x^2 - 3x + 4) \geq 7(ab + bc + ca),$$

where

$$x = \frac{a + b + c}{3}.$$

By virtue of the AM-GM inequality, we have  $x \geq 1$ . The third degree Schur's inequality states that

$$(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca),$$

which is equivalent to

$$ab + bc + ca \leq \frac{3(3x^3 + 1)}{4x}.$$

Therefore, it suffices to show that

$$3(6x^2 - 3x + 4) \geq \frac{21(3x^3 + 1)}{4x}.$$

This inequality reduces to

$$(x - 1)(3x^2 - 9x + 7) \geq 0,$$

which is true because

$$3x^2 - 9x + 7 = 3\left(x - \frac{3}{2}\right)^2 + \frac{1}{4} > 0.$$

The equality holds for  $a = b = c = 1$ .

**Second Solution.** Use the mixing variables technique. Let

$$F(a, b, c) = a^2 + b^2 + c^2 + 6 - \frac{3}{2} \left( a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right).$$

Assume that

$$a = \min\{a, b, c\}$$

and show that

$$F(a, b, c) \geq F(a, x, x) \geq 0,$$

where

$$x = \sqrt{bc}, \quad x \geq 1.$$

We have

$$\begin{aligned} F(a, b, c) - F(a, x, x) &= (b - c)^2 - \frac{3}{2} \left( b + c - 2\sqrt{bc} + \frac{1}{b} + \frac{1}{c} - \frac{2}{\sqrt{bc}} \right) \\ &= \frac{1}{2}(\sqrt{b} - \sqrt{c})^2 \left[ 2(\sqrt{b} + \sqrt{c})^2 - 3 - \frac{3}{bc} \right] \\ &\geq \frac{1}{2}(\sqrt{b} - \sqrt{c})^2 \left( 8\sqrt{bc} - 3 - \frac{3}{bc} \right) \\ &\geq \frac{1}{2}(\sqrt{b} - \sqrt{c})^2 (8 - 3 - 3) \geq 0 \end{aligned}$$

and

$$\begin{aligned}
 F(a, x, x) &= F\left(\frac{1}{x^2}, x, x\right) \\
 &= \frac{x^6 - 6x^5 + 12x^4 - 6x^3 - 3x^2 + 2}{2x^4} \\
 &= \frac{(x-1)^2(x^4 - 4x^3 + 3x^2 + 4x + 2)}{2x^4} \\
 &= \frac{(x-1)^2[(x^2 - 2x - 1)^2 + x^2 + 1]}{2x^4} \geq 0.
 \end{aligned}$$

□

**P 2.22.** If  $a, b, c$  are real numbers, then

$$(1 + a^2)(1 + b^2)(1 + c^2) + 8abc \geq \frac{1}{4}(1 + a)^2(1 + b)^2(1 + c)^2.$$

**Solution.** It is easy to check that

$$(1 + a^2)(1 + b^2)(1 + c^2) + 8abc = (1 + abc)^2 + (a + bc)^2 + (b + ca)^2 + (c + ab)^2.$$

Thus, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
 (1 + a^2)(1 + b^2)(1 + c^2) + 8abc &\geq \frac{[(1 + abc) + (a + bc) + (b + ca) + (c + ab)]^2}{4} \\
 &= \frac{1}{4}(1 + a)^2(1 + b)^2(1 + c)^2.
 \end{aligned}$$

The equality holds for  $b = c = 1$  (or any cyclic permutation), and also for  $a = b = c = -1$ .

□

**P 2.23.** Let  $a, b, c$  be real numbers such that

$$a + b + c = 0.$$

Prove that

$$a^{12} + b^{12} + c^{12} \geq \frac{2049}{8}a^4b^4c^4.$$

**Solution.** Consider only the nontrivial case  $abc \neq 0$ , and rewrite the inequality as follows

$$\begin{aligned}
 a^{12} + b^{12} + (a + b)^{12} &\geq \frac{2049}{8}a^4b^4(a + b)^4, \\
 (a^6 + b^6)^2 - 2a^6b^6 + (a^2 + b^2 + 2ab)^6 &\geq \frac{2049}{8}a^4b^4(a^2 + b^2 + 2ab)^2.
 \end{aligned}$$

Let us denote

$$d = \frac{a^2 + b^2}{ab}, \quad |d| \geq 2.$$

Since

$$a^6 + b^6 = (a^2 + b^2)^3 - 3a^2b^2(a^2 + b^2),$$

the inequality can be restated as

$$(d^3 - 3d)^2 - 2 + (d + 2)^6 \geq \frac{2049}{8}(d + 2)^2,$$

which is equivalent to

$$(d - 2)(2d + 5)^2(4d^3 + 12d^2 + 87d + 154) \geq 0.$$

Since this inequality is obvious for  $d \geq 2$ , we only need to show that

$$4d^3 + 12d^2 + 87d + 154 \leq 0$$

for  $d \leq -2$ . Indeed,

$$\begin{aligned} 4d^3 + 12d^2 + 87d + 154 &< 4d^3 + 12d^2 + 85d + 154 \\ &= (d + 2)[(2d + 1)^2 + 76] \leq 0. \end{aligned}$$

The equality holds for  $a = b = -c/2$  (or any cyclic permutation).

□

**P 2.24.** *If  $a, b, c$  are real numbers such that  $abc \geq 0$ , then*

$$a^2 + b^2 + c^2 + 2abc + 4 \geq 2(a + b + c) + ab + bc + ca.$$

(Vasile Cîrtoaje, 2012)

**Solution.** Let us denote

$$x = a(b - 1)(c - 1), \quad y = b(c - 1)(a - 1), \quad z = c(a - 1)(b - 1).$$

Since

$$xyz = abc(a - 1)^2(b - 1)^2(c - 1)^2 \geq 0,$$

at least one of  $x, y, z$  is nonnegative; let

$$a(1 - b)(1 - c) \geq 0.$$

Since

$$abc \geq a(b + c - 1),$$

it suffices to show that

$$a^2 + b^2 + c^2 + 2a(b + c - 1) + 4 \geq 2(a + b + c) + ab + bc + ca,$$

which is equivalent to

$$a^2 - (4 - b - c)a + b^2 + c^2 - bc - 2(b + c) + 4 \geq 0,$$

$$\left(a - 2 + \frac{b + c}{2}\right)^2 + \frac{3}{4}(b - c)^2 \geq 0.$$

The equality holds for  $a = b = c = 1$ , and also for  $a = 0$  and  $b = c = 2$  (or any cyclic permutation).

□

**P 2.25.** Let  $a, b, c$  be real numbers such that

$$a + b + c = 3.$$

(a) If  $a, b, c \geq -3$ , then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

(b) If  $a, b, c \geq -7$ , then

$$\frac{1 - a}{(1 + a)^2} + \frac{1 - b}{(1 + b)^2} + \frac{1 - c}{(1 + c)^2} \geq 0.$$

(Vasile Cîrtoaje, 2012)

**Solution.** Assume that

$$a = \min\{a, b, c\},$$

and denote

$$t = \frac{b + c}{2},$$

$$E(a, b, c) = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c},$$

$$F(a, b, c) = \frac{1 - a}{(1 + a)^2} + \frac{1 - b}{(1 + b)^2} + \frac{1 - c}{(1 + c)^2}.$$

(a) From  $a, b, c \geq -3$  and  $a + b + c = 3$ , it follows that

$$-3 \leq a \leq \frac{a + b + c}{3} = 1.$$

We will show that

$$E(a, b, c) \geq E(a, t, t) \geq 0.$$

We have

$$\begin{aligned}
 E(a, b, c) - E(a, t, t) &= \frac{1}{b^2} + \frac{1}{c^2} - \frac{2}{t^2} - \left( \frac{1}{b} + \frac{1}{c} - \frac{2}{t} \right) \\
 &= \frac{(b-c)^2(b^2 + c^2 + 4bc)}{b^2c^2(b+c)^2} - \frac{(b-c)^2}{bc(b+c)} \\
 &= \frac{(b-c)^2[(b+c)^2 - bc(b+c-2)]}{b^2c^2(b+c)^2}.
 \end{aligned}$$

Since

$$\begin{aligned}
 (b+c)^2 - bc(b+c-2) &= (b+c)^2 - bc(1-a) \\
 &\geq (b+c)^2 - \frac{(b+c)^2(1-a)}{4} \\
 &= \frac{(b+c)^2(3+a)}{4} \geq 0,
 \end{aligned}$$

we have

$$E(a, b, c) - E(a, t, t) \geq 0.$$

Also,

$$E(a, t, t) = \frac{1-a}{a^2} + \frac{2(1-t)}{t^2} = \frac{3(a-1)^2(a+3)}{a^2(3-a)^2} \geq 0.$$

The equality holds for  $a = b = c = 1$ , and also for  $a = -3$  and  $b = c = 3$  (or any cyclic permutation).

(b) From

$$t \geq \frac{a+b+c}{3} = 1, \quad t = \frac{3-a}{2} \leq 5,$$

it follows that

$$t \in [1, 5].$$

We will show that

$$F(a, b, c) \geq F(a, t, t) \geq 0.$$

Write the left inequality as follows:

$$\begin{aligned}
 &\left[ \frac{1-b}{(1+b)^2} - \frac{1-t}{(1+t)^2} \right] + \left[ \frac{1-c}{(1+c)^2} - \frac{1-t}{(1+t)^2} \right] \geq 0, \\
 &(b-c) \left[ \frac{(b-1)t - b - 3}{(1+b)^2} - \frac{(c-1)t - c - 3}{(1+c)^2} \right] \geq 0, \\
 &(b-c)^2[(3+b+c-bc)t + 3(b+c) + bc] \geq 0, \\
 &(b-c)^2[2t^2 + 9t + 5 - bc(t-1)] \geq 0.
 \end{aligned}$$

The last inequality is true since

$$2t^2 + 9t + 5 - bc(t-1) \geq 2t^2 + 9t + 5 - t^2(t-1) = (5-t)(1+t)^2 \geq 0.$$



Also, we have

$$\begin{aligned} F(a, t, t) &= \frac{1-a}{(1+a)^2} + \frac{2(1-t)}{(1+t)^2} \\ &= \frac{t-1}{2(2-t)^2} + \frac{2(1-t)}{(1+t)^2} \\ &= \frac{3(1-t)^2(5-t)}{2(2-t)^2(1+t)^2} \geq 0. \end{aligned}$$

The proof is completed. The equality occurs for  $a = b = c = 1$ , and also for  $a = -7$  and  $b = c = 5$  (or any cyclic permutation). □

**P 2.26.** If  $a, b, c$  are real numbers, then

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$

(Sungyoon Kim, 2006)

**Solution.** Applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned} a^6 + b^6 + c^6 - 3a^2b^2c^2 &= \frac{1}{2}(a^2 + b^2 + c^2)[(b^2 - c^2)^2 + (c^2 - a^2)^2 + (a^2 - b^2)^2] \\ &\geq \frac{1}{2}[a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)]^2 \\ &= \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2. \end{aligned}$$

Thus, the proof is completed. The equality holds for  $a = b = c$ , and also for  $a = 0$  and  $b + c = 0$  (or any cyclic permutation). □

**P 2.27.** If  $a, b, c$  are real numbers, then

$$\left(\frac{a^2 + b^2 + c^2}{3}\right)^3 \geq a^2b^2c^2 + \frac{1}{16}(a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2011)

**Solution** (by Vo Quoc Ba Can). Without loss of generality, assume that  $b$  and  $c$  have the same sign; that is

$$bc \geq 0.$$

Let

$$x = \sqrt{\frac{b^2 + c^2}{2}}.$$

Since

$$\begin{aligned} \left(\frac{a^2 + b^2 + c^2}{3}\right)^3 - a^2b^2c^2 &= \left(\frac{a^2 + 2x^2}{3}\right)^3 - a^2x^4 + a^2(x^4 - b^2c^2) \\ &= \frac{(a^2 - x^2)^2(a^2 + 8x^2)}{27} + a^2(x^4 - b^2c^2) \\ &= \frac{(2a^2 - b^2 - c^2)^2(a^2 + 4b^2 + 4c^2)}{108} + \frac{a^2(b^2 - c^2)^2}{4}, \end{aligned}$$

the desired inequality can be rewritten as

$$(2a^2 - b^2 - c^2)^2(a^2 + 4b^2 + 4c^2) \geq \frac{27}{4}(b - c)^2[(a - b)^2(a - c)^2 - 4a^2(b + c)^2].$$

According to the inequalities

$$x^2 - y^2 \leq 2x(x + y)$$

and

$$2xy \leq \frac{1}{2}(x + y)^2,$$

we have

$$\begin{aligned} (a - b)^2(a - c)^2 - 4a^2(b + c)^2 &\leq 2(a - b)(a - c)[(a - b)(a - c) + 2a(b + c)] \\ &= 2(a^2 - b^2)(a^2 - c^2) \leq \frac{1}{2}(2a^2 - b^2 - c^2)^2. \end{aligned}$$

Therefore, it suffices to show that

$$8(a^2 + 4b^2 + 4c^2) \geq 27(b - c)^2,$$

which is equivalent to the obvious inequality

$$8a^2 + 5b^2 + 5c^2 + 54bc \geq 0.$$

The equality holds for  $a = b = c$ , and for  $-a = b = c$  (or any cyclic permutation).

□

**P 2.28.** *If  $a, b, c$  are real numbers, then*

$$(a^2 + b^2 + c^2)^3 \geq \frac{108}{5}a^2b^2c^2 + 2(a - b)^2(b - c)^2(c - a)^2.$$

(Vo Quoc Ba Can and Vasile Cîrtoaje, 2011)

**Solution.** Write the inequality as  $f(a, b, c) \geq 0$ , where

$$f(a, b, c) = (a^2 + b^2 + c^2)^3 - \frac{108}{5}a^2b^2c^2 - 2(a-b)^2(b-c)^2(c-a)^2.$$

Without loss of generality, assume that  $b$  and  $c$  have the same sign. Since  $f(-a, -b, -c) = f(a, b, c)$ , we may consider  $b \geq 0, c \geq 0$ . In addition, for  $a > 0$ , we have

$$f(a, b, c) - f(-a, b, c) = 8a(b+c)(a^2+bc)(b-c)^2 \geq 0.$$

Therefore, it suffices to prove the desired inequality for  $a \leq 0, b \geq 0, c \geq 0$ . For  $b = c = 0$ , the inequality is trivial. Otherwise, due to homogeneity, we may assume that  $b + c = 1$ . Denoting  $x = bc$ , we can write the desired inequality as follows:

$$(a^2 + 1 - 2x)^3 \geq \frac{108}{5}a^2x^2 + 2(1 - 2x)(a^2 - a + x)2,$$

$$\frac{2}{5}(4a - 5)^2x^2 + 2(a + 1)(a^3 - 9a^2 + 5a - 3)x + (a + 1)^2(a^4 - 2a^3 + 4a^2 - 2a + 1) \geq 0.$$

This inequality holds if

$$\frac{2}{5}(4a - 5)^2(a^4 - 2a^3 + 4a^2 - 2a + 1) \geq (a^3 - 9a^2 + 5a - 3)^2.$$

Since

$$10(a^4 - 2a^3 + 4a^2 - 2a + 1) = (a + 1)^2 + (3a^2 - 4a + 3)^2 \geq (3a^2 - 4a + 3)^2,$$

it suffices to prove that

$$(4a - 5)^2(3a^2 - 4a + 3)^2 \geq 25(a^3 - 9a^2 + 5a - 3)^2.$$

This is true for  $a \leq 0$  if

$$(5 - 4a)(3a^2 - 4a + 3) \geq 5(-a^3 + 9a^2 - 5a + 3),$$

which reduces to

$$a(a + 1)^2 \leq 0.$$

Thus, the proof is completed. The equality holds for  $a = 0$  and  $b + c = 0$  (or any cyclic permutation). □

**P 2.29.** If  $a, b, c$  are real numbers, then

$$2(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \geq (a - b)^2(b - c)^2(c - a)^2.$$

(Vasile Cîrtoaje, 2011)

**First Solution.** Since

$$2(a^2 + b^2) = (a - b)^2 + (a + b)^2$$

and

$$(b^2 + c^2)(c^2 + a^2) = (ab + c^2)^2 + (bc - ac)^2,$$

by virtue of the Cauchy-Schwarz inequality, we have

$$\begin{aligned} 2(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) &\geq [(a - b)(ab + c^2) + (a + b)(bc - ac)]^2 \\ &= (a^2b + b^2c + c^2a - ab^2 - bc^2 - ca^2)^2 \\ &= (a - b)^2(b - c)^2(c - a)^2. \end{aligned}$$

This completes the proof. The equality holds for  $(a - b)(bc - ac) = (a + b)(ab + c^2)$ , which is equivalent to

$$(a + b + c)(ab + bc + ca) = 5abc.$$

**Second Solution.** Making the substitution

$$x = \sum ab^2 = ab^2 + bc^2 + ca^2, \quad y = \sum a^2b = a^2b + b^2c + c^2a,$$

we have

$$\begin{aligned} \sum a^2b^4 &= (\sum ab^2)^2 - 2abc \sum a^2b = x^2 - 2abcy, \\ \sum a^4b^2 &= (\sum a^2b)^2 - 2abc \sum ab^2 = y^2 - 2abcx, \end{aligned}$$

hence

$$\begin{aligned} \prod(a^2 + b^2) &= \sum a^2b^4 + \sum a^4b^2 + 2a^2b^2c^2 \\ &= x^2 + y^2 - 2abc(x + y) + 2a^2b^2c^2. \end{aligned}$$

Then, the desired inequality is equivalent to

$$\begin{aligned} 2[x^2 + y^2 - 2abc(x + y) + 2a^2b^2c^2] &\geq (x - y)^2, \\ (x + y)^2 - 4abc(x + y) + 4a^2b^2c^2 &\geq 0, \\ (x + y - 2abc)^2 &\geq 0. \end{aligned}$$

□

**P 2.30.** If  $a, b, c$  are real numbers, then

$$32(a^2 + bc)(b^2 + ca)(c^2 + ab) + 9(a - b)^2(b - c)^2(c - a)^2 \geq 0.$$

(Vasile Cîrtoaje, 2011)

**Solution** (by Vo Quoc Ba Can). For  $a, b, c \geq 0$ , the inequality is trivial. Otherwise, since the inequality is symmetric and does not change by substituting  $-a, -b, -c$  for  $a, b, c$ , we may assume that  $a \leq 0$  and  $b, c \geq 0$ . Substituting  $-a$  for  $a$ , we need to prove that

$$32(a^2 + bc)(b^2 - ac)(c^2 - ab) + 9(a + b)^2(a + c)^2(b - c)^2 \geq 0$$

for all  $a, b, c \geq 0$ . By the AM-GM inequality, we have

$$(a + b)^2(a + c)^2 = [a(b + c) + (a^2 + bc)]^2 \geq 4a(b + c)(a^2 + bc).$$

Thus, it suffices to prove that

$$32(a^2 + bc)(b^2 - ac)(c^2 - ab) + 36a(b + c)(a^2 + bc)(b - c)^2 \geq 0,$$

which is true if

$$8(b^2 - ac)(c^2 - ab) + 9a(b + c)(b - c)^2 \geq 0.$$

Since

$$\begin{aligned} (b^2 - ac)(c^2 - ab) &= bc(bc + a^2) - a(b^3 + c^3) \\ &\geq 2abc\sqrt{bc} - a(b^3 + c^3) = -a(b\sqrt{b} - c\sqrt{c})^2, \end{aligned}$$

it is enough to show that

$$9(b + c)(b - c)^2 - 8(b\sqrt{b} - c\sqrt{c})^2 \geq 0.$$

Using the notation

$$\sqrt{b} = x, \quad \sqrt{c} = y,$$

this inequality can be rewritten as

$$(x - y)^2[9(x^2 + y^2)(x + y)^2 - 8(x^2 + xy + y^2)^2] \geq 0,$$

It is true because, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} 9(x^2 + y^2)(x + y)^2 &= 9[(x - y)^2 + 2xy][(x - y)^2 + 4xy] \\ &\geq 9[(x - y)^2 + 2\sqrt{2}xy]^2 \geq 9 \left[ \frac{2\sqrt{2}}{3}(x - y)^2 + 2\sqrt{2}xy \right]^2 \\ &= 8(x^2 + xy + y^2)^2 \geq 0. \end{aligned}$$

The equality occurs when two of  $a, b, c$  are zero, and when  $-a = b = c$  (or any cyclic permutation).

□

**P 2.31.** If  $a, b, c$  are real numbers, then

$$a^4(b-c)^2 + b^4(c-a)^2 + c^4(a-b)^2 \geq \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2011)

**Solution.** Since

$$\begin{aligned} b^4(c-a)^2 + c^4(a-b)^2 &\geq \frac{1}{2}[b^2(c-a) + c^2(a-b)]^2 \\ &= \frac{1}{2}(b-c)^2(bc-ca-ab)^2, \end{aligned}$$

it suffices to prove that

$$2a^4 + (ab-bc+ca)^2 \geq (a-b)^2(a-c)^2,$$

which is equivalent to

$$a^2(a^2 - 2bc + 2ca + 2ab) \geq 0.$$

Therefore, the desired inequality is true if

$$a^2 - 2bc + 2ca + 2ab \geq 0.$$

Indeed, from

$$\sum (a^2 - 2bc + 2ca + 2ab) = (a+b+c)^2 \geq 0,$$

due to symmetry, we may assume that  $a^2 - 2bc + 2ca + 2ab \geq 0$ . Thus, the proof is completed. The equality occurs when  $a = b = c$ , when two of  $a, b, c$  are equal to zero, and when  $a = 0$  and  $b + c = 0$  (or any cyclic permutation). □

**P 2.32.** If  $a, b, c$  are real numbers, then

$$a^2(b-c)^4 + b^2(c-a)^4 + c^2(a-b)^4 \geq \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2011)

**Solution.** Let us denote

$$x = \sum ab^2 = ab^2 + bc^2 + ca^2, \quad y = \sum a^2b = a^2b + b^2c + c^2a.$$

Since

$$\begin{aligned} \sum a^2b^4 &= (\sum ab^2)^2 - 2abc \sum a^2b = x^2 - 2abcy, \\ \sum a^4b^2 &= (\sum a^2b)^2 - 2abc \sum ab^2 = y^2 - 2abcx, \end{aligned}$$

$$\sum a^2b^2(a^2 + b^2) = x^2 + y^2 - 2abc(x + y),$$

we have

$$\begin{aligned} \sum a^2(b - c)^4 &= \sum a^2(b^4 - 4b^3c + 6b^2c^2 - 4bc^3 + c^4) \\ &= \sum a^2b^2(a^2 + b^2) - 4abc(\sum ab^2 + \sum a^2b) + 18a^2b^2c^2 \\ &= x^2 + y^2 - 6abc(x + y) + 18a^2b^2c^2. \end{aligned}$$

Therefore, we can write the desired inequality as

$$x^2 + y^2 - 6abc(x + y) + 18a^2b^2c^2 \geq \frac{1}{2}(x - y)^2,$$

which is equivalent to the obvious inequality

$$(x + y - 6abc)^2 \geq 0.$$

The equality holds for

$$a(b - c)^2 + b(c - a)^2 + c(a - b)^2 = 0.$$

□

**P 2.33.** If  $a, b, c$  are real numbers, then

$$a^2(b^2 - c^2)^2 + b^2(c^2 - a^2)^2 + c^2(a^2 - b^2)^2 \geq \frac{3}{8}(a - b)^2(b - c)^2(c - a)^2.$$

(Vasile Cîrtoaje, 2011)

**Solution.** We see that the inequality remains unchanged and the product

$$(a + b)(b + c)(c + a)$$

changes its sign by replacing  $a, b, c$  with  $-a, -b, -c$ , respectively. Thus, without loss of generality, we may assume that

$$(a + b)(b + c)(c + a) \geq 0.$$

According to this condition, at least one of  $a, b, c$  is nonnegative. So, we may consider  $a \geq 0$ , and hence

$$a(a + b)(b + c)(c + a) \geq 0.$$

Since

$$b^2(c^2 - a^2)^2 + c^2(a^2 - b^2)^2 \geq \frac{1}{2} [b(c^2 - a^2) + c(a^2 - b^2)]^2 = \frac{1}{2}(b - c)^2(a^2 + bc)^2,$$

it suffices to show that

$$2a^2(b + c)^2 + (a^2 + bc)^2 \geq \frac{3}{4}(a - b)^2(a - c)^2,$$

which is equivalent to

$$\begin{aligned}(a+b)(a+c)[a^2 + 5a(b+c) + bc] &\geq 0, \\ (a+b)(a+c)[(a+b)(a+c) + 4a(b+c)] &\geq 0, \\ (a+b)^2(a+c)^2 + 4a(a+b)(b+c)(c+a) &\geq 0.\end{aligned}$$

Since the last inequality is clearly true, the proof is completed. The equality holds for  $a = b = c$ , for  $-a = b = c$  (or any cyclic permutation), and for  $b = c = 0$  (or any cyclic permutation). □

**P 2.34.** If  $a, b, c$  are real numbers such that

$$ab + bc + ca = 3,$$

then

$$(a) \quad (a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq 3(a+b+c)^2;$$

$$(b) \quad (a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq \frac{3}{2}(a^2 + b^2 + c^2).$$

(Vasile Cîrtoaje, 1995)

**Solution.** Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

We have

$$\begin{aligned}\prod(b^2 + bc + c^2) &= \prod[(b+c)^2 - bc] \\ &= \prod(b+c)^2 - \sum bc(a+b)^2(a+c)^2 + abc \sum a(b+c)^2 - a^2b^2c^2.\end{aligned}$$

Since

$$\begin{aligned}\prod(b+c)^2 &= (pq - r)^2 = r^2 - 2pqr + p^2q^2, \\ \sum bc(a+b)^2(a+c)^2 &= \sum bc(a^2 + q)^2 = r \sum a^3 + 2pqr + q^2 \\ &= r(3r + p^3 - 3pq) + 2pqr + q^2 = 3r^2 + (p^3 - pq)r + q^3\end{aligned}$$

and

$$abc \sum a(b+c)^2 = r(3r + pq) = 3r^2 + pqr,$$

we get

$$\prod(b^2 + bc + c^2) = (p^2 - q)q^2 - p^3r.$$



(a) Write the inequality as follows

$$3 \prod (b^2 + bc + c^2) \geq (a + b + c)^2 (ab + bc + ca)^2,$$

$$(2p^2 - 3q)q^2 - 3p^3r \geq 0,$$

$$q^2(p^2 - 3q) + p^2(q^2 - 3pr) \geq 0,$$

$$q^2 \sum (b - c)^2 + p^2 \sum a^2(b - c)^2 \geq 0.$$

Clearly, the last inequality holds for all real  $a, b, c$ . The equality holds when  $a = b = c = \pm 1$ .

(b) Write the inequality in the homogeneous forms

$$2 \prod (b^2 + bc + c^2) \geq (a^2 + b^2 + c^2)(ab + bc + ca)^2,$$

$$2(p^2 - q)q^2 - 2p^3r - (p^2 - 2q)q^2 \geq 0,$$

$$p^2(q^2 - 2pr) \geq 0,$$

$$(a + b + c)^2(a^2b^2 + b^2c^2 + c^2a^2) \geq 0.$$

The equality holds when  $a + b + c = 0$  and  $ab + bc + ca = 3$ .

□

**P 2.35.** If  $a, b, c$  are real numbers, then

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq 3(ab + bc + ca)(a^2b^2 + b^2c^2 + c^2a^2).$$

(Vasile Cîrtoaje, 2011)

**Solution.** As we have shown in the proof of the preceding P 2.34,

$$\prod (b^2 + bc + c^2) = (p^2 - q)q^2 - p^3r,$$

where

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

Thus, we can write the desired inequality as

$$(p^2 - q)q^2 - p^3r \geq 3q(q^2 - 2pr),$$

$$q^2(p^2 - 4q) + p(6q - p^2)r \geq 0.$$

Consider further two cases:  $6q - p^2 \geq 0$  and  $6q - p^2 \leq 0$ .

*Case 1:*  $6q - p^2 \geq 0$ . By Schur's inequality of degree four, we have

$$pr \geq \frac{1}{6}(p^2 - q)(4q - p^2).$$

Therefore, it suffices to show that

$$q^2(p^2 - 4q) + \frac{1}{6}(6q - p^2)(p^2 - q)(4q - p^2) \geq 0,$$

which is equivalent to the obvious inequality

$$(p^2 - 4q)^2(p^2 - 3q) \geq 0.$$

*Case 2:*  $6q - p^2 \leq 0$ . Since

$$pr \leq \frac{1}{3}q^2,$$

it suffices to show that

$$q^2(p^2 - 4q) + \frac{1}{3}(6q - p^2)q^2 \geq 0,$$

which is equivalent to the obvious inequality

$$q^2(p^2 - 3q) \geq 0.$$

This completes the proof. The inequality is an equality for  $a = b = c$ , for  $a = 0$  and  $b = c$  (or any cyclic permutation), and for  $b = c = 0$  (or any cyclic permutation). □

**P 2.36.** *If  $a, b, c$  are real numbers, not all of the same sign, then*

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq 3(ab + bc + ca)^3.$$

(Vasile Cîrtoaje, 2011)

**Solution.** Since the inequality is symmetric and does not change by substituting  $-a, -b, -c$  for  $a, b, c$ , we may assume that  $a \leq 0$  and  $b, c \geq 0$ . Substituting  $-a$  for  $a$ , we need to prove that

$$(a^2 - ab + b^2)(b^2 + bc + c^2)(c^2 - ca + a^2) \geq 3(bc - ab - ac)^3$$

for  $a, b, c \geq 0$ . Since the left hand side of this inequality is nonnegative, consider further the nontrivial case

$$bc - ab - ac > 0.$$

Since

$$b^2 + bc + c^2 - 3(bc - ab - ac) = (b - c)^2 + 3a(b + c) \geq 0,$$

it suffices to show that

$$(a^2 - ab + b^2)(a^2 - ac + c^2) \geq (bc - ab - ac)^2.$$

**First Solution.** From  $bc - ab - ac > 0$ , it follows that  $a = \min\{a, b, c\}$ . Since

$$a^2 - ab + b^2 \geq (b - a)^2, \quad a^2 - ac + c^2 \geq (c - a)^2,$$

it suffices to show that

$$(b-a)^2(c-a)^2 \geq (bc-ab-ac)^2.$$

This is true if  $(b-a)(c-a) \geq bc-ab-ac$ ; indeed,

$$(b-a)(c-a) - (bc-ab-ac) = a^2 \geq 0.$$

The original inequality is an equality when two of  $a, b, c$  are zero, and when  $a = 0$  and  $b = c$  (or any cyclic permutation).

**Second Solution.** Since

$$4(a^2 - ab + b^2) = (a+b)^2 + 3(a-b)^2,$$

$$4(a^2 - ac + c^2) = (a+c)^2 + 3(a-c)^2,$$

we can apply the Cauchy-Schwarz inequality as follows

$$16(a^2 - ab + b^2)(b^2 + bc + c^2) \geq [(a+b)(a+c) + 3(a-b)(a-c)]^2.$$

Thus, we only need to show that

$$(a+b)(a+c) + 3(a-b)(a-c) \geq 4(bc-ab-ac),$$

which is equivalent to the obvious inequality  $a(2a+b+c) \geq 0$ .

□

**P 2.37.** If  $a, b, c$  are real numbers, then

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq \frac{3}{8}(a^2 + b^2)(b^2 + c^2)(c^2 + a^2).$$

(Vasile Cîrtoaje, 2011)

**Solution.** If  $a, b, c$  have the same sign, then the inequality follows from

$$a^2 + ab + b^2 \geq a^2 + b^2, \quad b^2 + bc + c^2 \geq b^2 + c^2, \quad c^2 + ca + a^2 \geq c^2 + a^2.$$

Consider now that  $a, b, c$  have not the same sign. Since the inequality is symmetric and does not change by substituting  $-a, -b, -c$  for  $a, b, c$ , we may assume that  $a \leq 0$  and  $b, c \geq 0$ . Substituting  $-a$  for  $a$ , we need to prove that

$$(a^2 - ab + b^2)(a^2 - ac + c^2)(b^2 + bc + c^2) \geq \frac{3}{8}(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)$$

for  $a, b, c \geq 0$ . Write this inequality in the form

$$\begin{aligned} & [(a^2 + b^2) + (a-b)^2][(a^2 + c^2) + (a-c)^2][2(b^2 + c^2) + 2bc] \geq \\ & \geq 3(a^2 + b^2)(b^2 + c^2)(c^2 + a^2). \end{aligned}$$

It suffices to show that

$$\begin{aligned} 2(b^2 + c^2)[(a - b)^2(a^2 + c^2) + (a - c)^2(a^2 + b^2)] + 2bc(a^2 + b^2)(a^2 + c^2) &\geq \\ &\geq (a^2 + b^2)(b^2 + c^2)(c^2 + a^2), \end{aligned}$$

which is equivalent to

$$2(b^2 + c^2)[(a - b)^2(a^2 + c^2) + (a - c)^2(a^2 + b^2)] \geq (b - c)^2(a^2 + b^2)(a^2 + c^2).$$

For the nontrivial case where

$$a^2 + b^2 \neq 0, \quad b^2 + c^2 \neq 0, \quad c^2 + a^2 \neq 0,$$

we rewrite the inequality in the form

$$\frac{(a - b)^2}{a^2 + b^2} + \frac{(a - c)^2}{a^2 + c^2} \geq \frac{(b - c)^2}{2(b^2 + c^2)}.$$

Consider further two cases.

*Case 1:*  $2a^2 \leq b^2 + c^2$ . By the Cauchy-Schwarz inequality, we have

$$\frac{(a - b)^2}{a^2 + b^2} + \frac{(a - c)^2}{a^2 + c^2} \geq \frac{[(b - a) + (a - c)]^2}{(a^2 + b^2) + (a^2 + c^2)} = \frac{(b - c)^2}{2a^2 + b^2 + c^2}.$$

Thus, it suffices to show that

$$\frac{1}{2a^2 + b^2 + c^2} \geq \frac{1}{2(b^2 + c^2)},$$

which reduces to  $b^2 + c^2 \geq 2a^2$ .

*Case 2:*  $2a^2 \geq b^2 + c^2$ . By the Cauchy-Schwarz inequality, we have

$$\frac{(a - b)^2}{a^2 + b^2} + \frac{(a - c)^2}{a^2 + c^2} \geq \frac{[c(b - a) + b(a - c)]^2}{c^2(a^2 + b^2) + b^2(a^2 + c^2)} = \frac{a^2(b - c)^2}{a^2(b^2 + c^2) + 2b^2c^2}.$$

Therefore, it suffices to prove that

$$\frac{a^2}{a^2(b^2 + c^2) + 2b^2c^2} \geq \frac{1}{2(b^2 + c^2)}.$$

This reduces to

$$a^2(b^2 + c^2) \geq 2b^2c^2,$$

which is true because

$$2a^2(b^2 + c^2) - 4b^2c^2 \geq (b^2 + c^2)^2 - 4b^2c^2 = (b^2 - c^2)^2 \geq 0.$$

Thus, the proof is completed. The equality holds when two of  $a, b, c$  are zero, and when  $-a = b = c$  (or any cyclic permutation).

□

**P 2.38.** If  $a, b, c$  are real numbers, then

$$2(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \geq (a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2).$$

(Vasile Cîrtoaje, 2014)

**Solution.** If  $a, b, c$  have the same sign, then the inequality follows from

$$a^2 + b^2 \geq a^2 - ab + b^2, \quad b^2 + c^2 \geq b^2 - bc + c^2, \quad c^2 + a^2 \geq c^2 - ca + a^2.$$

Consider now that  $a, b, c$  have not the same sign. Since the inequality is symmetric and does not change by substituting  $-a, -b, -c$  for  $a, b, c$ , we may assume that  $a \leq 0$  and  $b, c \geq 0$ . Substituting  $-a$  for  $a$ , we need to prove that

$$2(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \geq (b^2 - bc + c^2)(c^2 + ca + a^2)(a^2 + ab + b^2)$$

for  $a, b, c \geq 0$ . Using the notation

$$A = b^2 + c^2, \quad B = c^2 + a^2, \quad C = a^2 + b^2,$$

we can write the inequality as follows:

$$2ABC \geq (A - bc)(B + ca)(C + ab),$$

$$ABC + a^2b^2c^2 \geq ab(AB - c^2C) + ac(AC - b^2B) - bc(BC - a^2A),$$

$$ABC + a^2b^2c^2 \geq ab(c^4 + a^2b^2) + ac(b^4 + a^2c^2) - bc(a^4 + b^2c^2).$$

It suffices to show that

$$ABC + a^2b^2c^2 \geq ab(c^4 + a^2b^2) + ac(b^4 + a^2c^2) + bc(a^4 + b^2c^2).$$

Moreover, since

$$2ab \leq a^2 + b^2, \quad 2ac \leq a^2 + c^2, \quad 2bc \leq b^2 + c^2,$$

it is enough to prove that

$$2ABC + 2a^2b^2c^2 \geq (a^2 + b^2)(c^4 + a^2b^2) + (a^2 + c^2)(b^4 + a^2c^2) + (b^2 + c^2)(a^4 + b^2c^2).$$

Indeed, this inequality reduces to the obvious inequality

$$6a^2b^2c^2 \geq 0.$$

The equality holds when two of  $a, b, c$  are zero.

□

**P 2.39.** If  $a, b, c$  are real numbers, then

$$9(1 + a^4)(1 + b^4)(1 + c^4) \geq 8(1 + abc + a^2b^2c^2)^2.$$

(Vasile Cîrtoaje, 2004)

**Solution.** Substituting  $a, b, c$  by  $|a|, |b|, |c|$ , respectively, the left side of the inequality remains unchanged, while the right side either remains unchanged or increases. Therefore, it suffices to prove the inequality only for  $a, b, c \geq 0$ . If  $a = b = c$ , then the inequality reduces to

$$9(1 + a^4)^3 \geq 8(1 + a^3 + a^6)^2,$$

$$9(a^2 + \frac{1}{a^2})^3 \geq 8(a^3 + \frac{1}{a^3} + 1)^2.$$

Setting

$$x = a + \frac{1}{a},$$

this inequality can be written as follows

$$9(x^2 - 2)^3 \geq 8(x^3 - 3x + 1)^2,$$

$$x^6 - 6x^4 - 16x^3 + 36x^2 + 48x - 80 \geq 0,$$

$$(x - 2)^2[x(x^3 - 8) + 4(x^3 - 5) + 6x^2] \geq 0.$$

Since  $x \geq 2$ , the last inequality is clearly true. Multiplying now the inequalities

$$9(1 + a^4)^3 \geq 8(1 + a^3 + a^6)^2,$$

$$9(1 + b^4)^3 \geq 8(1 + b^3 + b^6)^2,$$

$$9(1 + c^4)^3 \geq 8(1 + c^3 + c^6)^2,$$

we get

$$[9(1 + a^4)(1 + b^4)(1 + c^4)]^3 \geq 8^3(1 + a^3 + a^6)^2(1 + b^3 + b^6)^2(1 + c^3 + c^6)^2.$$

From this inequality and Hölder's inequality

$$(1 + a^3 + a^6)(1 + b^3 + b^6)(1 + c^3 + c^6) \geq (1 + abc + a^2b^2c^2)^3,$$

the conclusion follows. The equality holds for  $a = b = c = 1$ .

□

**P 2.40.** If  $a, b, c$  are real numbers, then

$$2(1 + a^2)(1 + b^2)(1 + c^2) \geq (1 + a)(1 + b)(1 + c)(1 + abc).$$

(Vasile Cîrtoaje, 2001)

**Solution.** Substituting  $a, b, c$  by  $|a|, |b|, |c|$ , respectively, the left side of the inequality remains unchanged, while the right side either remains unchanged or increases. Therefore, it suffices to prove the inequality only for  $a, b, c \geq 0$ .

**First Solution.** For  $a = b = c$ , the inequality reduces to

$$2(1 + a^2)^3 \geq (1 + a)^3(1 + a^3).$$

This is true since

$$2(1 + a^2)^3 - (1 + a)^3(1 + a^3) = (1 - a)^4(1 + a + a^2) \geq 0.$$

Multiplying the inequalities

$$2(1 + a^2)^3 \geq (1 + a)^3(1 + a^3),$$

$$2(1 + b^2)^3 \geq (1 + b)^3(1 + b^3),$$

$$2(1 + c^2)^3 \geq (1 + c)^3(1 + c^3),$$

we get

$$8(1 + a^2)^3(1 + b^2)^3(1 + c^2)^3 \geq (1 + a)^3(1 + b)^3(1 + c)^3(1 + a^3)(1 + b^3)(1 + c^3).$$

Using this result, we still have to show that

$$(1 + a^3)(1 + b^3)(1 + c^3) \geq (1 + abc)^3,$$

which is just Hölder's inequality. We can also prove this inequality by adding the inequalities

$$a^3b^3 + b^3c^3 + c^3a^3 \geq 3a^2b^2c^2,$$

$$a^3 + b^3 + c^3 \geq 3abc.$$

The equality holds for  $a = b = c = 1$ .

**Second Solution.** We use the substitution

$$a = \frac{1 - x}{1 + x}, \quad b = \frac{1 - y}{1 + y}, \quad c = \frac{1 - z}{1 + z},$$

where  $x, y, z \in (-1, 1]$ . Since

$$\frac{1 + a^2}{1 + a} = \frac{1 + x^2}{1 + x}, \quad \frac{1 + b^2}{1 + b} = \frac{1 + y^2}{1 + y}, \quad \frac{1 + c^2}{1 + c} = \frac{1 + z^2}{1 + z}$$

and

$$1 + abc = \frac{2(1 + xy + yz + zx)}{(1 + x)(1 + y)(1 + z)},$$

the required inequality becomes

$$(1 + x^2)(1 + y^2)(1 + z^2) \geq 1 + xy + yz + zx,$$

$$x^2y^2 + y^2z^2 + z^2x^2 + x^2 + y^2 + z^2 \geq xy + yz + zx,$$

$$x^2y^2 + y^2z^2 + z^2x^2 + \frac{1}{2}(x - y)^2 + \frac{1}{2}(y - z)^2 + \frac{1}{2}(z - x)^2 \geq 0.$$

□

**P 2.41.** Let  $a, b, c$  be real numbers. Prove that

$$(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) \geq 1$$

for

$$\begin{aligned} (a) \quad & a^3b^3 + b^3c^3 + c^3a^3 = 3; \\ (b) \quad & a^4b^4 + b^4c^4 + c^4a^4 = 3. \end{aligned} \quad (\text{Titu Andreescu, 2006})$$

**Solution.** (a) Substituting  $a, b, c$  by  $|a|, |b|, |c|$ , respectively, the left side of the inequality remains unchanged or decreases, while the right side remains unchanged or increases. Therefore, it suffices to prove the inequality for  $a, b, c \geq 0$ . If  $a = 0$ , then the inequality reduces to  $b^2c^2(b - c)^2 \geq 0$ . Consider further then  $a, b, c > 0$ . We first show that

$$3(a^2 - ab + b^2)^3 \geq a^6 + a^3b^3 + b^6.$$

Indeed, setting

$$x = \frac{a}{b} + \frac{b}{a}, \quad x \geq 2,$$

we can write this inequality as

$$\begin{aligned} 3(x - 1)^3 &\geq x^3 - 3x + 1, \\ (x - 2)^2(2x - 1) &\geq 0. \end{aligned}$$

Using this result, we have

$$\begin{aligned} 27(a^2 - ab + b^2)^3(b^2 - bc + c^2)^3(c^2 - ca + a^2)^3 &\geq \\ &\geq (a^6 + a^3b^3 + b^6)(b^6 + b^3c^3 + c^6)(c^6 + c^3a^3 + a^6). \end{aligned}$$

Therefore, it suffices to show that

$$(a^6 + a^3b^3 + b^6)(b^6 + b^3c^3 + c^6)(c^6 + c^3a^3 + a^6) \geq 27.$$

Writing this inequality in the form

$$(a^3b^3 + b^6 + a^6)(b^6 + b^3c^3 + c^6)(a^6 + c^6 + c^3a^3) \geq (a^3b^3 + b^3c^3 + c^3a^3)^3,$$

we see that it is just Hölder's inequality. The equality holds when  $a = b = c = 1$ .

(b) Since

$$2(a^2 - ab + b^2)^2 - (a^4 + b^4) = (a - b)^4 \geq 0,$$

it suffices to show that

$$(a^4 + b^4)(b^4 + c^4)(c^4 + a^4) \geq 8.$$

In addition, since

$$9(a^4 + b^4)(b^4 + c^4)(c^4 + a^4) \geq 8(a^4 + b^4 + c^4)(a^4b^4 + b^4c^4 + c^4a^4) = 24(a^4 + b^4 + c^4),$$



we only need to show that

$$a^4 + b^4 + c^4 \geq 3.$$

This is true because

$$\begin{aligned} (a^4 + b^4 + c^4)^2 - 9 &= (a^4 + b^4 + c^4)^2 - 3(a^4b^4 + b^4c^4 + c^4a^4) \\ &= \frac{1}{2} [(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2] \geq 0. \end{aligned}$$

The equality holds when  $a = b = c = 1$ .

□

**P 2.42.** If  $a, b, c$  are nonzero real numbers, then

$$\sum \frac{b^2 - bc + c^2}{a^2} + 2 \sum \frac{a^2}{bc} \geq \left( \sum a \right) \left( \sum \frac{1}{a} \right).$$

(Vasile Cîrtoaje, 2010)

**Solution.** We have

$$\begin{aligned} \sum \frac{b^2 - bc + c^2}{a^2} + 2 \sum \frac{a^2}{bc} &= \sum \left( \frac{b^2 - bc + c^2}{a^2} + \frac{b^2}{ca} + \frac{c^2}{ab} \right) \\ &= \sum \frac{(b^2 - bc + c^2)(ab + bc + ca)}{a^2bc} \\ &= \frac{ab + bc + ca}{a^2b^2c^2} \sum bc(b^2 - bc + c^2). \end{aligned}$$

Then, we can write the inequality as

$$(ab + bc + ca) \left[ \sum bc(b^2 - bc + c^2) - abc \sum a \right] \geq 0.$$

Since

$$\begin{aligned} \sum bc(b^2 - bc + c^2) - abc \sum a &= \left( \sum bc \right) \left( \sum a^2 \right) - \sum b^2c^2 - 2abc \sum a \\ &= \left( \sum bc \right) \left( \sum a^2 \right) - \left( \sum bc \right)^2 \\ &= \left( \sum bc \right) \left( \sum a^2 - \sum bc \right), \end{aligned}$$

the inequality is equivalent to

$$(ab + bc + ca)^2(a^2 + b^2 + c^2 - ab - bc - ca) \geq 0,$$

which is true. The equality holds for  $a = b = c$ , and also for  $ab + bc + ca = 0$ .

□

**P 2.43.** Let  $a, b, c$  be real numbers. Prove that

(a) if  $a, b, c \in [0, 1]$ , then

$$abc - (b + c - a)(c + a - b)(a + b - c) \leq 1;$$

(b) if  $a, b, c \in [-1, 1]$ , then

$$abc - (b + c - a)(c + a - b)(a + b - c) \leq 4.$$

(Vasile Cîrtoaje, 2011)

**Solution.** We will show that if  $a, b, c \in [m, M]$ , where  $M \geq 0$ , then

$$abc - (b + c - a)(c + a - b)(a + b - c) \leq M(M - m)^2.$$

Without loss of generality, assume that

$$M \geq a \geq b \geq c \geq m.$$

We have two cases to consider.

*Case 1:*  $a \leq 0$ . The required inequality is true, since

$$abc - (b + c - a)(c + a - b)(a + b - c) \leq 0 \leq M(M - m)^2.$$

Indeed, substituting  $-a, -b, -c$  for  $a, b, c$ , respectively, the left inequality can be restated as

$$abc \geq (b + c - a)(c + a - b)(a + b - c),$$

where  $a, b, c \geq 0$ . This is just the well-known Schur's inequality of degree three.

*Case 2:*  $a > 0$ . Since  $(M - m)^2 \geq (a - c)^2$  and  $M \geq a$ , we have

$$M(M - m)^2 \geq a(a - c)^2.$$

Therefore, it suffices to show that

$$abc - (b + c - a)(c + a - b)(a + b - c) \leq a(a - c)^2,$$

which is equivalent to

$$(b - c)[a^2 + (b - 2c)a - b^2 + c^2] \geq 0.$$

This is true since  $b - c \geq 0$  and

$$\begin{aligned} a^2 + (b - 2c)a - b^2 + c^2 &= (a - b)(a + 2b - 2c) + (b - c)^2 \\ &\geq 2(a - b)(b - c) + (b - c)^2 \geq 0. \end{aligned}$$

Thus, the proof is completed. The equality holds for  $a = M$  and  $b = c = m$  (or any cyclic permutation).

□

**P 2.44.** Let  $a, b, c$  be real numbers. Prove that

(a) if  $a, b, c \in [0, 1]$ , then

$$\sum a^2(a-b)(a-c) \leq 1;$$

(b) if  $a, b, c \in [-1, 1]$ , then

$$\sum a^2(a-b)(a-c) \leq 4.$$

(Vasile Cîrtoaje, 2011)

**Solution.** We will show that if  $a, b, c \in [m, M]$ , then

$$\sum a^2(a-b)(a-c) \leq (M-m)^2 \cdot \max\{m^2, M^2\}.$$

Without loss of generality, assume that

$$M \geq a \geq b \geq c \geq m.$$

Since

$$b^2(b-c)(b-a) \leq 0, \quad (a-c)^2 \leq (M-m)^2, \quad \max\{a^2, c^2\} \leq \max\{m^2, M^2\},$$

it suffices to show that

$$a^2(a-b)(a-c) + c^2(c-a)(c-b) \leq (a-c)^2 \cdot \max\{a^2, c^2\}.$$

This is equivalent to

$$(a-c)^2 [a^2 + c^2 + ac - ab - bc - \max\{a^2, c^2\}] \leq 0,$$

that is true if

$$a^2 + c^2 + ac - ab - bc - \max\{a^2, c^2\} \leq 0.$$

*Case 1:*  $a^2 \geq c^2$ . From

$$a^2 - c^2 = (a-c)(a+c) \geq 0,$$

it follows that  $a+c \geq 0$ . Then,

$$a^2 + c^2 + ac - ab - bc - \max\{a^2, c^2\} = (a+c)(c-b) \leq 0.$$

*Case 2:*  $a^2 \leq c^2$ . From

$$a^2 - c^2 = (a-c)(a+c) \leq 0,$$

it follows that  $a+c \leq 0$ . Then,

$$a^2 + c^2 + ac - ab - bc - \max\{a^2, c^2\} = (a+c)(a-b) \leq 0.$$

Thus, the proof is completed. For  $M^2 \geq m^2$ , the equality holds when  $a = M$  and  $b = c = m$  (or any cyclic permutation). For  $M^2 \leq m^2$ , the equality holds when  $a = m$  and  $b = c = M$  (or any cyclic permutation).

□

**P 2.45.** Let  $a, b, c$  be real numbers such that

$$ab + bc + ca = abc + 2.$$

Prove that

$$a^2 + b^2 + c^2 - 3 \geq (2 + \sqrt{3})(a + b + c - 3).$$

(Vasile Cîrtoaje, 2011)

**Solution.** Substituting  $a + 1, b + 1, c + 1$  for  $a, b, c$ , respectively, we need to prove that

$$a + b + c = abc$$

implies

$$a^2 + b^2 + c^2 \geq \sqrt{3}(a + b + c).$$

This inequality is true if

$$(a^2 + b^2 + c^2)^2 \geq 3(a + b + c)^2,$$

which is equivalent to the homogeneous inequality

$$(a^2 + b^2 + c^2)^2 \geq 3abc(a + b + c).$$

Since

$$(ab + bc + ca)^2 - 3abc(a + b + c) = \frac{1}{2} \sum a^2(b - c)^2 \geq 0,$$

it suffices to prove that

$$(a^2 + b^2 + c^2)^2 \geq (ab + bc + ca)^2,$$

which is equivalent to

$$(a^2 + b^2 + c^2 - ab - bc - ca)(a^2 + b^2 + c^2 + ab + bc + ca) \geq 0.$$

This inequality is true since

$$2(a^2 + b^2 + c^2 - ab - bc - ca) = (a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0,$$

$$2(a^2 + b^2 + c^2 + ab + bc + ca) = (a + b)^2 + (b + c)^2 + (c + a)^2 \geq 0.$$

The equality holds for  $a = b = c = 1$ , and for  $a = b = c = 1 + \sqrt{3}$ .

□

**P 2.46.** Let  $a, b, c$  be real numbers such that

$$(a + b)(b + c)(c + a) = 10.$$

Prove that

$$(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) + 12a^2b^2c^2 \geq 30.$$

(Vasile Cîrtoaje, 2011)

**Solution.** Since

$$2(b^2 + c^2) = (b + c)^2 + (b - c)^2$$

and

$$(a^2 + b^2)(a^2 + c^2) = (a^2 + bc)^2 + a^2(b - c)^2,$$

by virtue of the Cauchy-Schwarz inequality, we have

$$\begin{aligned} 2(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) &\geq [(b + c)(a^2 + bc) + a(b - c)^2]^2 \\ &= [(a + b)(b + c)(c + a) - 4abc]^2 \\ &= 4(5 - 2abc)^2. \end{aligned}$$

Thus, it suffices to show that

$$(5 - 2abc)^2 + 6a^2b^2c^2 \geq 15,$$

which is equivalent to

$$(abc - 1)^2 = 0.$$

Notice that the homogeneous inequality

$$10(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) + 120a^2b^2c^2 \geq 3(a + b)^2(b + c)^2(c + a)^2$$

becomes an equality for  $\frac{a}{k} = b = c$  (or any cyclic permutation), where

$$k + \frac{1}{k} = 3.$$

□

**P 2.47.** Let  $a, b, c$  be real numbers such that

$$(a + b)(b + c)(c + a) = 5.$$

Prove that

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) + 12a^2b^2c^2 \geq 15.$$

(Vasile Cîrtoaje, 2011)

**Solution.** Since

$$b^2 + bc + c^2 = \frac{3}{4}(b + c)^2 + \frac{1}{4}(b - c)^2$$

and

$$(a^2 + ab + b^2)(a^2 + ac + c^2) = \frac{1}{4}(2a^2 + ab + ac + 2bc)^2 + \frac{3}{4}a^2(b - c)^2,$$

by the Cauchy-Schwarz inequality, we have

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq$$

$$\begin{aligned}
&\geq \frac{3}{16}[(b+c)(2a^2+ab+ac+2bc)+a(b-c)^2]^2 \\
&= \frac{3}{4}[(a+b)(b+c)(c+a)-2abc]^2 = \frac{3}{4}(5-2abc)^2.
\end{aligned}$$

Thus, it suffices to show that

$$\frac{3}{4}(5-2abc)^2 + 12a^2b^2c^2 \geq 15,$$

which is equivalent to

$$(2abc-1)^2 = 0.$$

The homogeneous inequality

$$5(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) + 60a^2b^2c^2 \geq 3(a+b)^2(b+c)^2(c+a)^2$$

becomes an equality for  $\frac{a}{k} = b = c$  (or any cyclic permutation), where

$$k + \frac{1}{k} = 3.$$

□

**P 2.48.** Let  $a, b, c$  be real numbers such that

$$a + b + c = 1, \quad a^3 + b^3 + c^3 = k.$$

Prove that

(a) if  $k = 25$ , then  $|a| \leq 1$  or  $|b| \leq 1$  or  $|c| \leq 1$ ;

(b) if  $k = -11$ , then  $1 < a \leq 2$  or  $1 < b \leq 2$  or  $1 < c \leq 2$ .

(Vasile Cîrtoaje, 2011)

**Solution.** Without loss of generality, assume that  $a \leq b \leq c$ . If  $b = 1$ , then  $a + c = 0$ , and hence

$$k = a^3 + b^3 + c^3 = 1 + a^3 + c^3 = 1,$$

which is false in (a) and (b). From  $(b-a)(b-c) \leq 0$ , we get

$$b^2 - (a+c)b + ac \leq 0,$$

which is equivalent to

$$2b^2 - b + ac \leq 0.$$

(a) It suffices to show that  $|b| \leq 1$ . We have

$$25 - b^3 = a^3 + c^3 = (a+c)^3 - 3ac(a+c) = (1-b)^3 - 3ac(1-b),$$

which yields

$$ac = \frac{8 + b - b^2}{b - 1}.$$

Thus, from

$$2b^2 - b + ac = \frac{2(b+1)(4-3b+b^2)}{b-1} \leq 0,$$

we get  $|b| \leq 1$ . The equality  $|b| = 1$  holds for  $a = b = -1$  and  $c = 3$ .

(b) It suffices to show that  $1 < b \leq 2$ . We have

$$-11 - b^3 = a^3 + c^3 = (a+c)^3 - 3ac(a+c) = (1-b)^3 - 3ac(1-b),$$

which yields

$$ac = \frac{b^2 - b + 4}{1 - b}.$$

Thus, the inequality  $2b^2 - b + ac \leq 0$  is equivalent to

$$\frac{(b-2)(b^2+1)}{1-b} \geq 0,$$

which involves  $1 < b \leq 2$ . The equality  $b = 2$  holds for  $a = -3$  and  $b = c = 2$ .

□

**P 2.49.** Let  $a, b, c$  be real numbers such that

$$a + b + c = a^3 + b^3 + c^3 = 2.$$

Prove that  $a, b, c \notin \left[\frac{5}{4}, 2\right]$ .

(Vasile Cîrtoaje, 2011)

**Solution.** If  $a = 2$ , then we get a contradiction because

$$b + c = 2 - a = 0, \quad b^3 + c^3 = 2 - a^3 = -6, \quad b^3 + c^3 = (b+c)(b^2 - bc + c^2) = 0.$$

From

$$2 = a^3 + b^3 + c^3 = a^3 + (b+c)^2 - 3bc(b+c) = a^3 + (2-a)^2 - 3bc(2-a),$$

we obtain

$$bc = \frac{2(1-a)^2}{2-a}.$$

Thus, the inequality  $(b+c)^2 \geq 4bc$  involves

$$(2-a)^2 \geq \frac{8(1-a)^2}{2-a},$$

$$\frac{a(4 - 2a - a^2)}{2 - a} \geq 0,$$

$$a \in (-\infty, -1 - \sqrt{5}] \cup [0, -1 + \sqrt{5}] \cup (2, \infty).$$

Since

$$-1 + \sqrt{5} < \frac{5}{4},$$

it follows that  $a \notin \left[\frac{5}{4}, 2\right]$ . Similarly, we have

$$b, c \notin \left[\frac{5}{4}, 2\right].$$

□

**P 2.50.** If  $a, b, c$  and  $k$  are real numbers, then

$$\sum (a - b)(a - c)(a - kb)(a - kc) \geq 0.$$

(Vasile Cîrtoaje, 2005)

**Solution.** For  $a = b = c$ , the equality holds. Otherwise, using the substitution

$$m = k + 2, \quad u = (1 - k)a, \quad b = a + x, \quad c = a + y,$$

the inequality can be written as

$$Au^2 + Bu + C \geq 0,$$

where

$$A = x^2 - xy + y^2,$$

$$B = (x + y)(2A - mxy),$$

$$C = (x + y)^2(A - mxy) + m^2x^2y^2.$$

The quadratic  $Au^2 + Bu + C$  has the discriminant

$$D = B^2 - 4AC = -3m^2x^2y^2(x - y)^2.$$

Since  $A > 0$  and  $D \leq 0$ , the conclusion follows. The equality holds for  $a = b = c$ , and for  $a/k = b = c$  (or any cyclic permutation).

**Remark 1.** The inequality is equivalent to

$$\sum a^4 + k(k + 2) \sum a^2b^2 + (1 - k^2)abc \sum a \geq (k + 1) \sum ab(a^2 + b^2).$$



For  $k = 0$ , we get Schur's inequality of degree four

$$a^4 + b^4 + c^4 + abc(a + b + c) \geq \sum ab(a^2 + b^2).$$

For  $k = 1$ , we get the inequality

$$a^4 + b^4 + c^4 + 3(a^2b^2 + b^2c^2 + c^2a^2) \geq 2 \sum ab(a^2 + b^2),$$

with equality for  $a = b = c$ .

For  $k = 2$ , we get the inequality

$$a^4 + b^4 + c^4 + 8(a^2b^2 + b^2c^2 + c^2a^2) \geq 3(ab + bc + ca)(a^2 + b^2 + c^2),$$

which can be rewritten as

$$9(a^4 + b^4 + c^4) + 126(a^2b^2 + b^2c^2 + c^2a^2) \geq 5(a + b + c)^4,$$

with equality for  $a = b = c$ , and for  $a/2 = b = c$  (or any cyclic permutation).

**Remark 2.** The inequality in P 2.50 is equivalent to

$$\sum (a - b)^2(a + b - c - kc)^2 \geq 0.$$

□

**P 2.51.** If  $a, b, c$  are real numbers, then

$$\sum a^2(a - b)(a - c) \geq \frac{(a - b)^2(b - c)^2(c - a)^2}{a^2 + b^2 + c^2 + ab + bc + ca}.$$

**Solution** (by Michael Rozenberg). Since

$$\sum a^2(a - b)(a - c) = \frac{1}{2} \sum (b - c)^2(b + c - a)^2,$$

we can write the inequality in the form

$$\left[ \sum (b + c)^2 \right] \left[ \sum (b - c)^2(b + c - a)^2 \right] \geq 4(a - b)^2(b - c)^2(c - a)^2.$$

Using now the Cauchy-Schwarz inequality, it suffices to show that

$$\left[ \sum (b + c)(b - c)(b + c - a) \right]^2 \geq 4(a - b)^2(b - c)^2(c - a)^2,$$

which is an identity. The equality holds for  $a = b = c$ , for  $a = 0$  and  $b = c$  (or any cyclic permutation), and for  $a = 0$  and  $b + c = 0$  (or any cyclic permutation).

□

**P 2.52.** If  $a, b, c$  are real numbers, then

$$3(a^2 + 3)(b^2 + 3)(c^2 + 3) \geq 16(a + b + c)^2 + 48.$$

(Nguyen Van Huyen, 2011)

**Solution.** Denote

$$x = \frac{(b^2 + 3)(c^2 + 3)}{16}, \quad s = \frac{b + c}{2}, \quad p = bc,$$

and write the inequality as follows:

$$3(a^2 + 3)x \geq (a + 2s)^2 + 3,$$

$$(3x - 1)a^2 - 4sa + 9x - 4s^2 - 3 \geq 0,$$

$$(3x - 1) \left( a - \frac{2s}{3x - 1} \right)^2 + 3(3x - 1) - \frac{12xs^2}{3x - 1}.$$

Since

$$3x - 1 \geq \frac{27}{16} - 1 > 0,$$

it suffices to show that

$$(3x - 1)^2 \geq 4xs^2,$$

which is equivalent to

$$9x^2 - 2(2s^2 + 3)x + 1 \geq 0,$$

hence

$$x \geq \frac{2s^2 + 3 + 2\sqrt{s^4 + 3s^2}}{9}.$$

Since

$$x = \frac{b^2c^2 + 3(b^2 + c^2) + 9}{16} = \frac{(3 - p)^2 + 12s^2}{16},$$

we need to show that

$$9(3 - p)^2 + 76s^2 \geq 48 + 32\sqrt{s^4 + 3s^2}.$$

*Case 1:*  $s^2 \geq 3$ . It suffices to show that

$$64s^2 \geq 48 + 32\sqrt{s^4 + 3s^2},$$

i.e.  $4s^2 - 3 \geq 2\sqrt{s^4 + 3s^2}$ . Indeed

$$(4s^2 - 3)^2 - 4(s^4 + 3s^2) = 12s^2(s^2 - 3) + 9 > 0.$$

*Case 2:*  $s^2 \leq 3$ . Since  $p \leq s^2 \leq 3$ , we have  $3 - p \geq 3 - s^2 \geq 0$ . So it suffices to show that

$$9(3 - s^2)^2 + 76s^2 \geq 48 + 32\sqrt{s^4 + 3s^2},$$

i.e.

$$\begin{aligned} 9s^4 + 22s^2 + 33 &\geq 32\sqrt{s^4 + 3s^2}, \\ 9(s^2 - 1)^2 + 8 \left( 5s^2 + 3 - 4\sqrt{s^4 + 3s^2} \right) &\geq 0, \\ 9(s^2 - 1)^2 + 8 \left( 2|s| - \sqrt{s^2 + 3} \right)^2 &\geq 0. \end{aligned}$$

The equality occurs for  $a = b = c = 1$  or  $a = b = c = -1$ .

□

**P 2.53.** Let  $a \geq b \geq c$  be real numbers. Prove that for fixed

$$a + b + c = p, \quad ab + bc + ca = q,$$

the product

$$r = abc$$

has the minimum value only when  $a = b$ , and the maximum value only when  $b = c$ .

**Solution.** If  $a = c$ , then  $a = b = c$ . Assume further  $a > c$ .

**First Solution.** Consider  $a$ ,  $c$  and  $r$  as functions of  $b$ . By differentiating the equality constraints, we get

$$a' + c' + 1 = 0, \quad a'(b + c) + c'(a + b) + a + c = 0,$$

hence

$$a'(b) = \frac{-(b - c)}{a - c} \leq 0, \quad c'(b) = \frac{-(a - b)}{a - c} \leq 0.$$

Since  $a'(b) \leq 0$  and  $c'(b) \leq 0$ , the functions  $a(b)$  and  $c(b)$  are decreasing. From

$$\begin{aligned} r'(b) &= ac + bca' + abc' = ac - \frac{bc(b - c)}{a - c} - \frac{ab(a - b)}{a - c} \\ &= ac - b(a + c - b) = -(a - b)(b - c) \leq 0, \end{aligned}$$

it follows that  $r(b)$  is decreasing. Thus,  $r(b)$  has the minimum value when  $b$  is maximum ( $a$  is minimum), i.e. when  $b = a$ , and has the maximum value when  $b$  is minimum ( $c$  is maximum), i.e. when  $b = c$ .

**Second Solution.** From

$$(b - a)(b - c) \leq 0,$$

which is equivalent to

$$b^2 + bc - b(a + c) \leq 0, \quad b^2 + q - 2b(a + c) \leq 0, \quad b^2 + q - 2b(p - b) \leq 0,$$

we get

$$3b^2 - 2pb + q \leq 0,$$

hence

$$b \in [b_1, b_2],$$

where

$$b_1 = \frac{p - \sqrt{p^2 - 3q}}{3}, \quad b_2 = \frac{p + \sqrt{p^2 - 3q}}{3}, \quad b_1 < b_2.$$

We have  $b = b_1$  and  $b = b_2$  when  $(b-a)(b-c) = 0$ ; that is, when  $b = c$  and  $b = a$ , respectively. On the other hand, from

$$abc = b[q - b(a+c)] = bq - b^2(p-b) = b^3 - pb^2 + qb,$$

we get

$$r(b) = b^3 - pb^2 + qb.$$

Since

$$r'(b) = 3b^2 - 2pb + q = (b-a)(b-c) \leq 0,$$

$r(b)$  is strictly decreasing on  $[b_1, b_2]$ , therefore  $r(b)$  has the minimum value only for  $b = b_2$ , when  $b = a$ , and has the maximum value only for  $b = b_1$ , when  $b = c$ .

**Remark 1.** The statement remains valid by replacing "fixed  $ab + bc + ca = q$ " with "fixed  $a^2 + b^2 + c^2 = S$ ". Thus, we can prove the following generalization:

- If  $a_1, a_2, \dots, a_n$  are real numbers such that

$$a_1 + a_2 + \dots + a_n = p, \quad a_1^2 + a_2^2 + \dots + a_n^2 = S,$$

where  $p$  and  $S$  are fixed real numbers satisfying  $p^2 \leq nS$ , then the product

$$r = a_1 a_2 \dots a_n$$

has the minimum value and the maximum value when  $n-1$  of the numbers  $a_1, a_2, \dots, a_n$  are equal.

Assume, by the sake of contradiction, that the product  $r$  has the minimum (maximum) value when three of  $a_1, a_2, \dots, a_n$  are distinct, for example, when  $a_1 < a_2 < a_3$ . According to P 2.53, the product  $r$  can be increased (decreased) by replacing the set  $(a_1, a_2, a_3)$  with a suitable set  $(b_1, b_2, b_3)$  such that

$$b_1 + b_2 + b_3 = a_1 + a_2 + a_3, \quad b_1^2 + b_2^2 + b_3^2 = a_1^2 + a_2^2 + a_3^2;$$

this is a contradiction.

**Remark 2.** If  $M \geq a \geq b \geq c \geq m$ , according to First Solution,  $r(b)$  has the minimum value when  $b$  is maximum ( $a$  and  $c$  are minimum), i.e. when  $b = a$  or  $c = m$ , and has the maximum value when  $b$  is minimum ( $a$  and  $c$  are maximum), i.e. when  $b = c$  or  $a = M$ . Thus, the following statement holds:

• If  $m, M, p$  and  $q$  are fixed real numbers such that there exist three nonnegative real numbers  $a, b, c$  satisfying

$$M \geq a \geq b \geq c \geq m, \quad a + b + c = p, \quad ab + bc + ca = q,$$

then the product

$$r = abc$$

has the minimum value when  $a = b \geq c$  or  $c = m$ , and the maximum value when  $a \geq b = c$  or  $a = M$ .

This result can be generalized as follows:

• If  $m, M, p$  and  $S$  are fixed real numbers such that there exist  $n$  nonnegative real numbers  $a_1, a_2, \dots, a_n$  satisfying

$$M \geq a_1 \geq a_2 \geq \dots \geq a_n \geq m, \quad a_1 + a_2 + \dots + a_n = p, \quad a_1^2 + a_2^2 + \dots + a_n^2 = S,$$

then the product

$$r = a_1 a_2 \dots a_n$$

has the minimum value for  $a_1 = \dots = a_{n-1} \geq a_n$  or  $a_n = m$ , and the maximum value for  $a_1 \geq a_2 = \dots = a_n$  or  $a_1 = M$ .

□

**P 2.54.** Let  $a, b, c$  be real numbers. Prove that for fixed

$$a + b + c = p, \quad abc = r,$$

the sum

$$q = ab + bc + ca$$

has the maximum value only when two of  $a, b, c$  are equal.

(Vasile Cîrtoaje, 2017)

**Solution.** Assume that  $a \geq b \geq c$ . If  $b = 0$ , then  $a + c = p$ ,  $q = ac$  and  $p^2 - 4q = (a - c)^2 \geq 0$ . So,  $q$  has the maximum value when  $a = c$ . Consider further  $b \neq 0$  and  $a > c$ .

**First Solution.** Assume that  $a, c$  and  $q$  as functions of  $b$ . By differentiating the equality constraints, we get

$$a' + c' + 1 = 0, \quad a'bc + c'ab + ac = 0,$$

hence

$$a'(b) = \frac{-a(b-c)}{b(a-c)} \leq 0, \quad c'(b) = \frac{-c(a-b)}{b(a-c)} \leq 0.$$

Since  $a'(b) \leq 0$  and  $c'(b) \leq 0$ , the functions  $a(b)$  and  $c(b)$  are decreasing. From

$$q'(b) = a + c + (b+c)a' + (a+b)c' = a + c - \frac{a(b^2 - c^2)}{b(a-c)} - \frac{c(a^2 - b^2)}{b(a-c)}$$

$$= a + c - \frac{ac + b^2}{b} = \frac{(a - b)(b - c)}{b}.$$

For  $b < 0$ , we have  $q'(b) \leq 0$ ,  $q(b)$  is strictly decreasing, hence  $q(b)$  has the maximum value for  $b = c$ .

For  $b > 0$ , we have  $q'(b) \geq 0$ ,  $q(b)$  is strictly increasing, hence  $q(b)$  has the maximum value for  $b = a$ .

**Second Solution.** From

$$q = ac + b(a + c) = \frac{r}{b} + b(p - b),$$

we get

$$q(b) = \frac{r}{b} + pb - b^2,$$

with

$$q'(b) = \frac{-r}{b^2} + p - 2b = \frac{(a - b)(b - c)}{b}.$$

For  $b < 0$ , we have  $q'(b) \leq 0$ ,  $q(b)$  is strictly decreasing, hence  $q(b)$  has the maximum value for  $b = c$ .

For  $b > 0$ , we have  $q'(b) \geq 0$ ,  $q(b)$  is strictly increasing, hence  $q(b)$  has the maximum value for  $b = a$ .

**Remark 1.** By replacing  $a, b$  and  $c$  with  $1/a, 1/b$  and  $1/c$ , respectively, we get the following statement:

- Let  $a, b, c$  be real numbers. For fixed

$$ab + bc + ca = q, \quad abc = r \neq 0,$$

the product

$$p_1 = abc(a + b + c)$$

has the maximum value only when two of  $a, b, c$  are equal.

**Remark 2.** The statement of P 2.54 remains valid by replacing "sum  $q = ab + bc + ca$  has the maximum value when two of  $a, b, c$  are equal" with "sum  $p_1 = a^2 + b^2 + c^2$  has the minimum value when two of  $a, b, c$  are equal". Thus, the statement can be generalized as follows:

- Let  $a_1, a_2, \dots, a_n$  be real numbers. For fixed

$$a_1 + a_2 + \dots + a_n = p, \quad a_1 a_2 \dots a_n = r,$$

the sum

$$p_1 = a_1^2 + a_2^2 + \dots + a_n^2$$

has the minimum value when  $n - 1$  of  $a_1, a_2, \dots, a_n$  are equal.

□

**P 2.55.** Let  $a, b, c$  be real numbers such that  $a + b + c = 3$ . Prove that

- (a)  $(ab + bc + ca)^2 \geq 9abc;$   
 (b)  $(ab + bc + ca)^2 + 9 \geq 18abc;$   
 (c)  $(ab + bc + ca - 3)^2 \geq 27(abc - 1).$

(Vasile Cîrtoaje, 2011)

**Solution.** Let  $q = ab + bc + ca$ . According to P 2.53, for fixed  $q$ , the product  $abc$  has the maximum value when two of  $a, b, c$  are equal. Therefore, it suffices to prove the desired inequalities for  $b = c$ , when  $a + 2b = 3$ .

The inequality (a) is equivalent to

$$(2ab + b^2)^2 \geq 9ab^2,$$

$$b^2(b - 1)^2 \geq 0.$$

The equality holds for  $a = b = c = 1$ , and also for  $a = 3$  and  $b = c = 0$  (or any cyclic permutation).

The inequality (b) is equivalent to

$$(2ab + b^2 - 3)^2 + 9 \geq 18ab^2,$$

$$(b - 1)^2(b + 1)^2 \geq 0.$$

The equality holds for  $a = b = c = 1$ , and also for  $a = 5$  and  $b = c = -1$  (or any cyclic permutation).

The inequality (c) is equivalent to

$$(2ab + b^2 - 3)^2 \geq 27(ab^2 - 1),$$

$$(b - 1)^2(b + 2)^2 \geq 0.$$

The equality holds for  $a = b = c = 1$ , and also for  $a = 7$  and  $b = c = -2$  (or any cyclic permutation).

**Remark 1.** Another solution for the inequality (b) is the following. Using the substitution

$$a = x + 1, \quad b = y + 1, \quad c = z + 1,$$

the inequality becomes

$$(xy + yz + zx)^2 \geq 12(xy + yz + zx) + 18xyz,$$

where  $x, y, z$  are real numbers such that

$$x + y + z = 0.$$

Assume that  $y$  and  $z$  have the same sign, hence  $yz \geq 0$ . Substituting  $x$  by  $-y - z$ , the inequality can be rewritten in the form

$$(y^2 + yz + z^2)^2 + 12(y^2 + yz + z^2) + 18yz(y + z) \geq 0.$$

Since

$$y^2 + yz + z^2 \geq \frac{3}{4}(y + z)^2 \geq 3yz,$$

it suffices to show that

$$9y^2z^2 + 9(y + z)^2 + 18yz(y + z) \geq 0,$$

which is equivalent to

$$9(yz + y + z)^2 \geq 0.$$

**Remark 2.** Another solution for the inequality (c) is the following. Assume that  $a = \max\{a, b, c\}$ ,  $a \geq 1$ . Since

$$3 - ab - bc - ca \geq 3 - a(b + c) - \frac{1}{4}(b + c)^2 = 3 - a(3 - a) - \frac{1}{4}(3 - a)^2 = \frac{3}{4}(a - 1)^2$$

and

$$abc - 1 \leq \frac{1}{4}a(b + c)^2 - 1 = \frac{1}{4}a(3 - a)^2 - 1 = \frac{1}{4}(a - 1)^2(a - 4),$$

it suffices to prove that

$$\frac{9}{16}(a - 1)^4 \geq \frac{27}{4}(a - 1)^2(a - 4),$$

which is equivalent to

$$(a - 1)^2(a - 7)^2 \geq 0.$$

□

**P 2.56.** Let  $a, b, c$  be real numbers such that

$$ab + bc + ca + abc = 4.$$

Prove that

(a) if  $abc > 0$ , then

$$2(a + b + c) + ab + bc + ca \leq \frac{9}{abc};$$

(b) if  $abc < 0$ , then

$$2(a + b + c) + ab + bc + ca \geq \frac{9}{abc}.$$

(Vasile Cîrtoaje, 2018)



**Solution.** We write both inequalities in the unique form

$$2abc(a + b + c) + abc(ab + bc + ca) \leq 9.$$

According to Remark 1 from P 2.54, for fixed  $ab + bc + ca$  and  $abc$ , the product  $abc(a + b + c)$  has the maximum value when two of  $a, b, c$  are equal. Therefore, it suffices to prove the inequality for  $b = c$ ; that is, to show that

$$2ab^2(a + 2b + 2ab + b^2) \leq 9$$

for  $2ab + b^2 + ab^2 = 4$ . Write this hypothesis as

$$(b + 2)(b - 2 + ab) = 0.$$

For  $b = -2$ , the required inequality is equivalent to  $8a^2 + 3 \geq 0$ , while for  $a = \frac{2-b}{b}$ , it is equivalent to  $(b^2 - 1)^2 \geq 0$ .

The inequality (a) is an equality for  $a = b = c = 1$ , while the inequality (b) is an equality for  $a = -3$  and  $b = c = -1$  (or any cyclic permutation).

□

**P 2.57.** If  $a, b, c$  are real numbers such that

$$a + b + c + abc = 4,$$

then

$$a^2 + b^2 + c^2 + 3 \geq 2(ab + bc + ca).$$

(Vasile Cîrtoaje, 2011)

**First Solution.** Write the inequality in the form

$$(a + b + c)^2 + 3 \geq 4(ab + bc + ca).$$

According to P 2.54, for fixed  $a + b + c$  and  $abc$ , the sum  $ab + bc + ca$  has the maximum value when two of  $a, b, c$  are equal. Therefore, considering  $b = c$ , we need to show that  $a + 2b + ab^2 = 4$  involves

$$a^2 + 2b^2 + 3 \geq 2(2ab + b^2),$$

that is

$$a^2 + 3 \geq 4ab.$$

We have

$$\begin{aligned} a^2 + 3 - 4ab &= \left( \frac{4 - 2b}{b^2 + 1} \right)^2 + 3 - \frac{4b(4 - 2b)}{b^2 + 1} \\ &= \frac{11b^4 - 16b^3 + 18b^2 - 32b + 19}{(b^2 + 1)^2} = \frac{(b - 1)^2(11b^2 + 6b + 19)}{(b^2 + 1)^2} \geq 0. \end{aligned}$$

The equality holds for  $a = b = c = 1$ .

**Second Solution.** Without loss of generality, assume that  $a \geq b \geq c$ . The case  $a \leq 0$  is not possible, since it involves  $a + b + c + abc \leq 0 < 4$ . If  $a \geq 0 \geq b \geq c$ , then

$$a^2 + b^2 + c^2 + 3 - 2(ab + bc + ca) \geq a^2 + (b - c)^2 + 3 > 0.$$

Also, if  $a \geq b \geq 0 \geq c$ , then

$$a^2 + b^2 + c^2 + 3 - 2(ab + bc + ca) \geq (a - b)^2 + c^2 + 3 > 0.$$

Consider further that  $a, b, c \geq 0$  and denote

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

We need to show that

$$p^2 + 3 \geq 4q$$

for  $p + r = 4$ . By Schur's inequality of degree three, we have

$$p^3 + 9r \geq 4pq.$$

Therefore, we get

$$p(p^2 + 3 - 4q) \geq p^3 + 3p - (p^3 + 9r) = 12(p - 3).$$

To complete the proof, we need to show that  $p \geq 3$ . By virtue of the AM-GM inequality, we have

$$p^3 \geq 27r,$$

$$p^3 \geq 27(4 - p),$$

$$(p - 3)(p^2 + 3p + 36) \geq 0,$$

$$p \geq 3.$$

□

**P 2.58.** If  $a, b, c$  are real numbers such that

$$ab + bc + ca = 3abc,$$

then

$$4(a^2 + b^2 + c^2) + 9 \geq 7(ab + bc + ca).$$

(Vasile Cîrtoaje, 2011)

**Solution.** If one of  $a, b, c$  is 0, then the inequality is trivial. Otherwise, write the inequality in the homogeneous form

$$4(a^2 + b^2 + c^2) + \frac{81a^2b^2c^2}{(ab + bc + ca)^2} \geq 7(ab + bc + ca),$$

or

$$81r^2 \geq q^2(15q - 4p^2),$$

where

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

**First Solution.** For fixed  $p$  and  $q$ ,  $r^2$  has the minimum value when  $r$  is either minimum (when  $r \geq 0$ ) or maximum (when  $r \leq 0$ ). According to P 2.53,  $r$  has the minimum and maximum value when two of  $a, b, c$  are equal. For  $a = b$ , the inequality becomes

$$(a - c)^2(4a - c)^2 \geq 0.$$

The equality holds for  $a = b = c = 1$ , and for  $a = b = 2$  and  $c = 1/2$  (or any cyclic permutation).

**Second Solution** (by *Vo Quoc Ba Can*). Consider the nontrivial case  $15q - 4p^2 > 0$ . Substituting  $a, b, c$  by  $|a|, |b|, |c|$ , respectively, the left side of the inequality remains unchanged, while the right side remains unchanged or increases. Therefore, it suffices to prove the inequality only for  $a, b, c > 0$  and  $15q - 4p^2 > 0$ . Assume that  $a \geq b \geq c > 0$ . There are two cases to consider.

*Case 1:*  $4b^2 \leq 3ab + 3bc + ca$ . Since

$$4q^2(15q - 4p^2) \leq \left[ \frac{q^2}{b} + b(15q - 4p^2) \right]^2,$$

it suffices to show that

$$18r \geq \frac{q^2}{b} + b(15q - 4p^2),$$

which is equivalent to the obvious inequality

$$(a - b)(b - c)(3ab + 3bc + ca - 4b^2) \geq 0.$$

*Case 2:*  $4b^2 > 3ab + 3bc + ca$ . Since

$$4q^2(15q - 4p^2) \leq \left[ \frac{q^2}{a} + a(15q - 4p^2) \right]^2,$$

it suffices to show that

$$18r \geq \frac{q^2}{a} + a(15q - 4p^2),$$

which is equivalent to

$$(a - b)(a - c)(4a^2 - 3ab - bc - 3ca) \geq 0.$$

This is true, since

$$4a^2 - 3ab - bc - 3ca = (4b^2 - 3ab - 3bc - ca) + 2(a - b)(2a + 2b - c) > 0.$$

□

**P 2.59.** Let  $a, b, c \leq \frac{6}{5}$  be real numbers such that  $a^2 + b^2 + c^2 = 4$ . If

$$k = \frac{16(2 + 15\sqrt{2})}{125} \approx 2.97,$$

then

$$ab + bc + ca + k \geq abc.$$

(Vasile Cîrtoaje, 2018)

**Solution.** According to Remark 2 from P 2.53, for  $a^2 + b^2 + c^2 = 4$  and  $ab + bc + ca = \text{constant}$ , the product  $abc$  has the maximum value when  $a = \frac{6}{5}$  or  $a \geq b = c$ . Therefore, it is enough to consider these cases.

*Case 1:*  $a = \frac{6}{5}$ . We need to prove that if

$$b^2 + c^2 = \frac{64}{25},$$

then

$$bc - 6(b + c) \leq 5k.$$

We have

$$bc - 6(b + c) \leq \frac{b^2 + c^2}{2} + 6\sqrt{2(b^2 + c^2)} = \frac{32}{25} + \frac{48\sqrt{2}}{5} = 5k.$$

*Case 2:*  $a \geq b = c$ . We need to show that if

$$a^2 + 2b^2 = 4,$$

then

$$k \geq (a - 1)b^2 - 2ab.$$

For  $0 > a \geq b$ , the inequality is true since its right side is negative. Consider next that  $a \geq 0$ . We get the required inequality by adding the inequalities

$$\frac{32}{125} \geq (a - 1)b^2$$

and

$$\frac{48\sqrt{2}}{25} \geq -2ab.$$

The first inequality is true because, for the non-trivial case  $1 \leq a \leq \frac{6}{5}$ , we have we have

$$\begin{aligned} 64 - 250(a-1)b^2 &= 64 - 125(a-1)(4-a^2) \\ &= 564 - 500a - 125a^2 + 125a^3 = (6-5a)(94-5a-25a^2) \geq 0. \end{aligned}$$

The second inequality is true if

$$\left(\frac{48}{25}\right)^2 \geq 2a^2b^2.$$

Indeed, since  $25a^2 \leq 36$ , we have

$$\begin{aligned} \left(\frac{48}{25}\right)^2 - 2a^2b^2 &= \left(\frac{48}{25}\right)^2 - a^2(4-a^2) \\ &= \frac{2304 - 2500a^2 + 625a^4}{625} = \frac{(36-25a^2)(64-25a^2)}{625} \geq 0. \end{aligned}$$

The equality occurs for  $a = \frac{6}{5}$  and  $b = c = \frac{-4\sqrt{2}}{5}$  (or any cyclic permutation).

□

**P 2.60.** Let  $f_4(a, b, c)$  be a symmetric homogeneous polynomial of degree four. Prove that the inequality  $f_4(a, b, c) \geq 0$  holds for all real numbers  $a, b, c$  if and only if  $f_4(a, 1, 1) \geq 0$  for all real  $a$ .

**Solution.** Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

Any symmetric homogeneous polynomial  $f_4(a, b, c)$  can be written as

$$f_4(a, b, c) = Apr + Bp^4 + Cp^2q + Dq^2,$$

where  $A, B, C, D$  are real constants. For fixed  $p$  and  $q$ , the linear function

$$g(r) = Apr + Bp^4 + Cp^2q + Dq^2$$

has the minimum value when  $r$  is either minimum or maximum. By P 2.53,  $r$  has the minimum and maximum value when two of  $a, b, c$  are equal. Since  $f_4(a, b, c)$  is symmetric, homogeneous and satisfies  $f_4(-a, -b, -c) = f_4(a, b, c)$ , it follows that the inequality  $f_4(a, b, c) \geq 0$  holds for all real numbers  $a, b, c$  if and only if  $f_4(a, 1, 1) \geq 0$  and  $f_4(a, 0, 0) \geq 0$  for all real  $a$ . Notice that the condition " $f_4(a, 0, 0) \geq 0$  for all real  $a$ " is not necessary because it follows from the condition " $f_4(a, 1, 1) \geq 0$  for all real  $a$ " as follows:

$$f_4(a, 0, 0) = \lim_{t \rightarrow 0} f_4(a, t, t) = \lim_{t \rightarrow 0} t^4 f_4(a/t, 1, 1) \geq 0.$$

**Remark.** Similarly, we can prove the following statement, which is valid for the extended case where  $f_4(a, b, c)$  is only a symmetric polynomial (homogeneous or non-homogeneous).

• Let  $f_4(a, b, c)$  be a symmetric polynomial function of degree  $n = 4$ . The inequality  $f_4(a, b, c) \geq 0$  holds for all real numbers  $a, b, c$  if and only if  $f_4(a, b, b) \geq 0$  for all real numbers  $a$  and  $b$ .

Notice that a function  $f(a, b, c)$  is symmetric if it is unchanged by any permutation of its variables. A function  $f(a, b, c)$  is a polynomial function if it is a polynomial in one variable when the other two variables are fixed. In addition, a polynomial function  $f(a, b, c)$  is of degree  $n$  if  $f(a, a, a)$  is a polynomial of degree  $n$ .

□

**P 2.61.** If  $a, b, c$  are real numbers, then

$$10(a^4 + b^4 + c^4) + 64(a^2b^2 + b^2c^2 + c^2a^2) \geq 33 \sum ab(a^2 + b^2).$$

(Vasile Cîrtoaje, 2008)

**Solution.** According to P 2.60, it suffices to prove the required inequality for  $b = c = 1$ , when it becomes

$$\begin{aligned} 5a^4 - 33a^3 + 64a^2 - 33a + 9 &\geq 0, \\ (a - 3)^2(5a^2 - 3a + 1) &\geq 0. \end{aligned}$$

This is true since

$$5a^2 - 3a + 1 = 5\left(a - \frac{3}{10}\right)^2 + \frac{11}{20} > 0.$$

The equality holds for  $a/3 = b = c$  (or any cyclic permutation).

□

**P 2.62.** If  $a, b, c$  are real numbers such that

$$a + b + c = 3,$$

then

$$3(a^4 + b^4 + c^4) + 33 \geq 14(a^2 + b^2 + c^2).$$

(Vasile Cîrtoaje, 2009)

**First Solution.** Write the inequality as  $F(a, b, c) \geq 0$ , where

$$F(a, b, c) = 3(a^4 + b^4 + c^4) + 33 - 14(a^2 + b^2 + c^2).$$

Due to symmetry, we may assume that  $a \leq b \leq c$ . Let us denote

$$x = \frac{b+c}{2}, \quad x \geq 1.$$

To prove the desired inequality, we use the mixing variables method. We will show that

$$F(a, b, c) \geq F(a, x, x) \geq 0.$$

We have

$$\begin{aligned} F(a, b, c) - F(a, x, x) &= 3(b^4 + b^4 - 2x^4) - 14(b^2 + c^2 - 2x^2) \\ &= 3[(b^2 + c^2)^2 - 4x^4] + 6(x^4 - b^2c^2) - 14(b^2 + c^2 - 2x^2) \\ &= (b^2 + c^2 - 2x^2)[3(b^2 + c^2 + 2x^2) - 14] + 6(x^2 - bc)(x^2 + bc). \end{aligned}$$

Since

$$x^2 - bc = \frac{1}{4}(b - c)^2, \quad b^2 + c^2 - 2x^2 = 2(x^2 - bc) = \frac{1}{2}(b - c)^2,$$

we get

$$\begin{aligned} F(a, b, c) - F(a, x, x) &= \frac{1}{2}(b - c)^2[3(b^2 + c^2 + 2x^2) - 14 + 3(x^2 + bc)] \\ &= \frac{1}{2}(b - c)^2[3(x^2 - bc) + 18x^2 - 14] \geq 0. \end{aligned}$$

Also,

$$F(a, x, x) = F(3 - 2x, x, x) = 6(x - 1)^2(3x - 5)^2 \geq 0.$$

This completes the proof. The equality holds for  $a = b = c = 1$ , and for  $a = -1/3$  and  $b = c = 5/3$  (or any cyclic permutation).

**Second Solution.** Write the inequality in the homogeneous form

$$81(a^4 + b^4 + c^4) + 11(a + b + c)^4 \geq 42(a^2 + b^2 + c^2)(a + b + c)^2.$$

According to P 2.60, it suffices to prove this inequality for  $b = c = 1$ , when it becomes

$$25a^4 - 40a^3 + 6a^2 + 8a + 1 \geq 0,$$

$$(a - 1)^2(5a + 1)^2 \geq 0.$$

□

**P 2.63.** If  $a, b, c$  are real numbers such that

$$a^2 + b^2 + c^2 = 3,$$

then

$$a^4 + b^4 + c^4 + 3(ab + bc + ca) \leq 12.$$

**Solution.** Write the inequality in the homogeneous form

$$3(a^4 + b^4 + c^4) + 3(ab + bc + ca)(a^2 + b^2 + c^2) \leq 4(a^2 + b^2 + c^2)^2.$$

According to P 2.60, it suffices to prove this inequality for  $b = c = 1$ , when it becomes

$$a^4 - 6a^3 + 13a^2 - 12a + 4 \geq 0,$$

$$(a - 1)^2(a - 2)^2 \geq 0.$$

The equality holds for  $a = b = c = \pm 1$ , for  $a = \sqrt{2}$  and  $b = c = \sqrt{2}/2$  (or any cyclic permutation), and for  $a = -\sqrt{2}$  and  $b = c = -\sqrt{2}/2$  (or any cyclic permutation).  $\square$

**P 2.64.** Let  $\alpha, \beta, \gamma$  be real numbers such that

$$1 + \alpha + \beta = 2\gamma.$$

The inequality

$$\sum a^4 + \alpha \sum a^2b^2 + \beta abc \sum a \geq \gamma \sum ab(a^2 + b^2)$$

holds for any real numbers  $a, b, c$  if and only if

$$1 + \alpha \geq \gamma^2.$$

(Vasile Cîrtoaje, 2009)

**Solution.** Let

$$f_4(a, b, c) = \sum a^4 + \alpha \sum a^2b^2 + \beta abc \sum a - \gamma \sum ab(a^2 + b^2).$$

According to P 2.60, the inequality  $f_4(a, b, c) \geq 0$  holds for any real numbers  $a, b, c$  if and only if  $f_4(a, 1, 1) \geq 0$  for any real  $a$ . From

$$f_4(a, 1, 1) = (a - 1)^2[(a - \gamma + 1)^2 + 1 + \alpha - \gamma^2],$$

the conclusion follows. The equality holds for  $a = b = c$ . In addition, if  $1 + \alpha = \gamma^2$ , then the equality holds also for  $\frac{a}{\gamma - 1} = b = c$  (or any cyclic permutation).

**Remark.** For  $\gamma = k + 1$  and  $\alpha = k(k + 2)$  (which involves  $1 + \alpha = \gamma^2$ ), we get

$$\sum a^4 + k(k + 2) \sum a^2b^2 + (1 - k^2)abc \sum a \geq (k + 1) \sum ab(a^2 + b^2), \quad k \in \mathbb{R},$$

which is equivalent to the elegant inequality from P 2.50, namely

$$\sum (a - b)(a - c)(a - kb)(a - kc) \geq 0,$$

where the equality holds for  $a = b = c$ , and also for  $a/k = b = c$  (or any cyclic permutation). In addition, for  $k = 0$ , we get Schur's inequality of degree four

$$\sum a^2(a - b)(a - c),$$

with equality for  $a = b = c$ , and also for  $a = 0$  and  $b = c$  (or any cyclic permutation).  $\square$



**P 2.65.** If  $a, b, c$  are real numbers such that

$$a^2 + b^2 + c^2 = 2,$$

then

$$ab(a^2 - ab + b^2 - c^2) + bc(b^2 - bc + c^2 - a^2) + ca(c^2 - ca + a^2 - b^2) \leq 1.$$

**Solution.** Write the inequality in the homogeneous form

$$(a^2 + b^2 + c^2)^2 \geq 4 \sum ab(a^2 - ab + b^2 - c^2).$$

According to P 2.60, it suffices to prove this inequality for  $b = c = 1$ , when it can be written as

$$a^2(a - 4)^2 \geq 0.$$

The equality holds for

$$a^2 + b^2 + c^2 = 2(ab + bc + ca).$$

□

**P 2.66.** If  $a, b, c$  are real numbers, then

$$(a + b)^4 + (b + c)^4 + (c + a)^4 \geq \frac{4}{7}(a^4 + b^4 + c^4).$$

(Vietnam TST, 1996)

**Solution.** Denote the left side of the inequality by  $f_4(a, b, c)$ . According to P 2.60, it suffices to prove that  $f_4(a, 1, 1) \geq 0$  for all real  $a$ . Indeed,

$$f_4(a, 1, 1) = \frac{2}{7}(5a^4 + 28a^3 + 42a^2 + 28a + 59) > 0$$

since, for the nontrivial case  $a < 0$ , we have

$$5a^4 + 28a^3 + 42a^2 + 28a + 59 = (5a^2 - 2a)(a + 3)^2 + 9(a + \frac{23}{9})^2 + \frac{2}{9} > 0.$$

The equality holds for  $a = b = c = 0$ .

□

**P 2.67.** Let  $a, b, c$  be real numbers. If

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc,$$

then

$$(3 - p)r + \frac{p^2 + q^2 - pq}{3} \geq q.$$

(Vasile Cîrtoaje, 2011)

**First Solution.** Write the inequality as

$$(p^2 - 3q) + (q^2 - 3pr) \geq pq - 9r,$$

$$\frac{1}{2} \sum (b - c)^2 + \frac{1}{2} \sum a^2(b - c)^2 \geq \sum a(b - c)^2.$$

According to the AM-GM inequality, it suffices to prove that

$$\sqrt{\left[ \sum (b - c)^2 \right] \left[ \sum a^2(b - c)^2 \right]} \geq \sum a(b - c)^2.$$

Clearly, this inequality follows immediately from the Cauchy-Schwarz inequality. The equality holds for  $a = b = c$ , and for  $b = c = 1$  (or any cyclic permutation).

**Second Solution.** Write the inequality as  $f_4(a, b, c) \geq 0$ , where

$$f_4(a, b, c) = 3(3 - p)r + p^2 + q^2 - pq - 3q.$$

is a symmetric polynomial of degree four in  $a, b, c$ . According to Remark from the proof of P 2.60, it suffices to prove that  $f_4(a, b, b) \geq 0$  for all real numbers  $a$  and  $b$ . Indeed, we have

$$f_4(a, b, b) = (a - b)^2(b - 1)^2 \geq 0.$$

□

**P 2.68.** If  $a, b, c$  are real numbers, then

$$\frac{ab(a + b) + bc(b + c) + ca(c + a)}{(a^2 + 1)(b^2 + 1)(c^2 + 1)} \leq \frac{3}{4}.$$

(Vasile Cîrtoaje, 2011)

**First Solution.** We try to get a stronger homogeneous inequality. According to the AM-GM inequality, we have

$$(a^2 + 1)(b^2 + 1)(c^2 + 1) = (a^2b^2c^2 + 1) + (a^2b^2 + b^2c^2 + c^2a^2) + (a^2 + b^2 + c^2)$$

$$\geq 2abc + 2\sqrt{(a^2b^2 + b^2c^2 + c^2a^2)(a^2 + b^2 + c^2)}.$$

Therefore, it suffices to prove the homogeneous inequality

$$3abc + 3\sqrt{(a^2b^2 + b^2c^2 + c^2a^2)(a^2 + b^2 + c^2)} \geq 2 \sum ab(a + b).$$

Using the identity

$$9(a^2 + b^2 + c^2) = \sum (2a + 2b - c)^2$$

together with the Cauchy-Schwarz inequality, we get

$$3\sqrt{(a^2b^2 + b^2c^2 + c^2a^2)(a^2 + b^2 + c^2)} =$$

$$\begin{aligned}
&= \sqrt{\left[\sum a^2b^2\right] \left[\sum (2a+2b-c)^2\right]} \geq \sum ab(2a+2b-c) \\
&= 2 \sum ab(a+b) - 3abc.
\end{aligned}$$

The equality holds for  $a = b = c = 1$ .

**Second Solution.** Write the inequality as

$$3(abc - 1)^2 + f_4(a, b, c) \geq 0,$$

where

$$f_4(a, b, c) = \sum a^2b^2 + \sum a^2 + 2abc - \frac{4}{3} \sum ab(a+b)$$

is a symmetric polynomial of degree four. Clearly, it suffices to prove that  $f_4(a, b, c) \geq 0$ . According to Remark from P 2.60, it suffices to prove that  $f_4(a, b, b) \geq 0$  for all real numbers  $a$  and  $b$ . Indeed, we have

$$\begin{aligned}
3f_4(a, b, b) &= (6b^2 - 8b + 3)a^2 - 2b^2a + b^2(3b^2 - 8b + 6) \\
&= (6b^2 - 8b + 3) \left(a - \frac{b^2}{6b^2 - 8b + 3}\right)^2 + \frac{18b^2(b-1)^4}{6b^2 - 8b + 3} \geq 0.
\end{aligned}$$

**Remark.** The inequality is equivalent to

$$3(abc - 1)^2 + \sum (a-1)^2(b-c)^2 + (ab+bc+ca-a-b-c)^2 \geq 0.$$

□

**P 2.69.** If  $a, b, c$  are real numbers such that  $abc > 0$ , then

$$\left(a + \frac{1}{a} - 1\right) \left(b + \frac{1}{b} - 1\right) \left(c + \frac{1}{c} - 1\right) + 2 \geq \frac{1}{3}(a+b+c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

(Vasile Cîrtoaje, 2011)

**Solution.** Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

Multiplying by  $abc$ , we can rewrite the inequality as

$$r^2 + (4 - p - q)r + p^2 + q^2 - \frac{4pq}{3} - p - q + 1 \geq 0.$$

Since the equality holds for  $a = b = c = 1$ , when  $p = q = 3$  and  $r = 1$ , we write the inequality as

$$\left(r - 1 + \frac{p - q}{2}\right)^2 + f(p, q, r) \geq 0,$$

where

$$\begin{aligned} 12f(p, q, r) &= 24(3 - p)r + 9(p^2 + q^2) - 10pq - 24q \\ &\geq 24(3 - p)r + 8(p^2 + q^2 - pq) - 24q \end{aligned}$$

Thus, it suffices to prove that  $f_4(a, b, c) \geq 0$  for all real  $a, b, c$ , where

$$f_4(a, b, c) = 3(3 - p)r + p^2 + q^2 - pq - 3q \geq 0.$$

According to Remark from P 2.60, it suffices to prove that  $f_4(a, b, b) \geq 0$  for all real numbers  $a$  and  $b$ . Indeed, we have

$$f_4(a, b, b) = (b - 1)^2(a - b)^2 \geq 0.$$

The equality holds for  $a = b = c = 1$ .

**Remark.** The inequalities in P 2.68 and P 2.69 are particular cases of the following more general statement (*Vasile Cîrtoaje, 2011*).

- Let  $a, b, c$  be real numbers such that  $abc > 0$ . If

$$-2 \leq k \leq 1,$$

then

$$\begin{aligned} \left(a + \frac{1}{a} + k\right) \left(b + \frac{1}{b} + k\right) \left(c + \frac{1}{c} + k\right) + (1 - k)(2 + k)^2 &\geq \\ &\geq \frac{1}{3}(2 + k)^2(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right). \end{aligned}$$

□

**P 2.70.** If  $a, b, c$  are real numbers, then

$$\left(a^2 + \frac{1}{2}\right) \left(b^2 + \frac{1}{2}\right) \left(c^2 + \frac{1}{2}\right) \geq \left(a + b - \frac{1}{2}\right) \left(b + c - \frac{1}{2}\right) \left(c + a - \frac{1}{2}\right).$$

(*Vasile Cîrtoaje, 2011*)

**Solution.** It suffices to prove that  $f_4(a, b, c) \geq 0$ , where

$$f_4(a, b, c) = \prod \left(a^2 + \frac{1}{2}\right) - \prod \left(a + b - \frac{1}{2}\right) - \left(abc + \frac{1}{2} - \frac{a + b + c}{2}\right)^2$$

is a symmetric polynomial of degree four. According to Remark from P 2.60, it suffices to prove that  $f_4(a, b, b) \geq 0$  for all real  $a$  and  $b$ . Indeed, we have

$$2f_4(a, b, b) = [(2b - 1)a - b(2 - b)]^2 \geq 0.$$

The equality holds for  $a = b = c = 1$ .

**Remark.** The inequality is equivalent to

$$(2abc + 1 - a - b - c)^2 + 2(ab + bc + ca - a - b - c)^2 \geq 0.$$

□

**P 2.71.** If  $a, b, c$  are real numbers such that

$$a + b + c = 3,$$

then

$$\frac{a(a-1)}{8a^2+9} + \frac{b(b-1)}{8b^2+9} + \frac{c(c-1)}{8c^2+9} \geq 0.$$

(Vasile Cîrtoaje, 2013)

**Solution** (by Michael Rozenberg). Write the inequality as follow

$$\sum \frac{pa(a-1)}{8a^2+9} \geq 0, \quad p > 0,$$

$$\sum \left[ \frac{pa(a-1)}{8a^2+9} + 1 \right] \geq 3,$$

$$\sum \frac{(p+8)a^2 - pa + 9}{8a^2+9} \geq 3.$$

Choosing  $p = 18 + 6\sqrt{17}$ , the inequality can be written as

$$\sum \frac{(ka-3)^2}{8a^2+9} \geq 3, \quad k = 3 + \sqrt{17}.$$

Let  $m$  be a real constant. According to the Cauchy-Schwarz inequality, we have

$$\sum \frac{(ka-3)^2}{8a^2+9} \geq \frac{[\sum (ka-3)(ma+3)]^2}{\sum (ma+3)^2(8a^2+9)},$$

with equality for

$$\frac{ka-3}{(8a^2+9)(ma+3)} = \frac{kb-3}{(8b^2+9)(mb+3)} = \frac{kc-3}{(8c^2+9)(mc+3)}. \quad (*)$$

On the other hand, we can check that the original inequality becomes an equality for  $a = b = c$ , and also for  $a = 3/2$  and  $b = c = 3/4$  (or any cyclic permutation). It is easy to get that the equality conditions (\*) are satisfied for  $a = 3/2$  and  $b = c = 3/4$  if and only if  $m = k$ . For this value of  $m$ , it suffices to show that

$$\frac{[\sum (ka-3)(ka+3)]^2}{\sum (ka+3)^2(8a^2+9)} \geq 3,$$

$$\left[ \sum (k^2a^2 - 9) \right]^2 \geq 3 \sum (ka+3)^2(8a^2+9),$$

$$[k^2(a^2 + b^2 + c^2) - 27]^2 \geq 3 \sum (ka+3)^2(8a^2+9).$$

Write this inequality in the homogeneous form  $f_4(a, b, c) \geq 0$ , where

$$f_4(a, b, c) = [k^2(a^2 + b^2 + c^2) - 3(a+b+c)^2]^2$$

$$-3 \sum (ka + a + b + c)^2 [8a^2 + (a + b + c)^2].$$

According to P 2.60, it suffices to prove that  $f_4(a, 1, 1) \geq 0$  for all real  $a$ . Indeed, this inequality is equivalent to

$$(a - 1)^2(a - 2)^2 \geq 0.$$

□

**P 2.72.** If  $a, b, c$  are real numbers such that

$$a + b + c = 3,$$

then

$$\frac{(a - 11)(a - 1)}{2a^2 + 1} + \frac{(b - 11)(b - 1)}{2b^2 + 1} + \frac{(c - 11)(c - 1)}{2c^2 + 1} \geq 0.$$

(Vasile Cîrtoaje, 2013)

**Solution.** Write the inequality as

$$\sum \left[ \frac{(a - 11)(a - 1)}{2a^2 + 1} + 1 \right] \geq 3,$$

$$\sum \frac{(a - 2)^2}{2a^2 + 1} \geq 1.$$

According to the Cauchy-Schwarz inequality, we have

$$\sum \frac{(a - 2)^2}{2a^2 + 1} \geq \frac{[\sum (a - 2)^2]^2}{\sum (a - 2)^2(2a^2 + 1)}.$$

Therefore, it suffices to show that

$$\left[ \sum (a - 2)^2 \right]^2 \geq \sum (a - 2)^2(2a^2 + 1),$$

$$(a^2 + b^2 + c^2)^2 \geq 2 \sum a^4 - 8 \sum a^3 + 9 \sum a^2 - 4 \sum a + 12.$$

Write this inequality in the homogeneous form  $f_4(a, b, c) \geq 0$ , where

$$\begin{aligned} f_4(a, b, c) &= 3(a^2 + b^2 + c^2)^2 - 6 \sum a^4 + 8 \left( \sum a^3 \right) \left( \sum a \right) - 3 \left( \sum a^2 \right) \left( \sum a \right)^2 \\ &= 2 \left[ \sum a^4 + \sum ab(a^2 + b^2) - 3abc \sum a \right]. \end{aligned}$$

According to P 2.60, it suffices to prove that  $f_4(a, 1, 1) \geq 0$  for all real  $a$ . Indeed,

$$f_4(a, 1, 1) = 2(a - 1)^2(a + 2)^2 \geq 0.$$

The equality holds for  $a = b = c = 1$ .

□

**P 2.73.** If  $a, b, c$  are real numbers, then

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \geq 9(ab + bc + ca).$$

(Vasile Cîrtoaje, 1994)

**Solution.** We will prove the sharper inequality  $f_4(a, b, c) \geq 0$ , where

$$f_4(a, b, c) = (a^2 + 2)(b^2 + 2)(c^2 + 2) - 9(ab + bc + ca) - \left(abc - \frac{a + b + c}{3}\right)^2.$$

Since  $f_4(a, b, c)$  is a symmetric polynomial of degree four, according to Remark from P 2.60, it suffices to prove that  $f_4(a, b, b) \geq 0$  for all real numbers  $a$  and  $b$ . For fixed  $b$ , this inequality is equivalent to  $f(a) \geq 0$ , where

$$f(a) = 7(6b^2 + 5)a^2 + 2b(6b^2 - 83)a + 18b^4 - 13b^2 + 72.$$

It is true for all real  $a$  if and only if

$$7(6b^2 + 5)(18b^4 - 13b^2 + 72) \geq b^2(6b^2 - 83)^2.$$

Indeed, we have

$$7(6b^2 + 5)(18b^4 - 13b^2 + 72) - b^2(6b^2 - 83)^2 = 360(b^2 - 1)^2(2b^2 + 7) \geq 0.$$

The equality holds for  $a = b = c = 1$ .

**Remark.** A sharper inequality for all real  $a, b, c$  is the following:

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \geq 3(a + b + c)^2 + \frac{4}{9} [(a - b)^2 + (b - c)^2 + (c - a)^2],$$

with equality for  $a = b = c = 1$ .

The proof is similar, and the final inequality of the proof of the inequality

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \geq 3(a + b + c)^2 + \frac{4}{9} [(a - b)^2 + (b - c)^2 + (c - a)^2] + \left(abc - \frac{a + b + c}{3}\right)^2,$$

has the form

$$b^2(b^2 - 1)^2 \geq 0.$$

□

**P 2.74.** If  $a, b, c$  are real numbers such that

$$ab + bc + ca = 3,$$

then

$$4(a^4 + b^4 + c^4) + 11abc(a + b + c) \geq 45.$$

(Vasile Cîrtoaje, 2014)

**Solution.** Write the inequality in the homogeneous form

$$4(a^4 + b^4 + c^4) + 11abc(a + b + c) \geq 5(ab + bc + ca)^2.$$

It suffices to prove that there exists a positive number  $k$  such that  $f_4(a, b, c) \geq 0$ , where

$$f_4(a, b, c) = 4(a^4 + b^4 + c^4) + 11abc(a + b + c) - 5(ab + bc + ca)^2 - k(ab + bc + ca)(a^2 + b^2 + c^2 - ab - bc - ca).$$

According to P 2.60, the inequality  $f_4(a, b, c) \geq 0$  holds for all real  $a, b, c$  if and only if  $f_4(a, 1, 1) \geq 0$  for all real  $a$ . We have

$$\begin{aligned} f_4(a, 1, 1) &= (a - 1)^2(2a + 1)(2a + 3) - k(2a + 1)(a - 1)^2 \\ &= (a - 1)^2(2a + 1)(2a + 3 - k). \end{aligned}$$

Setting  $k = 2$ , we get

$$f_4(a, 1, 1) = (a - 1)^2(2a + 1)^2 \geq 0.$$

The equality holds for  $a = b = c = \pm 1$ .

□

**P 2.75.** Any sixth degree symmetric homogeneous polynomial  $f_6(a, b, c)$  can be written in the form

$$f_6(a, b, c) = Ar^2 + B(p, q)r + C(p, q),$$

where  $A$  is called the highest coefficient of  $f_6$ , and

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

In the case  $A \leq 0$ , prove that the inequality  $f_6(a, b, c) \geq 0$  holds for all real numbers  $a, b, c$  if and only if  $f_6(a, 1, 1) \geq 0$  for all real  $a$ .

(Vasile Cîrtoaje, 2006)

**Solution.** For  $A \leq 0$  and fixed  $p$  and  $q$ ,

$$g(r) = Ar^2 + B(p, q)r + C(p, q)$$

is a concave quadratic function of  $r$ . Therefore,  $g(r)$  has the minimum value when  $r$  has the minimum or maximum value. By P 2.53,  $r$  has the minimum and maximum value when two of  $a, b, c$  are equal. Since  $f_6(a, b, c)$  is symmetric, homogeneous and satisfies  $f_6(-a, -b, -c) = f_6(a, b, c)$ , it follows that the inequality  $f_6(a, b, c) \geq 0$  holds for all real numbers  $a, b, c$  if and only if  $f_6(a, 1, 1) \geq 0$  and  $f_6(a, 0, 0) \geq 0$  for all real  $a$ . Notice that the condition " $f_6(a, 0, 0) \geq 0$  for all real  $a$ " is not necessary because it follows from the condition " $f_6(a, 1, 1) \geq 0$  for all real  $a$ " as follows:

$$f_6(a, 0, 0) = \lim_{t \rightarrow 0} f_6(a, t, t) = \lim_{t \rightarrow 0} t^6 f_6(a/t, 1, 1) \geq 0.$$



**Remark 1.** A symmetric homogeneous polynomial of degree six in three variables has the form

$$(A) \quad f_6(a, b, c) = A_1 \sum a^6 + A_2 \sum ab(a^4 + b^4) + A_3 \sum a^2 b^2 (a^2 + b^2) \\ + A_4 \sum a^3 b^3 + A_5 abc \sum a^3 + A_6 abc \sum ab(a + b) + 3A_7 a^2 b^2 c^2,$$

where  $A_1, \dots, A_7$  are real constants. In order to write this polynomial as a function of  $p, q$  and  $r$ , we can use the following relations:

$$\begin{aligned} \sum a^3 &= 3r + p^3 - 3pq, \\ \sum ab(a + b) &= -3r + pq, \\ \sum a^3 b^3 &= 3r^2 - 3pqr + q^3, \\ \sum a^2 b^2 (a^2 + b^2) &= -3r^2 - 2(p^3 - 2pq)r + p^2 q^2 - 2q^3, \\ \sum ab(a^4 + b^4) &= -3r^2 - 2(p^3 - 7pq)r + p^4 q - 4p^2 q^2 + 2q^3, \\ \sum a^6 &= 3r^2 + 6(p^3 - 2pq)r + p^6 - 6p^4 q + 9p^2 q^2 - 2q^3. \end{aligned}$$

According to these relations, the highest coefficient  $A$  of the polynomial  $f_6(a, b, c)$  has the expression

$$(B) \quad A = 3(A_1 - A_2 - A_3 + A_4 + A_5 - A_6 + A_7).$$

**Remark 2.** The polynomial

$$P_1(a, b, c) = \sum (A_1 a^2 + A_2 bc)(B_1 a^2 + B_2 bc)(C_1 a^2 + C_2 bc)$$

has the highest coefficient

$$A = 3(A_1 + A_2)(B_1 + B_2)(C_1 + C_2) = P_1(1, 1, 1).$$

Indeed, since

$$P_1(a, b, c) = A_1 B_1 C_1 \sum a^6 + A_2 B_2 C_2 \sum b^3 c^3 + \left( \sum A_1 B_1 C_2 \right) abc \sum a^3 \\ + 3 \left( \sum A_1 B_2 C_2 \right) a^2 b^2 c^2,$$

we have

$$\begin{aligned} A &= 3A_1 B_1 C_1 + 3A_2 B_2 C_2 + 3 \sum A_1 B_1 C_2 + 3 \sum A_1 B_2 C_2 \\ &= 3(A_1 + A_2)(B_1 + B_2)(C_1 + C_2). \end{aligned}$$

Similarly, we can show that the polynomial

$$P_2(a, b, c) = \sum (A_1 a^2 + A_2 bc)(B_1 b^2 + B_2 ca)(C_1 c^2 + C_2 ab)$$

has the highest coefficient

$$A = 3(A_1 + A_2)(B_1 + B_2)(C_1 + C_2) = P_2(1, 1, 1),$$

and the polynomial

$$P_3(a, b, c) = \prod (A_1 a^2 + A_2 bc) = (A_1 a^2 + A_2 bc)(A_1 b^2 + A_2 ca)(A_1 c^2 + A_2 ab)$$

has the highest coefficient

$$A = (A_1 + A_2)^3 = P_3(1, 1, 1).$$

The polynomial

$$P_4(a, b, c) = \prod (a^2 + mab + b^2)$$

has the highest coefficient

$$A = (m - 1)^3.$$

Indeed, since

$$\prod (a^2 + mab + b^2) = \prod (p^2 - 2q - c^2 + mab),$$

$P_4(a, b, c)$  has the same highest coefficient as  $R_3(a, b, c) = \prod (-c^2 + mab)$ ; that is,

$$A = R_3(1, 1, 1) = (-1 + m)^3.$$

As a consequence,

$$P_5(a, b, c) = (a - b)^2(b - c)^2(c - a)^2 = \prod (a^2 - 2ab + b^2)$$

has the highest coefficient

$$A = (-2 - 1)^2 = -27.$$

**Remark 3.** We can extend the statement in P 2.75 as follows:

• Let  $f_6(a, b, c)$  be a sixth degree symmetric homogeneous polynomial having the highest coefficient  $A \leq 0$ , and let  $k_1, k_2$  be two fixed real numbers. The inequality  $f_6(a, b, c) \geq 0$  holds for all real numbers  $a, b, c$  satisfying

$$k_1(a + b + c)^2 + k_2(ab + bc + ca) \geq 0,$$

if and only if  $f_6(a, 1, 1) \geq 0$  for all real  $a$  satisfying  $k_1(a + 2)^2 + k_2(2a + 1) \geq 0$ .

Notice that the condition " $f_6(a, 0, 0) \geq 0$  for all real  $a$  satisfying  $k_1 a^2 \geq 0$ " is not necessary because it follows from the condition " $f_6(a, 1, 1) \geq 0$  for all real  $a$  satisfying  $k_1(a + 2)^2 + k_2(2a + 1) \geq 0$ ". Indeed, for the non-trivial case  $k_1 \geq 0$ , when the condition " $f_6(a, 0, 0) \geq 0$  for all real  $a$  satisfying  $k_1 a^2 \geq 0$ " becomes " $f_6(a, 0, 0) \geq 0$  for all real  $a$ ", we have

$$f_6(a, 0, 0) = \lim_{t \rightarrow 0} f_6(a, t, t) = \lim_{t \rightarrow 0} t^6 f_6(a/t, 1, 1) \geq 0.$$

**Remark 4.** The statement in P 2.75 and its extension in Remark 3 are also valid in the more general case when  $f_6(a, b, c)$  is a symmetric homogeneous function of the form

$$f_6(a, b, c) = Ar^2 + B(p, q)r + C(p, q),$$

where  $B(p, q)$  and  $C(p, q)$  are rational or nonrational functions.

□

**P 2.76.** If  $a, b, c$  are real numbers such that

$$ab + bc + ca = -1,$$

then

$$(a) \quad 5(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \geq 8;$$

$$(b) \quad (a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq 1.$$

(Vasile Cîrtoaje, 2011)

**Solution.** We use the *highest coefficient method* (see P 2.75). Let

$$p = a + b + c, \quad q = ab + bc + ca.$$

(a) Write the inequality in the homogeneous form  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = 5(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) + 8(ab + bc + ca)^3.$$

From

$$\prod(b^2 + c^2) = \prod(p^2 - 2q - a^2),$$

it follows that  $f_6(a, b, c)$  has the highest coefficient  $A = -5$ . Then, by P 2.75, it suffices to prove that  $f_6(a, 1, 1) \geq 0$  for all real  $a$ . Indeed, we have

$$f_6(a, 1, 1) = 2(a + 3)^2(5a^2 + 2a + 1) \geq 0.$$

The homogeneous inequality

$$f_6(a, b, c) \geq 0$$

is an equality for  $-a/3 = b = c$  (or any cyclic permutation), and for  $b = c = 0$  (or any cyclic permutation). The original inequality becomes an equality for  $a = -3/\sqrt{5}$  and  $b = c = 1/\sqrt{5}$  (or any cyclic permutation), and for  $a = 3/\sqrt{5}$  and  $b = c = -1/\sqrt{5}$  (or any cyclic permutation).

(b) Write the inequality in the homogeneous form  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = \prod(b^2 + bc + c^2) + (ab + bc + ca)^3.$$

According to Remark 2 from P 2.75,  $f_6(a, b, c)$  has the highest coefficient

$$A = (1 - 1)^3 = 0.$$

Then, by P 2.75, it suffices to prove that  $f_6(a, 1, 1) \geq 0$  for all real  $a$ . Indeed, we have

$$f_6(a, 1, 1) = (a + 2)^2(3a^2 + 2a + 1) \geq 0.$$

The homogeneous inequality  $f_6(a, b, c) \geq 0$  is an equality when  $a + b + c = 0$ , and when  $b = c = 0$  (or any cyclic permutation). The original inequality becomes an equality for

$$ab + bc + ca = -1, \quad a + b + c = 0.$$

**Remark.** As we have shown in the proof of P 2.34,

$$\prod (b^2 + bc + c^2) = (p^2 - q)q^2 - p^3abc.$$

Therefore,

$$f_6(a, b, c) = p^2(q^2 - pabc) = \frac{1}{2}p^2 \sum a^2(b - c)^2 \geq 0.$$

□

**P 2.77.** If  $a, b, c$  are real numbers, then

$$(a) \quad \sum a^2(a - b)(a - c)(a + 2b)(a + 2c) + (a - b)^2(b - c)^2(c - a)^2 \geq 0;$$

$$(b) \quad \sum a^2(a - b)(a - c)(a - 4b)(a - 4c) + 7(a - b)^2(b - c)^2(c - a)^2 \geq 0.$$

(Vasile Cîrtoaje, 2008)

**Solution.** Consider the more general inequality  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = f(a, b, c) + m(a - b)^2(b - c)^2(c - a)^2,$$

$$f(a, b, c) = \sum a^2(a - b)(a - c)(a - kb)(a - kc).$$

Since

$$f(a, b, c) = \sum a^2(a^2 + 2bc - q)[a^2 + (k + k^2)bc - kq], \quad q = ab + bc + ca,$$

$f(a, b, c)$  has the same highest coefficient as  $P_1(a, b, c)$ , where

$$P_1(a, b, c) = \sum a^2(a^2 + 2bc)[a^2 + (k + k^2)bc].$$

According to Remark 2 from P 2.75,  $f_6(a, b, c)$  has the highest coefficient

$$A = P_1(1, 1, 1) - 27m = 9(k^2 + k + 1 - 3m).$$

(a) For  $k = -2$  and  $m = 1$ , we get  $A = 0$ . Then, by P 2.75, it suffices to prove the original inequality for  $b = c = 1$ ; that is,

$$a^2(a - 1)^2(a + 2)^2 \geq 0.$$

The equality holds for  $a = b = c$ , for  $a + b + c = 0$ , and for  $a = 0$  and  $b = c$  (or any cyclic permutation).

(b) For  $k = 4$  and  $m = 7$ , we get  $A = 0$ . Then, by P 2.75, it suffices to prove the original inequality for  $b = c = 1$ ; that is,

$$a^2(a-1)^2(a-4)^2 \geq 0.$$

The equality holds for  $a = b = c$ , and for  $a^2 + b^2 + c^2 = 2(ab + bc + ca)$ .

**Remark.** The inequalities in P 2.77 are respectively equivalent to

$$\left(\sum a\right)^2 \left[\sum a^4 + abc \sum a - \sum ab(a^2 + b^2)\right] \geq 0$$

and

$$\left(\sum a^2 - \sum ab\right) \left(\sum a^2 - 2 \sum ab\right)^2 \geq 0.$$

□

**P 2.78.** If  $a, b, c$  are real numbers, then

$$(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) + (a - b)^2(b - c)^2(c - a)^2 \geq 0.$$

(Vasile Cîrtoaje, 2011)

**First Solution** (by Vo Quoc Ba Can). Without loss of generality, assume that  $b$  and  $c$  have the same sign. Since  $a^2 + 2bc \geq 0$  and

$$\begin{aligned} (a - b)^2(a - c)^2 &= \frac{1}{4}[(a^2 + 2bc) + (a^2 - 2ab - 2ac)]^2 \\ &\geq (a^2 + 2bc)(a^2 - 2ab - 2ac), \end{aligned}$$

it suffices to prove that

$$(b^2 + 2ca)(c^2 + 2ab) + (b - c)^2(a^2 - 2ab - 2ac) \geq 0.$$

This inequality is equivalent to

$$(b + c)^2 a^2 + 2bc(b + c)a + b^2 c^2 \geq 0,$$

$$[(b + c)a + bc]^2 \geq 0.$$

which is clearly true. The equality holds for  $ab + bc + ca = 0$ .

**Second Solution.** Denote the left side of the inequality by  $f_6(a, b, c)$ . According to Remark 2 from P 2.75,  $f_6(a, b, c)$  has the highest coefficient

$$A = (1 + 2)^3 - 27 = 0.$$

Then, by P 2.75, it suffices to prove that  $f_6(a, 1, 1) \geq 0$  for all real  $a$ . Indeed,

$$f_6(a, 1, 1) = (a^2 + 2)(2a + 1)^2 \geq 0.$$

**Remark 1.** The inequality is equivalent to

$$(a^2 + b^2 + c^2)(ab + bc + ca)^2 \geq 0.$$

**Remark 2.** The inequality in P 2.78 is a particular case of the following more general statement.

- If  $a, b, c$  are real numbers and

$$\alpha_k = \begin{cases} \frac{9k^2(k^2 - k + 1)}{4(k + 1)^3}, & 1 \leq k \leq 2 \\ \frac{k^2}{4}, & k \geq 2 \end{cases},$$

then

$$(a^2 + kbc)(b^2 + kca)(c^2 + kab) + \alpha_k(a - b)^2(b - c)^2(c - a)^2 \geq 0,$$

with equality for  $-ka = b = c$  (or any cyclic permutation), and also for  $b = c = 0$  (or any cyclic permutation).

□

**P 2.79.** If  $a, b, c$  are real numbers, then

$$(2a^2 + 5ab + 2b^2)(2b^2 + 5bc + 2c^2)(2c^2 + 5ca + 2a^2) + (a - b)^2(b - c)^2(c - a)^2 \geq 0.$$

(Vasile Cîrtoaje, 2011)

**Solution.** Denote the left side of the inequality by  $f_6(a, b, c)$ . Clearly,  $f_6(a, b, c)$  has the same highest coefficient as

$$(5ab - 2c^2)(5bc - 2a^2)(5ca - 2b^2) + (a - b)^2(b - c)^2(c - a)^2.$$

According to Remark 2 from P 2.75,  $f_6(a, b, c)$  has the highest coefficient

$$A = (5 - 2)^3 - 27 = 0.$$

Then, by P 2.75, it suffices to prove that  $f_6(a, 1, 1) \geq 0$  for all real  $a$ . Indeed,

$$f_6(a, 1, 1) = 9(2a^2 + 5a + 2)^2 \geq 0.$$

The equality holds for  $a + b + c = 0$ , and also for  $ab + bc + ca = 0$ .

**Remark 1.** The inequality in P 2.79 is equivalent to

$$(a + b + c)^2(ab + bc + ca)^2 \geq 0.$$

**Remark 2.** The following more general statement holds.

- Let  $a, b, c$  be real numbers. If  $k > -2$ , then

$$4 \prod (b^2 + kbc + c^2) \geq (2 - k)(a - b)^2(b - c)^2(c - a)^2.$$

Notice that this inequality is equivalent to

$$(k + 2)[(a + b + c)(ab + bc + ca) - (5 - 2k)abc]^2 \geq 0.$$

□

**P 2.80.** If  $a, b, c$  are real numbers, then

$$\left(a^2 + \frac{2}{3}ab + b^2\right) \left(b^2 + \frac{2}{3}bc + c^2\right) \left(c^2 + \frac{2}{3}ca + a^2\right) \geq \frac{64}{27}(a^2 + bc)(b^2 + ca)(c^2 + ab).$$

**Solution.** Write the inequality as  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = P_4(a, b, c) - \frac{64}{27}P_3(a, b, c),$$

$$P_4(a, b, c) = \left(a^2 + \frac{2}{3}ab + b^2\right) \left(b^2 + \frac{2}{3}bc + c^2\right) \left(c^2 + \frac{2}{3}ca + a^2\right),$$

$$P_3(a, b, c) = (a^2 + bc)(b^2 + ca)(c^2 + ab).$$

According to Remark 2 from P 2.75,  $f_6(a, b, c)$  has the highest coefficient

$$A = \left(\frac{2}{3} - 1\right)^3 - \frac{64}{27}(1 + 1)^3 < 0.$$

Then, by P 2.75, it suffices to prove that  $f_6(a, 1, 1) \geq 0$  for all real  $a$ . Indeed,

$$f_6(a, 1, 1) = \frac{8}{3} \left(a^2 + \frac{2}{3}a + 1\right)^2 - \frac{64}{27}(a^2 + 1)(a + 1)^2 = \frac{8}{27}(a - 1)^4 \geq 0.$$

The equality holds for  $a = b = c$ .

□

**P 2.81.** If  $a, b, c$  are real numbers, then

$$\sum a^2(a - b)(a - c) \geq \frac{2(a - b)^2(b - c)^2(c - a)^2}{a^2 + b^2 + c^2}.$$

**Solution.** Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc$$

and

$$f_6(a, b, c) = (a^2 + b^2 + c^2) \sum a^2(a - b)(a - c) - 2(a - b)^2(b - c)^2(c - a)^2.$$

Clearly,  $f_6(a, b, c)$  has the highest coefficient

$$A = -2(-27) = 54.$$

Since  $A > 0$ , we will use the *highest coefficient cancellation method*. It is easy to check that

$$f_6(1, 1, 1) = 0, \quad f_6(0, 1, 1) = 0.$$

Therefore, we define the symmetric homogeneous polynomial of degree three

$$P(a, b, c) = r + Bp^3 + Cpq$$

such that

$$P(1, 1, 1) = 0, \quad P(0, 1, 1) = 0;$$

that is,

$$P(a, b, c) = r + \frac{1}{9}p^3 - \frac{4}{9}pq.$$

We will prove the sharper inequality  $g_6(a, b, c) \geq 0$ , where

$$g_6(a, b, c) = f_6(a, b, c) - 54P^2(a, b, c).$$

Since  $g_6(a, b, c)$  has the highest coefficient  $A_1 = 0$ , it suffices to show that  $g_6(a, 1, 1) \geq 0$  for all real  $a$  (see P 2.75). Indeed, we have

$$f_6(a, 1, 1) = a^2(a^2 + 2)(a - 1)^2, \quad P(a, 1, 1) = \frac{1}{9}a(a - 1)^2,$$

hence

$$g_6(a, 1, 1) = f_6(a, 1, 1) - 54P^2(a, 1, 1) = \frac{1}{3}a^2(a - 1)^2(a + 2)^2 \geq 0.$$

The equality holds for  $a = b = c$ , for  $a = 0$  and  $b = c$  (or any cyclic permutation), and also for  $a = 0$  and  $b + c = 0$  (or any cyclic permutation).

**Remark.** In the same manner, we can prove the following generalization (Vasile Cîrtoaje, 2014).

- Let  $a, b, c$  be real numbers. If  $k \in [-1, 2)$ , then

$$\sum a^2(a - b)(a - c) \geq \frac{(2 - k)(a - b)^2(b - c)^2(c - a)^2}{a^2 + b^2 + c^2 + k(ab + bc + ca)},$$

with equality for  $a = b = c$ , and also for  $a = 0$  and  $b^2 = c^2$  (or any cyclic permutation).

□



**P 2.82.** If  $a, b, c$  are real numbers, then

$$\sum (a-b)(a-c)(a-2b)(a-2c) \geq \frac{8(a-b)^2(b-c)^2(c-a)^2}{a^2+b^2+c^2}.$$

**Solution.** Let

$$f_6(a, b, c) = (a^2 + b^2 + c^2) \sum (a-b)(a-c)(a-2b)(a-2c) - 8(a-b)^2(b-c)^2(c-a)^2.$$

Clearly,  $f_6(a, b, c)$  has the highest coefficient

$$A = (-8)(-27) = 216.$$

Since  $A > 0$ , we will use the *highest coefficient cancellation method*. Since

$$f_6(1, 1, 1) = 0, \quad f_6(2, 1, 1) = 0,$$

we define the symmetric homogeneous polynomial of degree three

$$P(a, b, c) = abc + B(a+b+c)^3 + C(a+b+c)(ab+bc+ca)$$

such that

$$P(1, 1, 1) = 0, \quad P(2, 1, 1) = 0.$$

We get  $B = 1/18$  and  $C = -5/18$ , hence

$$P(a, b, c) = abc + \frac{1}{18}(a+b+c)^3 - \frac{5}{18}(a+b+c)(ab+bc+ca).$$

Consider now the sharper inequality  $g_6(a, b, c) \geq 0$ , where

$$g_6(a, b, c) = f_6(a, b, c) - 216P^2(a, b, c).$$

Clearly,  $g_6(a, b, c)$  has the highest coefficient  $A_1 = 0$ . By P 2.75, it suffices to prove that  $g_6(a, 1, 1) \geq 0$  for all real  $a$ . We have

$$f_6(a, 1, 1) = (a^2 + 2)(a-1)^2(a-2)^2, \quad P(a, 1, 1) = \frac{1}{18}(a-1)^2(a-2),$$

hence

$$g_6(a, 1, 1) = f_6(a, 1, 1) - 216P^2(a, 1, 1) = \frac{1}{3}(a-1)^2(a^2-4)^2 \geq 0.$$

The equality holds for  $a = b = c$ , for  $a = 0$  and  $b + c = 0$  (or any cyclic permutation), and also for  $a/2 = b = c$  (or any cyclic permutation).

**Remark.** In the same manner, we can prove the following generalization (Vasile Cîrtoaje, 2014).

- Let  $a, b, c$  be real numbers. If  $k \in \mathbb{R}$ , then

$$\sum (a-b)(a-c)(a-kb)(a-kc) \geq \frac{(k+2)^2(a-b)^2(b-c)^2(c-a)^2}{2(a^2+b^2+c^2)},$$

with equality for  $a = b = c$ , for  $a/k = b = c$  (or any cyclic permutation) if  $k \neq 0$ , and for  $a = 0$  and  $b + c = 0$  (or any cyclic permutation). □

**P 2.83.** If  $a, b, c$  are real numbers, no two of which are zero, then

$$\frac{a^2 + 3bc}{b^2 + c^2} + \frac{b^2 + 3ca}{c^2 + a^2} + \frac{c^2 + 3ab}{a^2 + b^2} \geq 0.$$

(Vasile Cîrtoaje, 2014)

**Solution.** Write the inequality as  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = \sum (a^2 + 3bc)(a^2 + b^2)(a^2 + c^2).$$

Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

From

$$f_6(a, b, c) = \sum (a^2 + 3bc)(p^2 - 2q - c^2)(p^2 - 2q - b^2),$$

it follows that  $f_6(a, b, c)$  has the same highest coefficient  $A$  as  $f(a, b, c)$ , where

$$f(a, b, c) = \sum (a^2 + 3bc)b^2c^2 = 3r^2 + 3 \sum b^3c^3 = 12r^2 - 9pqr + 3q^3;$$

that is,

$$A = 12.$$

Since  $A > 0$ , we will use the *highest coefficient cancellation method*. It is easy to check that

$$f_6(-1, 1, 1) = 0.$$

So, we define the homogeneous polynomial

$$P(a, b, c) = r + Bp^3 + (B - 1)pq,$$

which satisfies the property  $P(-1, 1, 1) = 0$ . We will show that there is at least a real value of  $B$  such that the following sharper inequality holds

$$f_6(a, b, c) \geq 12P^2(a, b, c).$$

Let us denote

$$g_6(a, b, c) = f_6(a, b, c) - 12P^2(a, b, c).$$

Clearly,  $g_6(a, b, c)$  has the highest coefficient  $A_1 = 0$ . By P 2.75, it suffices to prove that  $g_6(a, 1, 1) \geq 0$  for all real  $a$ . We have

$$f_6(a, 1, 1) = (a + 1)^2(a^2 + 1)(a^2 - 2a + 7)$$

and

$$P(a, 1, 1) = (a + 1)[B(a + 2)(a + 5) - 2(a + 1)],$$

hence

$$g_6(a, 1, 1) = f_6(a, 1, 1) - 12P^2(a, 1, 1) = (a + 1)^2g(a),$$

where

$$g(a) = (a^2 + 1)(a^2 - 2a + 7) - 12[B(a + 2)(a + 5) - 2(a + 1)]^2.$$

Choosing  $B = 1/4$ , we get

$$4g(a) = a^2(a - 1)^2 + 4(4a^2 + a + 4) > 0,$$

hence  $g_6(a, 1, 1) \geq 0$  for all real  $a$ . The proof is completed. The equality holds for  $-a = b = c$  (or any cyclic permutation). □

**P 2.84.** If  $a, b, c$  are real numbers, no two of which are zero, then

$$\frac{a^2 + 6bc}{b^2 - bc + c^2} + \frac{b^2 + 6ca}{c^2 - ca + a^2} + \frac{c^2 + 6ab}{a^2 - ab + b^2} \geq 0.$$

(Vasile Cîrtoaje, 2014)

**Solution.** Write the inequality as  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = \sum (a^2 + 6bc)(a^2 - ab + b^2)(a^2 - ac + c^2).$$

Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

From

$$f_6(a, b, c) = \sum (a^2 + 6bc)(p^2 - 2q - c^2 - ab)(p^2 - 2q - b^2 - ac),$$

it follows that  $f_6(a, b, c)$  has the same highest coefficient  $A$  as  $f(a, b, c)$ , where

$$f(a, b, c) = \sum (a^2 + 6bc)(b^2 + ca)(c^2 + ab);$$

that is, according to Remark 2 from P 2.75,

$$A = f(1, 1, 1) = 84.$$

Since  $A > 0$ , we use the *highest coefficient cancellation method*. We will show that there are two real numbers  $B$  and  $C$  such that the following sharper inequality holds

$$f_6(a, b, c) \geq 84P^2(a, b, c),$$

where

$$P(a, b, c) = r + Bp^3 + Cpq.$$

Let us denote

$$g_6(a, b, c) = f_6(a, b, c) - 84P^2(a, b, c).$$

Clearly,  $g_6(a, b, c)$  has the highest coefficient equal to zero. Then, it suffices to prove that  $g_6(a, 1, 1) \geq 0$  for all real  $a$ . We have

$$g_6(a, 1, 1) = f_6(a, 1, 1) - 84P^2(a, 1, 1),$$

where

$$f_6(a, 1, 1) = (a^2 - a + 1)(a^2 + a + 1)(a^2 - 2a + 8)$$

and

$$P(a, 1, 1) = a + B(a + 2)^3 + C(a + 2)(2a + 1).$$

Let

$$g(a) = g_6(a, 1, 1).$$

Since  $g(-2) = 0$ , we can have  $g(a) \geq 0$  in the vicinity of  $a = -2$  only if  $g'(-2) = 0$ , which involves  $C = -61/168$ . On the other hand, from  $g(1) = 0$ , we get  $B = 155/1512$ . Using these values of  $B$  and  $C$ , the inequality  $g_6(a, 1, 1) \geq 0$  is equivalent to

$$\begin{aligned} & 27216(a^2 - a + 1)(a^2 + a + 1)(a^2 - 2a + 8) \geq \\ & \geq [155(a + 2)^3 - 549(a + 2)(2a + 1) + 1512a]^2; \end{aligned}$$

that is,

$$(a + 2)^2(a - 1)^2(3191a^2 - 8734a + 49391) \geq 0,$$

which is true for all real  $a$ . The proof is completed. The equality holds for  $a = b + c = 0$  (or any cyclic permutation). □

**P 2.85.** If  $a, b, c$  are real numbers such that

$$ab + bc + ca \geq 0,$$

then

$$\frac{4a^2 + 23bc}{b^2 + c^2} + \frac{4b^2 + 23ca}{c^2 + a^2} + \frac{4c^2 + 23ab}{a^2 + b^2} \geq 0.$$

(Vasile Cîrtoaje, 2014)

**Solution.** Write the inequality as  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = \sum (4a^2 + 23bc)(a^2 + b^2)(a^2 + c^2).$$

Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

From

$$f_6(a, b, c) = \sum (4a^2 + 23bc)(p^2 - 2q - c^2)(p^2 - 2q - b^2),$$

it follows that  $f_6(a, b, c)$  has the same highest coefficient  $A$  as  $f(a, b, c)$ , where

$$f(a, b, c) = \sum (4a^2 + 23bc)b^2c^2 = 12r^2 + 23 \sum b^3c^3 = 81r^2 - 69pqr + 23q^3;$$

that is,

$$A = 81.$$

Since  $A > 0$ , we will use the *highest coefficient cancellation method*. It is easy to check that

$$f(-1, 2, 2) = 0.$$

Therefore, define the homogeneous polynomial

$$P(a, b, c) = r + \frac{4}{27}p^3 + Cpq,$$

which satisfies  $P(-1, 2, 2) = 0$ . We will show that there is at least a real  $C$  such that the following sharper inequality holds for  $ab + bc + ca \geq 0$ :

$$f_6(a, b, c) \geq 81P^2(a, b, c).$$

Let us denote

$$g_6(a, b, c) = f_6(a, b, c) - 81P^2(a, b, c).$$

Clearly,  $g_6(a, b, c)$  has the highest coefficient  $A_1 = 0$ . Therefore, by Remark 3 from P 2.75, it suffices to prove that  $g_6(a, 1, 1) \geq 0$  for all real  $a$  such that  $2a + 1 \geq 0$ . We have

$$f_6(a, 1, 1) = (2a + 1)(a^2 + 1)(2a^3 - a^2 + 14a + 39),$$

$$P(a, 1, 1) = \frac{1}{27}(2a + 1)[2a^2 + (27C + 11)a + 54C + 32],$$

$$g_6(a, 1, 1) = f_6(a, 1, 1) - 81P^2(a, 1, 1).$$

From the condition  $g_6(1, 1, 1) = 0$ , we get  $C = -1/3$ . For this value of  $C$ , we find

$$P(a, 1, 1) = \frac{2}{27}(2a + 1)(a^2 + a + 7),$$

then

$$\begin{aligned} g_6(a, 1, 1) &= \frac{1}{9}(2a+1)(10a^5 - 29a^4 + 16a^3 + 170a^2 - 322a + 155) \\ &= \frac{1}{9}(2a+1)(a-1)^2(10a^3 - 9a^2 - 12a + 155). \end{aligned}$$

We only need to show that

$$10a^3 - 9a^2 - 12a + 155 \geq 0$$

for  $a \geq -1/2$ . This is clearly true for  $-1/2 \leq a \leq 0$ . Also, for  $a > 0$ , we have

$$10a^3 - 9a^2 - 12a + 155 = 10a(a^2 - a + 1) + (a - 11)^2 + 34 > 0.$$

The proof is completed. The equality holds for  $-2a = b = c$  (or any cyclic permutation).  $\square$

**P 2.86.** If  $a, b, c$  are real numbers such that

$$ab + bc + ca = 3,$$

then

$$20(a^6 + b^6 + c^6) + 43abc(a^3 + b^3 + c^3) \geq 189.$$

(Vasile Cîrtoaje, 2014)

**Solution.** Write the inequality in the homogeneous form  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = 20(a^6 + b^6 + c^6) + 43abc(a^3 + b^3 + c^3) - 7(ab + bc + ca)^3.$$

Since the highest coefficient of  $f_6(a, b, c)$  is positive, namely

$$A = 20 \cdot 3 + 43 \cdot 3 = 189,$$

we will use the *highest coefficient cancellation method*. From

$$f_6(a, 1, 1) = (2a+1)(a-1)^2(10a^3 + 15a^2 + 44a + 33),$$

it follows that

$$f_6(1, 1, 1) = 0, \quad f_6(-1/2, 1, 1) = 0.$$

Define the homogeneous function

$$P(a, b, c) = r + Bp^3 + Cpq, \quad p = a + b + c, \quad q = ab + bc + ca, \quad r = abc,$$

such that  $P(1, 1, 1) = P(-1/2, 1, 1) = 0$ ; that is,

$$P(a, b, c) = r + \frac{4}{27}p^3 - \frac{5}{9}pq,$$

$$P(a, 1, 1) = \frac{27a + 4(a+2)^3 - 15(a+2)(2a+1)}{27} = \frac{2(a-1)^2(2a+1)}{27}.$$

We will show that the following sharper inequality holds for  $ab + bc + ca \geq 0$ :

$$f_6(a, b, c) \geq 189P^2(a, b, c).$$

Let us denote

$$g_6(a, b, c) = f_6(a, b, c) - 189P^2(a, b, c).$$

Since the highest coefficient of  $g_6(a, b, c)$  is zero, it suffices to prove that  $g_6(a, 1, 1) \geq 0$  for  $2a + 1 \geq 0$  (see Remark 3 from P 2.75). We have

$$g_6(a, 1, 1) = f_6(a, 1, 1) - 189P^2(a, 1, 1) = (2a + 1)(a - 1)^2g(a),$$

where

$$g(a) = 10a^3 + 15a^2 + 44a + 33 - \frac{28}{27}(a - 1)^2(2a + 1).$$

Since

$$g(a) \geq 10a^3 + 15a^2 + 44a + 33 - 5(a - 1)^2(2a + 1) = 22(a + 1)^2 + 8a^2 + 6 > 0,$$

we have  $g_6(a, 1, 1) \geq 0$  for  $a \geq -1/2$ . Thus, the proof is completed. The equality holds for  $a = b = c = 1$ . □

**P 2.87.** If  $a, b, c$  are real numbers such that

$$ab + bc + ca \geq 0,$$

then

$$(a) \quad (a^2 + b^2 + c^2)(ab + bc + ca)^2 \geq abc(4a^3 + 4b^3 + 4c^3 + 15abc);$$

$$(b) \quad 4(a + b + c)^6 \geq 81abc(5a^3 + 5b^3 + 5c^3 + 21abc).$$

(Vasile Cîrtoaje and Nguyen Van Huyen, 2020)

**Solution.** Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

We have

$$a^3 + b^3 + c^3 = 3r + p^3 - 3pq.$$

(a) Write the inequality as  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = -27r^2 - (p^3 - 3pq)r + p^2q^2 - 2q^3.$$

Since  $f_6(a, b, c)$  has the highest coefficient  $A = -27$ , it suffices to prove the inequality for  $b = c = 1$  and  $2a + 1 \geq 0$  (see Remark 3 from P 2.75). Thus, we need to show that

$$(a^2 + 2)(2a + 1)^2 \geq a(4a^3 + 15a + 8),$$

which is equivalent to

$$(2a + 1)(a - 1)^2 \geq 0.$$

The equality occurs for  $a = b = c$  and for  $-2a = b = c$  (or any cyclic permutation).

(b) Write the inequality as  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = -2916r^2 - 405(p^3 - 3pq)r + 4p^6.$$

Since  $f_6(a, b, c)$  has the highest coefficient  $A = -2916$ , it suffices to prove the inequality for  $b = c = 1$  and  $2a + 1 \geq 0$  (see Remark 3 from P 2.75). Thus, we need to show that

$$4(a + 2)^6 \geq 81a(5a^3 + 21a + 10),$$

which is equivalent to

$$(2a + 1)(a - 1)^2(2a^3 + 27a^2 - 42a + 256) \geq 0.$$

This is true since

$$2a^3 + 27a^2 - 42a + 256 = a^2(2a + 1) + 26a^2 - 42a + 256 > 21a^2 - 42a + 21 \geq 0.$$

The equality occurs for  $a = b = c$  and for  $-2a = b = c$  (or any cyclic permutation).

□

**P 2.88.** If  $a, b, c$  are real numbers, then

$$4 \sum (a^2 + bc)(a - b)(a - c)(a - 3b)(a - 3c) \geq 7(a - b)^2(b - c)^2(c - a)^2.$$

(Vasile Cîrtoaje, 2014)

**Solution.** Write the inequality as  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = 4f(a, b, c) - 7(a - b)^2(b - c)^2(c - a)^2,$$

$$f(a, b, c) = \sum (a^2 + bc)(a - b)(a - c)(a - 3b)(a - 3c).$$

We have

$$f_6(a, 1, 1) = 4(a^2 + 1)(a - 1)^2(a - 3)^2.$$

Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$



Since

$$(a - b)(a - c) = a^2 + 2bc - q$$

and

$$(a - 3b)(a - 3c) = a^2 + 12bc - 3q,$$

$f(a, b, c)$  has the same highest coefficient  $A_0$  as  $g(a, b, c)$ , where

$$g(a, b, c) = \sum (a^2 + bc)(a^2 + 2bc)(a^2 + 12bc);$$

that is, according to Remark 2 from P 2.75,

$$A_0 = g(1, 1, 1) = 3 \cdot 2 \cdot 3 \cdot 13 = 234.$$

Therefore,  $f_6(a, b, c)$  has the highest coefficient

$$A = 4A_0 - 7(-27) = 1125.$$

Since the highest coefficient  $A$  is positive, we will use the *highest coefficient cancellation method*. There are two cases to consider:  $q \geq 0$  and  $q < 0$ .

*Case 1:*  $q \geq 0$ . Since

$$f_6(1, 1, 1) = f_6(3, 1, 1) = 0,$$

define the homogeneous function

$$P(a, b, c) = r + Bp^3 + Cpq$$

such that  $P(1, 1, 1) = P(3, 1, 1) = 0$ ; that is,

$$P(a, b, c) = r + \frac{2}{45}p^3 - \frac{11}{45}pq,$$

hence

$$P(a, 1, 1) = \frac{45a + 2(a + 2)^3 - 11(a + 2)(2a + 1)}{45} = \frac{2(a - 1)^2(a - 3)}{45}.$$

We will show that the following sharper inequality holds for  $q \geq 0$ :

$$f_6(a, b, c) \geq 1125P^2(a, b, c).$$

Let us denote

$$g_6(a, b, c) = f_6(a, b, c) - 1125P^2(a, b, c).$$

Since the highest coefficient of  $g_6(a, b, c)$  is zero, it suffices to prove that  $g_6(a, 1, 1) \geq 0$  for all real  $a$  such that  $2a + 1 \geq 0$  (see Remark 3 from P 2.75). We have

$$g_6(a, 1, 1) = f_6(a, 1, 1) - 1125P^2(a, 1, 1) = \frac{8(a - 1)^2(a - 3)^2(a + 2)(2a + 1)}{9} \geq 0.$$

Case 2:  $q < 0$ . Define the homogeneous polynomial

$$P(a, b, c) = r + Bp^3 - \left(3B + \frac{1}{9}\right)pq,$$

which satisfies  $P(1, 1, 1) = 0$ . We will show that there is a real number  $B$  such that the following sharper inequality holds

$$f_6(a, b, c) \geq 1125P^2(a, b, c).$$

Let us denote

$$g_6(a, b, c) = f_6(a, b, c) - 1125P^2(a, b, c).$$

Clearly,  $g_6(a, b, c)$  has the highest coefficient equal to zero. Then, by Remark 3 from P 2.75, it suffices to prove that  $g_6(a, 1, 1) \geq 0$  for  $2a + 1 < 0$ . We have

$$g_6(a, 1, 1) = f_6(a, 1, 1) - 1125P^2(a, 1, 1),$$

where

$$P^2(a, 1, 1) = \left[ a + B(a + 2)^3 - \left(3B + \frac{1}{9}\right)(a + 2)(2a + 1) \right]^2.$$

Let us denote  $g(a) = g_6(a, 1, 1)$ . Since  $g(-2) = 0$ , we can have  $g(a) \geq 0$  in the vicinity of  $a = -2$  only if  $g'(-2) = 0$ , which involves  $B = 8/135$ . Using this value of  $B$ , we get

$$P^2(a, 1, 1) = \frac{4(a - 1)^4(4a - 7)^2}{25 \cdot 729},$$

$$\begin{aligned} g_6(a, 1, 1) &= 4(a - 1)^2 \left[ (a^2 + 1)(a - 3)^2 - \frac{5}{81}(a - 1)^2(4a - 7)^2 \right] \\ &= \frac{4}{81}(a - 1)^2(a + 2)^2(a^2 - 50a + 121) \geq 0. \end{aligned}$$

The proof is completed. The equality holds for  $a = b = c$ , for  $a/3 = b = c$  (or any cyclic permutation), and for  $a = 0$  and  $b + c = 0$  (or any cyclic permutation). □

**P 2.89.** Let  $a, b, c$  be real numbers such that

$$ab + bc + ca \geq 0.$$

For any real  $k$ , prove that

$$\sum 4bc(a - b)(a - c)(a - kb)(a - kc) + (a - b)^2(b - c)^2(c - a)^2 \geq 0.$$

(Vasile Cîrtoaje, 2014)

**Solution.** Write the inequality as  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = 4f(a, b, c) + (a - b)^2(b - c)^2(c - a)^2,$$

$$f(a, b, c) = \sum bc(a - b)(a - c)(a - kb)(a - kc).$$

Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

Since

$$(a - b)(a - c) = a^2 + 2bc - q$$

and

$$(a - kb)(a - kc) = a^2 + (k + k^2)bc - kq,$$

$f(a, b, c)$  has the same highest coefficient  $A_0$  as  $P_1(a, b, c)$ , where

$$P_1(a, b, c) = \sum bc(a^2 + 2bc)[a^2 + (k + k^2)bc];$$

that is, according to Remark 2 from P 2.75,

$$A_0 = P_1(1, 1, 1) = 3(1 + 2)(1 + k + k^2) = 9(1 + k + k^2).$$

Therefore,  $f_6(a, b, c)$  has the highest coefficient

$$A = 4A_0 - 27 = 9(2k + 1)^2.$$

We have

$$f_6(a, 1, 1) = 4(a - 1)^2(a - k)^2.$$

Consider first that  $k = -1/2$ , when  $A = 0$ . By P 2.75, it suffices to show that  $f_6(a, 1, 1) \geq 0$  for all real  $a$ . Clearly, this is true. Consider further that  $k \neq -1/2$ , when the highest coefficient  $A$  is positive. We will use the *highest coefficient cancellation method*. Since

$$f_6(1, 1, 1) = f_6(k, 1, 1) = 0,$$

define the homogeneous function

$$P(a, b, c) = r + Cpq + D\frac{q^2}{p}$$

such that  $P(1, 1, 1) = P(k, 1, 1) = 0$ ; that is,

$$P(a, b, c) = r + \frac{pq}{3(2k + 1)} - \frac{2(k + 2)q^2}{3(2k + 1)p}.$$

We will show that the following sharper inequality holds for  $ab + bc + ca \geq 0$ :

$$f_6(a, b, c) \geq 9(2k + 1)^2 P^2(a, b, c).$$

Let us denote

$$g_6(a, b, c) = f_6(a, b, c) - 9(2k + 1)^2 P^2(a, b, c).$$

Clearly,  $g_6(a, b, c)$  has the highest coefficient  $A_1 = 0$ . Then, by Remark 4 from P 2.75, it suffices to prove that  $g_6(a, 1, 1) \geq 0$  for all real  $a$  such that  $2a + 1 \geq 0$ . We have

$$\begin{aligned} P(a, 1, 1) &= a + \frac{(a+2)(2a+1)}{3(2k+1)} - \frac{2(k+2)(2a+1)^2}{3(2k+1)(a+2)} \\ &= \frac{2(a-1)^2(a-k)}{3(2k+1)(a+2)}, \end{aligned}$$

then

$$\begin{aligned} g_6(a, 1, 1) &= f_6(a, 1, 1) - 9(2k+1)^2 P^2(a, 1, 1) \\ &= 4(a-1)^2(a-k)^2 - \frac{4(a-1)^4(a-k)^2}{(a+2)^2} \\ &= \frac{12(a-1)^2(a-k)^2(2a+1)}{(a+2)^2} \geq 0. \end{aligned}$$

The proof is completed. The equality holds for  $a = b = c$ , for  $a/k = b = c$  (or any cyclic permutation) - if  $k \neq 0$ , and for  $b = c = 0$  (or any cyclic permutation). □

**P 2.90.** If  $a, b, c$  are real numbers, then

$$[(a^2b + b^2c + c^2a) + (ab^2 + bc^2 + ca^2)]^2 \geq 4(ab + bc + ca)(a^2b^2 + b^2c^2 + c^2a^2).$$

**First Solution.** Consider the nontrivial case  $ab + bc + ca \geq 0$ , and write the inequality as  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = [(a^2b + b^2c + c^2a) + (ab^2 + bc^2 + ca^2)]^2 - 4(ab + bc + ca)(a^2b^2 + b^2c^2 + c^2a^2).$$

Since

$$(a^2b + b^2c + c^2a) + (ab^2 + bc^2 + ca^2) = (a + b + c)(ab + bc + ca) - 3abc,$$

$f_6$  has the highest coefficient

$$A = (-3)^2 = 9.$$

Since  $A > 0$ , we will use the *highest coefficient cancellation method*. Because

$$f(1, 1, 1) = f(0, 1, 1) = 0,$$

define the homogeneous function

$$P(a, b, c) = abc + C(a + b + c)(ab + bc + ca) + D \frac{(ab + bc + ca)^2}{a + b + c}$$

such that  $P(1, 1, 1) = P(0, 1, 1) = 0$ ; that is,

$$P(a, b, c) = abc + \frac{(a+b+c)(ab+bc+ca)}{3} - \frac{4(ab+bc+ca)^2}{3(a+b+c)}.$$

We will show that the following sharper inequality holds for  $ab+bc+ca \geq 0$ :

$$f_6(a, b, c) \geq 9P^2(a, b, c).$$

Let us denote

$$g_6(a, b, c) = f_6(a, b, c) - 9P^2(a, b, c).$$

Clearly,  $g_6(a, b, c)$  has the highest coefficient  $A_1 = 0$ . Then, by Remark 4 from P 2.75, it suffices to prove that  $g_6(a, 1, 1) \geq 0$  for all real  $a$  such that  $2a+1 \geq 0$ . We have

$$f_6(a, 1, 1) = 4a^2(a-1)^2,$$

$$P(a, 1, 1) = a + \frac{(a+2)(2a+1)}{3} - \frac{4(2a+1)^2}{3(a+2)} = \frac{2a(a-1)^2}{3(a+2)},$$

hence

$$f_6(a, 1, 1) - 9P^2(a, 1, 1) = \frac{12a^2(a-1)^2(2a+1)}{(a+2)^2} \geq 0.$$

The proof is completed. The equality holds for  $a = b = c$ , for  $a = 0$  and  $b = c$  (or any cyclic permutation), and for  $b = c = 0$  (or any cyclic permutation).

**Second Solution** (by *Nguyen Van Quy*). Since the inequality remains unchanged by replacing  $a, b, c$  with  $-a, -b, -c$ , we may assume that  $a+b+c \geq 0$ . In addition, consider the non-trivial case  $ab+bc+ca > 0$ , when the inequality can be rewritten as

$$(a^2b + b^2c + c^2a) + (ab^2 + bc^2 + ca^2) \geq 2\sqrt{(ab+bc+ca)(a^2b^2 + b^2c^2 + c^2a^2)}.$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} (a^2b + b^2c + c^2a) + (ab^2 + bc^2 + ca^2) + (a^3 + b^3 + c^3) &= (a^2 + b^2 + c^2)(a + b + c) \\ &= \sqrt{[a^4 + b^4 + c^4 + 2(a^2b^2 + b^2c^2 + c^2a^2)] [a^2 + b^2 + c^2 + 2(ab + bc + ca)]} \\ &\geq \sqrt{(a^4 + b^4 + c^4)(a^2 + b^2 + c^2)} + 2\sqrt{(a^2b^2 + b^2c^2 + c^2a^2)(ab + bc + ca)}. \end{aligned}$$

Thus, it suffices to show that

$$\sqrt{(a^4 + b^4 + c^4)(a^2 + b^2 + c^2)} \geq a^3 + b^3 + c^3,$$

which follows also from the Cauchy-Schwarz inequality.

□

**P 2.91.** If  $a, b, c$  are real numbers such that

$$a + b + c = 3,$$

then

$$\frac{(a-1)(a-25)}{a^2+23} + \frac{(b-1)(b-25)}{b^2+23} + \frac{(c-1)(c-25)}{c^2+23} \geq 0.$$

**Solution.** Denote

$$p = a + b + c, \quad q = ab + bc + ca$$

and write the inequality in the homogeneous form  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = \sum (3a - p)(3a - 25p)(9b^2 + 23p^2)(9c^2 + 23p^2).$$

Since the highest coefficient of  $f_6$  is positive, namely

$$A = 3 \cdot 9^3,$$

we use the *highest coefficient cancellation method*. Thus, we will prove that there exist two real numbers  $B$  and  $C$  such that  $g_6(a, b, c) \geq 0$ , where

$$g_6(a, b, c) = f_6(a, b, c) - A[abc + B(a + b + c)^3 + C(a + b + c)(ab + bc + ca)]^2.$$

Since  $g_6$  has the highest coefficient equal to zero, it suffices to show that  $g_6(a, 1, 1) \geq 0$  for all real  $a$  (see P 2.75). Notice that

$$f_6(a, 1, 1) = 12(a-1)^2(7a+11)^2[23(a+2)^2+9]$$

and

$$g_6(a, 1, 1) = f_6(a, 1, 1) - 3 \cdot 9^3[a + B(a+2)^3 + C(a+2)(2a+1)]^2.$$

Let us denote  $g(a) = g_6(a, 1, 1)$ . Since  $g(-2) = 0$ , we can have  $g(a) \geq 0$  in the vicinity of  $a = -2$  only if  $g'(-2) = 0$ ; this involves  $C = -13/9$ , hence

$$\begin{aligned} g_6(a, 1, 1) = & 12(a-1)^2(7a+11)^2[23(a+2)^2+9] \\ & - 27[9a + 9B(a+2)^3 - 13(a+2)(2a+1)]^2. \end{aligned}$$

There are two cases to consider:  $5p^2 + q \leq 0$  and  $5p^2 + q \geq 0$ .

*Case 1:*  $5p^2 + q \leq 0$ . By Remark 3 from the proof of P 2.75, we only need to show that there exist a real number  $B$  such that  $g_6(a, 1, 1) \geq 0$  for all real  $a$  satisfying  $5(a+2)^2 + 2a + 1 \leq 0$ ; that is, for  $a \in [-3, -7/5]$ . From  $g_6(-11/7, 1, 1) = 0$ , we get  $B = 28/9$ , then

$$\begin{aligned} g_6(a, 1, 1) = & 12(a-1)^2(7a+11)^2[23(a+2)^2+9] \\ & - 108(7a+11)^2(2a^2+7a+9)^2 \\ = & -12(a+2)^2(7a+11)^2(13a^2+154a+157) \geq 0. \end{aligned}$$

*Case 2:*  $5p^2 + q \geq 0$ . By Remark 3 from the proof of P 2.75, we only need to show that there exist a real number  $B$  such that  $g_6(a, 1, 1) \geq 0$  for all real  $a$  satisfying  $5(a+2)^2 + 2a + 1 \geq 0$ ; that is, for  $a \in (-\infty, -3] \cup [-7/5, \infty)$ . From  $g_6(1, 1, 1) = 0$ , we get  $B = 4/9$ , then

$$\begin{aligned} g_6(a, 1, 1) &= 12(a-1)^2(7a+11)^2[23(a+2)^2+9] - 108(a-1)^4(2a+3)^2 \\ &= 12(a+2)^2(a-1)^2(1091a^2+3650a+3035) \geq 0. \end{aligned}$$

The proof is completed. The equality holds for  $a = b = c = 1$ , and for  $a = -11$  and  $b = c = 7$  (or any cyclic permutation). □

**P 2.92.** If  $a, b, c$  are real numbers such that  $abc \neq 0$ , then

$$\left(\frac{b+c}{a}\right)^2 + \left(\frac{c+a}{b}\right)^2 + \left(\frac{a+b}{c}\right)^2 > 2.$$

(Michael Rozenberg, 2014)

**Solution.** Assume that

$$a^2 = \min\{a^2, b^2, c^2\}.$$

By the Cauchy-Schwarz inequality, we have

$$\left(\frac{c+a}{b}\right)^2 + \left(\frac{a+b}{c}\right)^2 \geq \frac{[(c+a) + (-a-b)]^2}{b^2 + c^2} = \frac{(b-c)^2}{b^2 + c^2}.$$

On the other hand,

$$\left(\frac{b+c}{a}\right)^2 \geq \frac{(b+c)^2}{b^2 + c^2}.$$

Therefore,

$$\left(\frac{b+c}{a}\right)^2 + \left(\frac{c+a}{b}\right)^2 + \left(\frac{a+b}{c}\right)^2 \geq \frac{(b+c)^2}{b^2 + c^2} + \frac{(b-c)^2}{b^2 + c^2} = 2.$$

The equality holds if and only if

$$\left(\frac{c+a}{b}\right)^2 + \left(\frac{a+b}{c}\right)^2 = \frac{(b-c)^2}{b^2 + c^2}$$

and  $b+c=0$ . Since these relations involves  $a=0$ , we conclude that the inequality is strict (the equality does not hold). □

**P 2.93.** If  $a, b, c$  are real numbers, then

$$(a) \quad (a^2 + 1)(b^2 + 1)(c^2 + 1) \geq \frac{8}{3\sqrt{3}} |(a - b)(b - c)(c - a)|;$$

$$(b) \quad (a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1) \geq |(a - b)(b - c)(c - a)|.$$

(Kwon Ji Mun, 2011)

**Solution.** (a) **First Solution.** Without loss of generality, assume that  $a \leq b \leq c$ , when

$$|(a - b)(b - c)(c - a)| = (a - b)(b - c)(c - a).$$

Denote

$$k = \frac{4}{3\sqrt{3}}$$

and write the inequality as

$$Aa^2 + 2Ba + C \geq 0,$$

where

$$A = (b^2 + 1)(c^2 + 1) + 2k(b - c),$$

$$B = -k(b^2 - c^2),$$

$$C = (b^2 + 1)(c^2 + 1) + 2kbc(b - c).$$

Substituting  $b = \frac{-x}{\sqrt{3}}$  and  $c = \frac{y}{\sqrt{3}}$ , by the Cauchy-Schwarz inequality, we get

$$\begin{aligned} 9A &= (x^2 + 1 + 2)(1 + y^2 + 2) - 8(x + y) \\ &\geq (x + y + 2)^2 - 8(x + y) = (x + y - 2)^2 \geq 0. \end{aligned}$$

We have  $A = 0$  only for  $b = -1/\sqrt{3}$  and  $c = 1/\sqrt{3}$ , when

$$Aa^2 + 2Ba + C = 2Ba + C = 64/27.$$

Otherwise, for  $A > 0$ , it suffices to prove that  $AC - B^2 \geq 0$ . Let us denote

$$E = b - c, \quad F = bc + 1.$$

Since

$$(b^2 + 1)(c^2 + 1) = (b - c)^2 + (bc + 1)^2 = E^2 + F^2,$$

$$(b + c)^2 = (b - c)^2 + 4(bc + 1) - 4 = E^2 + 4F - 4,$$

$$(b^2 - c^2)^2 = (b - c)^2(b + c)^2 = E^2(E^2 + 4F - 4),$$

we have

$$A = E^2 + F^2 + 2kE, \quad B^2 = k^2E^2(E^2 + 4F - 4), \quad C = E^2 + F^2 + 2kE(F - 1),$$



and hence

$$\begin{aligned} AC - B^2 &= (E^2 + F^2 + 2kE)(E^2 + F^2 + 2kEF - 2kE) - k^2E^2(E^2 + 4F - 4) \\ &= (E^2 + F^2)(E^2 + F^2 + 2kEF) - k^2E^4 = \frac{1}{27}(E + \sqrt{3}F)^2(11E^2 - 2\sqrt{3}EF + 9F^2) \geq 0. \end{aligned}$$

The equality holds for

$$b - c + \sqrt{3}(bc + 1) = 0, \quad a + \frac{b + c}{1 + 3bc} = 0$$

(or any permutation).

**Second Solution** (by *Vo Quoc Ba Can*). Substituting

$$a = x\sqrt{3}, \quad b = y\sqrt{3}, \quad c = z\sqrt{3},$$

the inequality becomes

$$(3x^2 + 1)(3y^2 + 1)(3z^2 + 1) \geq 8|(x - y)(y - z)(z - x)|.$$

It suffices to show that

$$E^2 \geq 64(x - y)^2(y - z)^2(z - x)^2,$$

where

$$E = (3x^2 + 1)(3y^2 + 1)(3z^2 + 1) = 27x^2y^2z^2 + 9 \sum x^2y^2 + 3 \sum x^2 + 1.$$

It is easy to check that the equality holds for  $x = -1$ ,  $y = 0$  and  $z = 1$  (or any cyclic permutation), when

$$x + y + z = 0, \quad xy + yz + zx = -1, \quad xyz = 0.$$

From

$$\left(9xyz + \sum x\right)^2 \geq 0$$

and

$$\left(\sum xy + 1\right)^2 \geq 0,$$

we get

$$81x^2y^2z^2 \geq -18xyz \sum x - \sum x^2 - 2 \sum xy$$

and

$$1 \geq - \sum x^2y^2 - 2xyx \sum x - 2 \sum xy,$$

respectively. Therefore,

$$\begin{aligned} 3E &\geq \left(-18xyz \sum x - \sum x^2 - 2 \sum xy\right) + 27 \sum x^2y^2 + 9 \sum x^2 \\ &\quad + 3 \left(- \sum x^2y^2 - 2xyx \sum x - 2 \sum xy\right) \\ &= 24 \left(\sum x^2y^2 - xyz \sum x\right) + 8 \left(\sum x^2 - \sum xy\right) \\ &= 12 \sum x^2(y - z)^2 + \frac{4}{3} \sum (2x - y - z)^2. \end{aligned}$$

By the AM-GM inequality, we have

$$3E \geq 8\sqrt{\left[\sum x^2(y-z)^2\right] \left[\sum (2x-y-z)^2\right]}.$$

In addition, by the Cauchy-Schwarz inequality, we get

$$\begin{aligned} E^2 &\geq \frac{64}{9} \left[\sum x(y-z)(2x-y-z)\right]^2 \\ &= 64 \left(\sum x^2y - \sum xy^2\right)^2 \\ &= 64(x-y)^2(y-z)^2(z-x)^2. \end{aligned}$$

(b) Write the inequality as

$$\left[\left(a - \frac{1}{2}\right)^2 + \frac{3}{4}\right] \left[\left(b - \frac{1}{2}\right)^2 + \frac{3}{4}\right] \left[\left(c - \frac{1}{2}\right)^2 + \frac{3}{4}\right] \geq (a-b)(b-c)(c-a).$$

Using the substitution

$$a - \frac{1}{2} = \frac{\sqrt{3}}{2}x, \quad b - \frac{1}{2} = \frac{\sqrt{3}}{2}y, \quad c - \frac{1}{2} = \frac{\sqrt{3}}{2}z,$$

the inequality turns into the inequality in (a). From the equality conditions in (a), namely

$$y - z + \sqrt{3}(yz + 1) = 0, \quad x + \frac{y+z}{1+3yz} = 0,$$

we get the following equality conditions

$$b = \frac{c-1}{c}, \quad a = \frac{1}{1-c}$$

(or any permutation).

□

**P 2.94.** If  $a, b, c$  are real numbers such that

$$a + b + c = 3,$$

then

$$(1 - a + a^2)(1 - b + b^2)(1 - c + c^2) \geq 1.$$

**First Solution.** Since

$$2(1 - a + a^2)(1 - b + b^2) - (a + b - 1)^2 - 1 = a^2(b - 1)^2 + b^2(a - 1)^2 \geq 0$$

and

$$2(1 - c + c^2) \geq 1 + c^2,$$

it is enough to show that

$$[(a + b - 1)^2 + 1](1 + c^2) \geq 4.$$

By the Cauchy-Schwartz inequality, we have

$$[(a + b - 1)^2 + 1](1 + c^2) \geq [(a + b - 1) + c]^2 = (a + b + c - 1)^2 = 4.$$

The equality holds for  $a = b = c = 1$ .

**Second Solution** (by *Marian Tetiva*). Assume that

$$a \leq b \leq c.$$

There are two cases to consider:  $a \geq 0$  and  $a < 0$ .

*Case 1:*  $a \geq 0$ . Among the numbers  $a, b, c$  always there exist two (let  $b$  and  $c$ ) which are either less than or equal to 1, or larger than or equal to 1. Then,

$$bc(b - 1)(c - 1) \geq 0$$

and

$$\begin{aligned} (1 - b + b^2)(1 - c + c^2) &= 1 + (b^2 - b) + (c^2 - c) + (b^2 - b)(c^2 - c) \\ &\geq 1 + (b^2 - b) + (c^2 - c) \geq 1 - (b + c) + \frac{1}{2}(b + c)^2 \\ &= 1 - (3 - a) + \frac{1}{2}(3 - a)^2 = \frac{1}{2}(5 - 4a + a^2). \end{aligned}$$

Therefore, it suffices to show that

$$(1 - a + a^2)(5 - 4a + a^2) \geq 2.$$

Indeed,

$$(1 - a + a^2)(5 - 4a + a^2) - 2 = (a - 1)^2(a^2 - 3a + 3) \geq 0.$$

*Case 2:*  $a < 0$ . We have

$$c > \frac{3}{2}$$

because

$$c \geq \frac{b + c}{2} > \frac{a + b + c}{2} = \frac{3}{2}.$$

The desired inequality is true since

$$1 - a + a^2 > 1,$$

$$1 - b + b^2 = \left(\frac{1}{2} - b\right)^2 + \frac{3}{4} \geq \frac{3}{4},$$

$$1 - c + c^2 > 1 - c + \frac{3c}{2} = 1 + \frac{c}{2} > 1 + \frac{3}{4} = \frac{7}{4}.$$

□

**P 2.95.** If  $a, b, c$  are real numbers such that

$$a + b + c = 0,$$

then

$$\frac{a(a-4)}{a^2+2} + \frac{b(b-4)}{b^2+2} + \frac{c(c-4)}{c^2+2} \geq 0.$$

**Solution.** Write the inequality as follows

$$\sum \left[ \frac{a(a-4)}{a^2+2} + 1 \right] \geq 3,$$

$$\sum \frac{(a-1)^2}{a^2+2} \geq \frac{3}{2}.$$

From

$$a^2 = (b+c)^2 \leq 2(b^2+c^2),$$

we get

$$3a^2 \leq 2(a^2+b^2+c^2).$$

Similarly,

$$3b^2 \leq 2(a^2+b^2+c^2), \quad 3c^2 \leq 2(a^2+b^2+c^2).$$

Therefore, we have

$$\begin{aligned} \sum \frac{(a-1)^2}{a^2+2} &= \sum \frac{3(a-1)^2}{3a^2+6} \geq \sum \frac{3(a-1)^2}{2(a^2+b^2+c^2)+6} \\ &= \frac{3}{2(a^2+b^2+c^2+3)} \sum (a-1)^2 = \frac{3}{2}. \end{aligned}$$

Thus, the proof is completed. The equality holds for  $a = b = c = 0$ , and also for  $a = -2$  and  $b = c = 1$  (or any cyclic permutation).

□

**P 2.96.** If  $a, b, c$  are real numbers such that

$$a, b, c \leq 1 + \sqrt{2}, \quad a + b + c \geq 0,$$

then

$$2abc + a^2 + b^2 + c^2 + 1 \geq 2(ab + bc + ca).$$

(Vasile Cîrtoaje, 2014)

**Solution.** Assume that

$$a \leq b \leq c \leq 1 + \sqrt{2}.$$

**First Solution.** There are three cases:  $0 \leq a \leq b \leq c$ ,  $a \leq 0 \leq b \leq c$  and  $a \leq b \leq 0 \leq c$ .

*Case 1:*  $0 \leq a \leq b \leq c$ . Among the numbers  $1 - a$ ,  $1 - b$  and  $1 - c$  there are two which have the same sign; let

$$(1 - b)(1 - c) \geq 0.$$

From

$$\begin{aligned} 2abc + a^2 + b^2 + c^2 + 1 - 2(ab + bc + ca) &= \\ &= (a - 1)^2 + (b - c)^2 + 2a + 2abc - 2a(b + c) \\ &= (a - 1)^2 + (b - c)^2 + 2a(1 - b)(1 - c) \geq 0, \end{aligned}$$

the conclusion follows.

*Case 2:*  $a \leq 0 \leq b \leq c$ . Denote

$$x = \frac{b + c}{2}, \quad 0 \leq x \leq 1 + \sqrt{2},$$

and write the inequality as

$$(a + 1)^2 + (b - c)^2 - 2a(1 + b + c - bc) \geq 0.$$

Clearly, it suffices to show that

$$1 + b + c - bc \geq 0.$$

Indeed,

$$1 + b + c - bc \geq 1 + 2x - x^2 = (\sqrt{2} - 1 + x)(\sqrt{2} + 1 - x) \geq 0.$$

*Case 3:*  $a \leq b \leq 0 \leq c$ . Write the inequality in the form

$$2abc + (a - b)^2 + c^2 + 1 - 2(a + b)c \geq 0,$$

which is clearly true.

The equality holds for  $a = b = c = 1$ , and also for  $a = -1$  and  $b = c = 1 + \sqrt{2}$  (or any cyclic permutation).

**Second Solution.** According to P 2.53, for fixed  $a + b + c$  and  $ab + bc + ca$ , the product  $abc$  has the minimum value when  $a \leq b = c$ . Therefore, we only need to prove that

$$2ab^2 + a^2 + 1 \geq 4ab$$

for  $a \leq b \leq 1 + \sqrt{2}$  and  $a + 2b \geq 0$ . Write the inequality as

$$(a - 1)^2 + 2a(b - 1)^2 \geq 0$$

or

$$(a + 1)^2 - 2a(1 + 2b - b^2) \geq 0.$$

For  $a \geq 0$ , the inequality is clearly true. For  $a \leq 0$ , it is enough to show that  $1 + 2b - b^2 \geq 0$ . This is true because  $2b \geq -a \geq 0$  and  $b \leq 1 + \sqrt{2}$ .

**Remark.** Actually, the original inequality holds for the extended conditions

$$a, b, c \leq 1 + \sqrt{2}, \quad a + b + c + 2\sqrt{2} - 1 \geq 0,$$

with equality for  $a = b = c = 1$ , for  $a = -1$  and  $b = c = 1 + \sqrt{2}$  (or any cyclic permutation), and for  $a = -1$  and  $b = c = 1 - \sqrt{2}$  (or any cyclic permutation). □

**P 2.97.** If  $a, b, c$  are real numbers such that  $a + b + c = 2$  and  $ab + bc + ca > 0$ , then

$$(a^2 + bc)(b^2 + ca)(c^2 + ab) + abc \leq 1.$$

(Nguyen Van Huyen, 2020)

**Solution.** Write the inequality in the homogeneous form  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = p^6 - 8abcp^3 - 64P_2(a, b, c),$$

with

$$p = a + b + c, \quad P_2(a, b, c) = (a^2 + bc)(b^2 + ca)(c^2 + ab).$$

According to Remark 2 from P 2.75,  $f_6(a, b, c)$  has the highest coefficient

$$A = -64P_2(1, 1, 1) = -512.$$

Thus, according to P 2.75 and its Remark 3, it is enough to show that  $f_6(a, 1, 1) \geq 0$  for  $2a + 1 \geq 0$ . We have

$$\begin{aligned} f_6(a, 1, 1) &= (a + 2)^6 - 8a(a + 2)^3 - 64(a^2 + 1)(a + 1)^2 \\ &= a^2(a^4 + 12a^3 - 12a^2 - 16a + 16) \\ &\geq a^2(a^4 + 12a^3 - 12a^2 - 16a + 15) = a^2(a - 1)^2(a^2 + 14a + 15) \geq 0. \end{aligned}$$

The equality occurs for  $a = 0$  and  $b = c = 1$  (or any cyclic permutation). □

**P 2.98.** If  $a, b, c$  are real numbers such that

$$a^2 + b^2 + c^2 = 3, \quad a \geq \frac{4}{3},$$

then

$$3(abc + 1) \geq 2(ab + bc + ca).$$

(Vasile Cîrtoaje, 2019)

**Solution.** Since

$$2bc = (b + c)^2 + a^2 - 3,$$

we write the inequality as follows:

$$\begin{aligned} (3a - 2)(2bc) - 4a(b + c) + 6 &\geq 0, \\ (3a - 2)(b + c)^2 - 4a(b + c) + 3a^3 - 2a^2 - 9a + 12 &\geq 0, \\ (3a - 2) \left( b + c - \frac{2a}{3a - 2} \right)^2 + \frac{3(3a^4 - 4a^3 - 9a^2 + 18a - 8)}{3a - 2} &\geq 0, \\ (3a - 2) \left( b + c - \frac{2a}{3a - 2} \right)^2 + \frac{3(a - 1)^2(3a - 4)(a + 2)}{3a - 2} &\geq 0. \end{aligned}$$

The equality holds for  $a = b = c = 1$ , and also for  $a = \frac{4}{3}$  and  $b, c = \frac{4 \pm \sqrt{6}}{6}$ .

**Remark.** The inequality is equivalent to

$$3 \left( a - \frac{4}{3} \right) \left( b - \frac{4}{3} \right) \left( c - \frac{4}{3} \right) + \left( a + b + c - \frac{8}{3} \right)^2 \geq 0.$$

□

**P 2.99.** If  $a, b, c$  are real numbers such that  $a \geq \frac{8}{7}$  and  $a^2 + b^2 + c^2 = 3$ , then

$$\frac{3 - a - b - c}{1 - abc} \geq \frac{49}{100}.$$

(Vasile Cîrtoaje, 2018)

**Solution.** From

$$3 = a^2 + b^2 + c^2 > 3\sqrt[3]{a^2b^2c^2},$$

it follows  $abc < 1$ . In addition, since

$$2bc = (b + c)^2 - b^2 - c^2 = (b + c)^2 + a^2 - 3,$$

the inequality can be written as follows:

$$\begin{aligned} 49(abc - 1) &\geq 100(a + b + c - 3), \\ 49a[(b + c)^2 + a^2 - 3] - 98 &\geq 200(a + b + c - 3), \\ 49a(b + c)^2 - 200(b + c) + 49a^3 - 347a + 502 &\geq 0, \\ 49a \left( b + c - \frac{100}{49a} \right)^2 + \frac{E}{49} &\geq 0, \end{aligned}$$

where

$$E = (7a)^4 - 347(7a)^2 + 3514(7a) - 10000.$$

Denoting  $x = 7a$ ,  $x \geq 8$ , we have

$$\begin{aligned} E &= x^4 - 347x^2 + 3514x - 10000 = (x - 8)(x^3 + 8x^2 - 283x + 1250) \\ &\geq (x - 8)(x^3 + 8x^2 - 283x + 1240) = (x - 8)^2(x^2 + 16x - 155) \\ &\geq (x - 8)^2(x^2 + 16x - 161) = (x - 8)^2(x - 7)(x + 21) \geq 0. \end{aligned}$$

The equality occurs for  $a = \frac{8}{7}$  and

$$b + c = \frac{100}{49a} = \frac{25}{14}, \quad 2bc = (b + c)^2 + a^2 - 3 = \frac{293}{196},$$

therefore for  $a = \frac{8}{7}$  and  $b, c = \frac{25 \pm \sqrt{39}}{28}$ .

□

**P 2.100.** If  $a, b, c \in [-1, 1]$ , then

$$a^3 + b^3 + c^3 + abc \leq \frac{15}{16}(a + b + c) + \frac{19}{16}.$$

(Vasile Cîrtoaje, 2018)

**Solution.** Since

$$a^3 + b^3 + c^3 = 3abc + \frac{1}{2}(a + b + c)[3(a^2 + b^2 + c^2) - (a + b + c)^2],$$

we may apply the following statement (see Remark 2 from P 2.53): *If  $a, b, c \in [-1, 1]$ , then for fixed  $a + b + c$  and  $a^2 + b^2 + c^2$ , the product  $abc$  has the maximum value when  $a \geq b = c$  or  $a = 1$ .* Thus, we only need to consider these cases.

*Case 1:*  $a = 1$ . We need to show that

$$16(b^3 + c^3 + bc) - 15(b + c) + 18 \leq 0.$$

Denote

$$s = b + c, \quad s \leq 2,$$

and write the inequality as

$$16s^3 - 16bc(3s - 1) - 15s - 18 \leq 0.$$

We have  $4bc \leq s^2$  and

$$(b - 1)(c - 1) \geq 0, \quad bc \geq s - 1.$$



For  $3s - 1 \leq 0$ , it is enough to show that

$$16s^3 - 4s^2(3s - 1) - 15s - 18 \leq 0,$$

which is equivalent to

$$\begin{aligned} 4s^3 + 4s^2 - 15s - 18 &\leq 0, \\ (s - 2)(2s + 3)^2 &\leq 0. \end{aligned}$$

For  $3s - 1 \geq 0$ , it is enough to show that

$$16s^3 - 16(s - 1)(3s - 1) - 15s - 18 \leq 0,$$

which is equivalent to

$$\begin{aligned} 16s^3 - 48s^2 + 49s - 34 &\leq 0, \\ (s - 2)(16s^2 - 16s + 17) &\leq 0, \\ (s - 2)[8(s - 1)^2 + 8s^2 + 9] &\leq 0. \end{aligned}$$

*Case 2:*  $a \geq b = c$ . We need to prove that

$$16(a^3 + 2b^2 + ab^2) \leq 15(a + 2b) + 19.$$

There are two sub-cases:  $a \geq 0$  and  $a \leq 0$ .

If  $a \geq 0$ , write the inequality as

$$a(16a^2 + 32b^2 - 15) + 16b^3 - 30b - 19 \leq 0.$$

Since

$$a(16a^2 + 32b^2 - 15) \leq a(16 + 32b^2 - 15) = a(1 + 32b^2) \leq 1 + 32b^2,$$

it is enough to show that

$$16b^3 + 32b^2 - 30b - 18 \leq 0,$$

which is equivalent to

$$2(b - 1)(4b + 3)^2 \leq 0.$$

If  $a \leq 0$ , hence  $b \leq a \leq 0$ , substituting  $a = -x$  and  $b = -y$  ( $y \geq x \geq 0$ ), the required inequality becomes

$$16(x^3 + 2y^3 + xy^2) + 19 \geq 15(x + 2y).$$

Since  $xy^2 \geq 0$ , it suffices to show that

$$(16x^3 - 15x + 6) + 2(16y^3 - 15y + 6) + 1 \geq 0.$$

This is true because, by the AM-GM inequality, we have

$$16x^3 - 15x + 6 = (16x^3 + 3 + 3) - 15x \geq 3(2\sqrt[3]{18} - 5)x \geq 0$$

and, similarly,

$$16y^3 - 15y + 6 > 0.$$

The equality occurs for  $a = b = c = 1$ , and also for  $a = 1$  and  $b = c = \frac{-3}{4}$  (or any cyclic permutation).

□

**P 2.101.** If  $a, b, c$  are real numbers, then

$$(a^3 + b^3 + c^3)^2 \geq (a^4 + b^4 + c^4)(ab + bc + ca).$$

(Vasile Cîrtoaje, 2018)

**Solution.** Since

$$(a^3 + b^3 + c^3)^2 = (a^4 + b^4 + c^4)(a^2 + b^2 + c^2) - \sum b^2 c^2 (b - c)^2,$$

the inequality can be rewritten as follows

$$(a^4 + b^4 + c^4)(a^2 + b^2 + c^2 - ab - bc - ca) \geq \sum b^2 c^2 (b - c)^2,$$

$$(a^4 + b^4 + c^4) \sum (b - c)^2 \geq 2 \sum b^2 c^2 (b - c)^2,$$

$$\sum S_a (b - c)^2 \geq 0,$$

where

$$S_a = a^4 + (b^2 - c^2)^2.$$

The equality occurs for  $a = b = c$ .

□

**P 2.102.** Let  $a_1, a_2, \dots, a_n$  be real numbers such that

$$a_1^2 + a_2^2 + \dots + a_n^2 = n.$$

Prove that:

(a) for  $n = 3$ ,

$$\frac{a_1 + a_2 + a_3}{3} + \min_{i \neq j} (a_i - a_j)^2 \leq \frac{5}{3};$$

(b) for  $n = 5$ ,

$$\frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} + \min_{i \neq j} (a_i - a_j)^2 \leq 1.$$

(Vasile Cîrtoaje, 2019)

**Solution.** (a) We need to show that

$$s + \min_{i \neq j} (a_i - a_j)^2 \leq \frac{5}{3},$$

where

$$s = \frac{a_1 + a_2 + a_3}{3}.$$

According to Lemma below, we have

$$\min_{i \neq j} (a_i - a_j)^2 \leq \frac{3(1 - s^2)}{2}.$$

Thus, we only need to show that

$$s + \frac{3(1 - s^2)}{2} \leq \frac{5}{3},$$

which is equivalent to

$$(3s - 1)^2 \geq 0.$$

For  $a_1 \geq a_2 \geq a_3$ , the equality holds when  $a_1 - a_2 = a_2 - a_3$  and  $a_1 + a_2 + a_3 = 1$ , i.e. when

$$a_1 = \frac{1 + 2\sqrt{3}}{3}, \quad a_2 = \frac{1}{3}, \quad a_3 = \frac{1 - 2\sqrt{3}}{3}.$$

(b) We need to show that

$$s + \min_{i \neq j} (a_i - a_j)^2 \leq 1,$$

where

$$s = \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5}.$$

According to Lemma below, we have

$$\min_{i \neq j} (a_i - a_j)^2 \leq \frac{3(1 - s^2)}{2}.$$

Thus, we only need to show that

$$s + \frac{1 - s^2}{2} \leq 1,$$

which is equivalent to

$$(s - 1)^2 \geq 0.$$

For  $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5$ , the equality holds when  $a_1 - a_2 = a_2 - a_3 = a_3 - a_4 = a_4 - a_5$  and  $a_1 + a_2 + a_3 + a_4 + a_5 = 5$ , i.e. when  $a_1 = a_2 = a_3 = a_4 = a_5 = 1$ .

**Lemma.** If  $a_1, a_2, \dots, a_n$  are real numbers satisfying

$$a_1^2 + a_2^2 + \dots + a_n^2 = n, \quad a_1 + a_2 + \dots + a_n = ns, \quad s \in [-1, 1],$$

then

$$\min_{i \neq j} (a_i - a_j)^2 \leq \frac{12(1 - s^2)}{n^2 - 1}.$$

*Proof.* Assume that

$$a_1 \geq a_2 \geq \dots \geq a_n,$$

and denote

$$x = \min_{i \neq j} (a_i - a_j)^2.$$

We have

$$\begin{aligned} n^2(1 - s^2) &= n(a_1^2 + a_2^2 + \cdots + a_n^2) - (a_1 + a_2 + \cdots + a_n)^2 = \sum_{i < j} (a_i - a_j)^2 \\ &= \sum_{i=1}^{n-1} (a_i - a_{i+1})^2 + \sum_{i=1}^{n-2} (a_i - a_{i+2})^2 + \sum_{i=1}^{n-3} (a_i - a_{i+3})^2 + \cdots + \sum_{i=1}^1 (a_i - a_n)^2 \\ &\geq (n-1)x + 2^2(n-2)x + 3^2(n-3)x + \cdots + (n-1)^2 \cdot [n - (n-1)]x \\ &= n[1^2 + 2^2 + 3^2 + \cdots + (n-1)^2]x - [1^3 + 2^3 + 3^3 + \cdots + (n-1)^3]x \\ &= \frac{n^2(n-1)(2n-1)x}{6} - \frac{n^2(n-1)^2x}{4} = \frac{n^2(n^2-1)x}{12}, \end{aligned}$$

hence

$$x \leq \frac{12(1 - s^2)}{n^2 - 1}.$$

For  $a_1 \geq a_2 \geq \cdots \geq a_n$ , the equality occurs when

$$a_1 - a_2 = a_2 - a_3 = \cdots = a_{n-1} - a_n.$$

**Remark.** The inequality (b) can be generalized as follows:

$$1 - \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \frac{n^2 - 1}{24} \min_{i \neq j} (a_i - a_j)^2.$$

□

**P 2.103.** Let  $a_1, a_2, \dots, a_7$  be real numbers such that

$$a_1^2 + a_2^2 + \cdots + a_7^2 = n.$$

Prove that:

$$(a) \quad \sqrt{\frac{|a_1 + a_2 + \cdots + a_7|}{7}} + \min_{i \neq j} (a_i - a_j)^2 \leq 1;$$

$$(b) \quad \sqrt{\frac{|a_1 + a_2 + \cdots + a_7|}{7}} + 8 \min_{i \neq j} (a_i - a_j)^2 \leq \frac{19}{8}.$$

(Vasile Cîrtoaje, 2019)

**Solution.** Let

$$s = \frac{a_1 + a_2 + \cdots + a_7}{7}, \quad |s| \leq 1.$$

(a) We need to show that

$$\sqrt{|s|} + \min_{i \neq j} (a_i - a_j)^2 \leq 1.$$

According to Lemma from the proof of the preceding P 2.102, we have

$$\min_{i \neq j} (a_i - a_j)^2 \leq \frac{1 - s^2}{4}.$$

Thus, we only need to show that

$$\sqrt{|s|} + \frac{1 - s^2}{4} \leq 1,$$

which is equivalent to

$$(1 - |s|)(1 + |s|) \leq 4 \left(1 - \sqrt{|s|}\right).$$

This is true if

$$\left(1 + \sqrt{|s|}\right) (1 + |s|) \leq 4,$$

which is obvious for  $|s| \leq 1$ . The equality holds for  $a_1 = a_2 = \cdots = a_7 = \pm 1$ .

(b) We need to show that

$$\sqrt{|s|} + 8 \min_{i \neq j} (a_i - a_j)^2 \leq \frac{19}{8}.$$

As shown at (a), we have

$$\min_{i \neq j} (a_i - a_j)^2 \leq \frac{1 - s^2}{4}.$$

Thus, we only need to show that

$$\sqrt{|s|} + 2(1 - |s|^2) \leq \frac{19}{8},$$

which is equivalent to

$$16|s|^2 + 3 \geq 8\sqrt{|s|}.$$

This follows immediately from the AM-GM inequality. For  $a_1 \geq a_2 \geq \cdots \geq a_7$ , the equality holds when  $a_1 - a_2 = a_2 - a_3 = \cdots = a_6 - a_7$  and  $a_1 + a_2 + \cdots + a_7 = \frac{\pm 7}{4}$ , i.e. when

$$a_k = \frac{2 + (4 - k)\sqrt{15}}{8}, \text{ and also when } a_k = \frac{-2 + (4 - k)\sqrt{15}}{8}.$$

□

**P 2.104.** Let  $f$  be a differentiable convex function on a closed interval  $\mathbb{I} = [a, b]$ . If  $a_1, a_2, \dots, a_n \in \mathbb{I}$ , then Jensen's difference

$$D = f(a_1) + f(a_2) + \dots + f(a_n) - nf\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)$$

has the maximum value when all  $a_i \in \{a, b\}$ .

(Vasile Cîrtoaje, 1990)

**Solution.** For fixed  $a_2, a_3, \dots, a_n$ , let

$$D(a_1) = f(a_1) + f(a_2) + \dots + f(a_n) - nf\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right).$$

Since  $f'$  is increasing, from

$$D'(a_1) = f'(a_1) - f'\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right),$$

it follows that  $D'(a_1) \leq 0$  for  $a_1 \in [a, c]$  and  $D'(a_1) \geq 0$  for  $a_1 \in [c, b]$ , where

$$c = \frac{a_2 + \dots + a_n}{n - 1}.$$

As a consequence,  $D(a_1)$  is decreasing on  $[a, c]$  and increasing on  $[c, b]$ , hence  $D(a_1)$  has the maximum value for  $a_1 = a$  or  $a_1 = b$ . Similarly, Jensen's difference  $D$  has the maximum value for  $a_i \in \{a, b\}$ ,  $i = 2, 3, \dots, n$ .

**Remark.** The following statement is also valid:

• Let  $f$  be a differentiable convex function on an interval  $\mathbb{I}$ , and let  $a_1, a_2, \dots, a_n \in \mathbb{I}$  such that  $a_1 \geq a_2 \geq \dots \geq a_n$ . For fixed  $a_1$  and  $a_n$ , Jensen's difference

$$D = f(a_1) + f(a_2) + \dots + f(a_n) - nf\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)$$

has the maximum value when  $a_i \in \{a_1, a_n\}$  for  $i = 2, 3, \dots, n - 1$ .

The proof is similar with the above one.

□

**P 2.105.** If  $a, b, c$  are real numbers, then

$$2(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1) \geq (abc - 1)^2.$$

(Vasile Cîrtoaje, 2018)

**First Solution.** If one of  $a, b, c$  is zero, the inequality is true. Thus, if  $a = 0$ , the inequality becomes

$$2(b^2 - b + 1)(c^2 - c + 1) \geq 1.$$

Indeed, we have

$$2(b^2 - b + 1)(c^2 - c + 1) \geq 2 \cdot \frac{3}{4} \cdot \frac{3}{4} > 1.$$

Assume now that  $abc \neq 0$ , denote

$$x = (b^2 - b + 1)(c^2 - c + 1), \quad y = bc,$$

and write the required inequality as  $A \geq 0$ , where

$$A = (2x - y^2)a^2 - 2(x - y)a + 2x - 1.$$

We have

$$2x - y^2 \geq 2 \cdot \frac{3b^2}{4} \cdot \frac{3c^2}{4} - b^2c^2 > 0$$

and

$$A = (2x - y^2) \left( a - \frac{x - y}{2x - y^2} \right)^2 + B,$$

where

$$B = 2x - 1 - \frac{(x - y)^2}{2x - y^2} = \frac{x[3x - 2(y^2 - y + 1)]}{2x - y^2}.$$

Since

$$3x - 2(y^2 - y + 1) \geq 0,$$

with equality for  $b = c = \frac{3 \pm \sqrt{5}}{2}$  (see P 2.5), we have  $B \geq 0$ , hence  $A \geq 0$ .

The equality occurs for  $a = \frac{x - y}{2x - y^2}$  and  $b = c = \frac{3 \pm \sqrt{5}}{2}$  (when  $b^2 - 3b + 1 = 0$ ). Since

$$x = (b^2 - b + 1)^2 = (2b)^2 = 4b^2,$$

$$x - y = 4b^2 - b^2 = 3b^2, \quad 2x - y^2 = 8b^2 - b^4,$$

$$a = \frac{x - y}{2x - y^2} = \frac{3b^2}{8b^2 - b^4} = \frac{3}{8 - b^2} = \frac{3}{8 - (3b - 1)} = \frac{1}{3 - b} = b,$$

the equality holds for

$$a = b = c = \frac{3 \pm \sqrt{5}}{2}.$$

**Second Solution** ( by KaiRain). We get the required inequality by multiplying the inequalities

$$3(b^2 - b + 1)(c^2 - c + 1) \geq 2(y^2 - y + 1),$$

$$4(a^2 - a + 1)(y^2 - y + 1) \geq 3(ay - 1)^2,$$

where  $y = bc$ . The first inequality is treated in P 2.5, while the second inequality is equivalent to

$$(ay - 2a - 2y + 1)^2 \geq 0.$$

□

**P 2.106.** *If  $a, b, c$  are real numbers, then*

$$(1 + \sqrt{2})(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1) \geq a^2b^2c^2 + 1.$$

(Vasile Cîrtoaje, 1992)

**Solution.** For  $a = b = c$ , the inequality becomes

$$(1 + \sqrt{2})(a^2 - a + 1)^3 \geq a^6 + 1.$$

This inequality is equivalent to

$$[a^2 - (1 + \sqrt{2})a + 1]^2[\sqrt{2}a^2 - (\sqrt{2} - 1)a + \sqrt{2}] \geq 0.$$

Based on this result, we have

$$(1 + \sqrt{2})^3(a^2 - a + 1)^3(b^2 - b + 1)^3(c^2 - c + 1)^3 \geq (a^6 + 1)(b^6 + 1)(c^6 + 1).$$

It suffices to show that

$$(a^6 + 1)(b^6 + 1)(c^6 + 1) \geq (a^2b^2c^2 + 1)^3,$$

which is just Hölder's inequality. The equality occurs for

$$a = b = c = \frac{1}{2} \left( 1 + \sqrt{2} \pm \sqrt{2\sqrt{2} - 1} \right).$$

□

**P 2.107.** *If  $a, b, c, d$  are real numbers, then*

$$(a) \quad 4(1 - a + a^2)(1 - b + b^2)(1 - c + c^2) \geq 1 + a^2b^2 + b^2c^2 + c^2a^2;$$

$$(b) \quad (1 - a + a^2)(1 - b + b^2)(1 - c + c^2)(1 - d + d^2) \geq \left( \frac{1 + abcd}{2} \right)^2.$$

(Vasile Cîrtoaje, 1992)



**Solution.** We will use the inequality

$$2(1 - a + a^2)^2 \geq 1 + a^4,$$

which is equivalent to  $(1 - a)^4 \geq 0$ .

(a) We have

$$8(1 - a + a^2)^2(1 - b + b^2)^2(1 - c + c^2)^2 \geq (1 + a^4)(1 + b^4)(1 + c^4).$$

Thus, it suffices to show that

$$2(1 + a^4)(1 + b^4)(1 + c^4) \geq (1 + a^2b^2 + b^2c^2 + c^2a^2)^2,$$

which follows from the Cauchy-Schwarz inequality

$$(1 + a^4 + a^4 + 1)(1 + b^4 + c^4 + b^4c^4) \geq (1 + a^2b^2 + a^2c^2 + b^4c^4)^2.$$

The equality holds for  $a = b = c = 1$ .

(b) We have

$$4(1 - a + a^2)^2(1 - b + b^2)^2 \geq (1 + a^4)(1 + b^4) \geq (1 + a^2b^2)^2,$$

hence

$$2(1 - a + a^2)(1 - b + b^2) \geq 1 + a^2b^2.$$

The desired inequality follows by multiplying the inequalities

$$2(1 - a + a^2)(1 - b + b^2) \geq 1 + a^2b^2,$$

$$2(1 - c + c^2)(1 - d + d^2) \geq 1 + c^2d^2,$$

$$(1 + a^2b^2)(1 + c^2d^2) \geq (1 + abcd)^2.$$

The equality holds for  $a = b = c = d = 1$ .

□

**P 2.108.** If  $a, b, c, d$  are real numbers, then

$$3(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1) \geq a^2b^2c^2d^2 - abcd + 1.$$

**Solution.** For fixed  $b, c, d$ , denote  $m = bcd$  and write the inequality as  $F(a) \geq 1$ , where

$$F(a) = 3f(a)(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1),$$

$$f(a) = \frac{a^2 - a + 1}{m^2a - ma + 1}.$$

If  $abcd = 0$  or  $abc = 1$  or  $bcd = 1$  or  $cda = 1$  or  $dab = 1$ , the inequality is true. Indeed, if  $a = 0$  or  $bcd = 1$ , then  $f(a) = 1$  and

$$F(a) = 3(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1) \geq 3 \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} > 1.$$

Consider now  $abcd \neq 0$ ,  $abc \neq 1$ ,  $bcd \neq 1$ ,  $cda \neq 1$ ,  $dab \neq 1$ . We have the derivative

$$f'(a) = \frac{(m-1)[ma^2 - 2(m+1)a + 1]}{(m^2a^2 - ma + 1)^2},$$

with  $f'(a_1) = f'(a_2) = 0$ , where

$$a_1 = \frac{m+1 - \sqrt{m^2 + m + 1}}{m}, \quad a_2 = \frac{m+1 + \sqrt{m^2 + m + 1}}{m}.$$

For  $a = a_1$  and  $a = a_2$ , we have

$$abcd = \frac{2a-1}{a-2}.$$

*Case 1:*  $m > 1$ .  $f(a)$  is increasing on  $(-\infty, a_1] \cup [a_2, \infty)$  and decreasing on  $[a_1, a_2]$ . Thus, it suffices to consider  $a = -\infty$  and  $a = a_2$ . For  $a = -\infty$ , we have

$$F(-\infty) = \frac{3}{b^2c^2d^2}(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1) \geq \frac{3}{b^2c^2d^2} \cdot \frac{3b^2}{4} \cdot \frac{3c^2}{4} \cdot \frac{3d^2}{4} = \frac{81}{64} > 1.$$

It remains the case  $a = a_2$ .

*Case 2:*  $0 < m < 1$ .  $f(a)$  is increasing on  $[a_1, a_2]$  and decreasing on  $(-\infty, a_1] \cup [a_2, \infty)$ . Thus, it suffices to consider  $a = a_1$  and  $a = \infty$ . For  $a = \infty$ , we have  $F(\infty) > 1$ . It remains the case  $a = a_1$ .

*Case 3:*  $m < 0$ .  $f(a)$  is increasing on  $(-\infty, a_2] \cup [a_1, \infty)$  and decreasing on  $[a_2, a_1]$ . Thus, it suffices to consider  $a = -\infty$  and  $a = a_1$ . For  $a = -\infty$ , we have  $F(-\infty) > 1$ . It remains the case  $a = a_1$ .

Due to symmetry, it suffices to consider the cases  $a \in \{a_1, a_2\}$ ,  $b \in \{b_1, b_2\}$ ,  $c \in \{c_1, c_2\}$  and  $d \in \{d_1, d_2\}$ , when

$$abcd = \frac{2a-1}{a-2} = \frac{2b-1}{b-2} = \frac{2c-1}{c-2} = \frac{2d-1}{d-2}.$$

From these relations, we get  $a = b = c = d = -1$ , or

$$a = b = c = d = \frac{1}{4} \left( 3 + \sqrt{5} \pm \sqrt{6\sqrt{5} - 2} \right)$$

(which follows from  $a + \frac{1}{a} = \frac{3 + \sqrt{5}}{2}$ ). In the first case, we have

$$F(a) = \frac{3(a^2 - a + 1)^4}{a^8 - a^4 + 1} = 243 > 1,$$

and in the second case

$$F(a) = \frac{3(a^2 - a + 1)^4}{a^8 - a^4 + 1} = 1.$$

Therefore, the equality occurs for

$$a = b = c = d = \frac{1}{4} \left( 3 + \sqrt{5} \pm \sqrt{6\sqrt{5} - 2} \right).$$

□

**P 2.109.** If  $a, b, c, d$  are real numbers, then

$$(a^2 - a + 2)(b^2 - b + 2)(c^2 - c + 2)(d^2 - d + 2) \geq (a + b + c + d)^2.$$

(Vasile Cîrtoaje, 1994)

**Solution.** We denote

$$x = a + b - 1, \quad y = c + d - 1,$$

and use the inequality

$$(a^2 - a + 2)(b^2 - b + 2) \geq (a + b - 1)^2 + 3,$$

which is equivalent to

$$(2ab - a - b)^2 + 3(a - b)^2 \geq 0.$$

Thus, it suffices to show that

$$(x^2 + 3)(y^2 + 3) \geq (x + y + 2)^2,$$

which is equivalent to

$$x^2y^2 + 2(x^2 + y^2 - xy) - 4(x + y) + 5 \geq 0,$$

$$(xy - 1)^2 + 2(x^2 + y^2) - 4(x + y) + 4 \geq 0,$$

$$(xy - 1)^2 + (x - y)^2 + (x + y)^2 - 4(x + y) + 4 \geq 0,$$

$$(xy - 1)^2 + (x - y)^2 + (x + y - 2)^2 \geq 0,$$

The equality occurs for  $a = b = c = d = 1$ .

**Remark.** From the given proof, the following stronger inequality holds:

$$(a^2 - a + 2)(b^2 - b + 2)(c^2 - c + 2)(d^2 - d + 2) \geq (a + b + c + d)^2 + (x + y - 2)^2,$$

that is

$$(a^2 - a + 2)(b^2 - b + 2)(c^2 - c + 2)(d^2 - d + 2) \geq (a + b + c + d)^2 + (a + b + c + d - 4)^2.$$

□

**P 2.110.** If  $a, b, c, d$  are real numbers, then

$$(a^2 + 3)(b^2 + 3)(c^2 + 3)(d^2 + 3) \geq 16(a + b + c + d)^2.$$

(Vasile Cîrtoaje, 2011)

**Solution 1.** Denote

$$x = \frac{(b^2 + 3)(c^2 + 3)(d^2 + 3)}{64}, \quad s = \frac{b + c + d}{3},$$

and write the inequality as follows:

$$\begin{aligned} 4(a^2 + 3)x &\geq (a + 3s)^2, \\ (4x - 1)a^2 - 6sa + 12x - 9s^2 &\geq 0, \\ (4x - 1)\left(a - \frac{3s}{4x - 1}\right)^2 + 12x\left(1 - \frac{3s^2}{4x - 1}\right) &\geq 0. \end{aligned}$$

Since

$$4x - 1 \geq \frac{27}{16} - 1 > 0,$$

it suffices to show that

$$1 - \frac{3s^2}{4x - 1} \geq 0,$$

which is equivalent to

$$3(b^2 + 3)(c^2 + 3)(d^2 + 3) \geq 16(b + c + d)^2 + 48.$$

This inequality was proved at P 2.52.

The equality occurs for  $a = b = c = d = 1$  or  $a = b = c = d = -1$ .

**Solution 2.** Using the Cauchy-Schwarz inequality, we have

$$(a^2 + 3) \left[ 1 + \frac{(b + c + d)^2}{3} \right] \geq (a + b + c + d)^2.$$

So, it suffices to prove that

$$(a^2 + 3)(b^2 + 3)(c^2 + 3)(d^2 + 3) \geq 16(a^2 + 3) \left[ 1 + \frac{(b + c + d)^2}{3} \right],$$

which reduces to the inequality in P 2.52.

□

**P 2.111.** Let  $a, b, c, d$  be real numbers such that  $abcd > 0$ . Prove that

$$\left(a + \frac{1}{a}\right) \left(b + \frac{1}{b}\right) \left(c + \frac{1}{c}\right) \left(d + \frac{1}{d}\right) \geq (a + b + c + d) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right).$$

(Vasile Cîrtoaje, 2011)

**First Solution.** Write the inequality as  $A \geq B$ , where

$$\begin{aligned} A &= (a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1) \\ &= (1 + a^2c^2)(1 + b^2d^2) + \sum a^2 + \sum a^2b^2 + \sum a^2b^2c^2, \\ B &= \left(\sum a\right) \left(\sum abc\right) = 4abcd + \sum a^2(bc + cd + bd). \end{aligned}$$

Then,

$$\begin{aligned} A - B &= (1 - abcd)^2 + (ac - bd)^2 + \frac{1}{2} \sum a^2(1 - bc)^2 + \frac{1}{2} \sum a^2(1 - cd)^2 \\ &\quad + \sum a^2b^2 - \sum a^2bd, \end{aligned}$$

and hence

$$A - B \geq \sum a^2b^2 - \sum a^2bd = \frac{1}{2} \sum a^2(b - d)^2 \geq 0.$$

The equality holds for  $a = b = c = d = 1$ .

**Second Solution.** Since

$$(a + b)(b + c)(c + d)(d + a) - (a + b + c + d)(bcd + cda + dab + abc) = (ac - bd)^2 \geq 0,$$

it suffices to show that

$$(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1) \geq (a + b)(b + c)(c + d)(d + a).$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} (a^2 + 1)(1 + b^2) &\geq (a + b)^2, \\ (b^2 + 1)(1 + c^2) &\geq (b + c)^2, \\ (c^2 + 1)(1 + d^2) &\geq (c + d)^2, \\ (d^2 + 1)(1 + a^2) &\geq (d + a)^2. \end{aligned}$$

Multiplying these inequalities, we get

$$\begin{aligned} (a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1) &\geq |(a + b)(b + c)(c + d)(d + a)| \\ &\geq (a + b)(b + c)(c + d)(d + a). \end{aligned}$$

□

**P 2.112.** Let  $a, b, c, d$  be real numbers such that

$$a + b + c + d = 4, \quad a^2 + b^2 + c^2 + d^2 = 7.$$

Prove that

$$a^3 + b^3 + c^3 + d^3 \leq 16.$$

(Vasile Cîrtoaje, 2010)

**First Solution.** Assume that  $a \leq b \leq c \leq d$ , and denote

$$s = a + b, \quad p = ab, \quad s \leq 2, \quad 4p \leq s^2.$$

Since

$$\begin{aligned} 2(a^3 + b^3) &= 2(s^3 - 3ps), \\ c + d &= 4 - s, \quad c^2 + d^2 = 7 - (a^2 + b^2) = 7 - s^2 + 2p, \\ 2(c^3 + d^3) &= (c + d)[3(c^2 + d^2) - (c + d)^2] = (4 - s)(-4s^2 + 8s + 5 + 6p), \end{aligned}$$

we have

$$\begin{aligned} 2(a^3 + b^3 + c^3 + d^3 - 16) &= 12p(2 - s) + 6s^3 - 24s^2 + 27s - 12 \\ &\leq 3s^2(2 - s) + 6s^3 - 24s^2 + 27s - 12 \\ &= 3(s - 1)^2(s - 4) \leq 0. \end{aligned}$$

This completes the proof. The equality holds for  $a = b = c = 1/2$  and  $d = 5/2$  (or any cyclic permutation).

**Second Solution** (by Vo Quoc Ba Can). From

$$7 = a^2 + b^2 + c^2 + d^2 \geq a^2 + \frac{1}{3}(b + c + d)^2 = a^2 + \frac{1}{3}(4 - a)^2,$$

it follows that

$$a \in \left[ \frac{-1}{2}, \frac{5}{2} \right].$$

Similarly, we have

$$b, c, d \in \left[ \frac{-1}{2}, \frac{5}{2} \right].$$

On the other hand,

$$\begin{aligned} a^3 + b^3 + c^3 + d^3 &= \frac{5}{2} \sum a^2 + \sum (a^3 - \frac{5}{2}a^2) \\ &= \frac{35}{2} - \frac{1}{2} \sum a^2(5 - 2a) \end{aligned}$$

and, by virtue of the Cauchy-Schwarz inequality,

$$\sum a^2(5 - 2a) \geq \frac{[\sum a(5 - 2a)]^2}{\sum (5 - 2a)} = \frac{(5 \sum a - 2 \sum a^2)^2}{20 - 2 \sum a} = 3.$$

Therefore,

$$a^3 + b^3 + c^3 + d^3 \leq \frac{35}{2} - \frac{3}{2} = 16.$$

**Remark.** In the same manner as in the second solution, we can prove the following generalization.

- If  $a_1, a_2, \dots, a_n$  are real numbers such that

$$a_1 + a_2 + \dots + a_n = 2n, \quad a_1^2 + a_2^2 + \dots + a_n^2 = n(n+3),$$

then

$$a_1^3 + a_2^3 + \dots + a_n^3 \leq n(n^2 + 3n + 4),$$

with equality for  $a_1 = \dots = a_{n-1} = 1$  and  $a_n = n + 1$  (or any cyclic permutation).

□

**P 2.113.** Let  $a, b, c, d$  be real numbers such that

$$a + b + c + d = 0.$$

Prove that

$$12(a^4 + b^4 + c^4 + d^4) \leq 7(a^2 + b^2 + c^2 + d^2)^2.$$

(Vasile Cîrtoaje, 2010)

**Solution.** Assume that  $a^2 = \max\{b^2, c^2, d^2\}$  and denote

$$x = \sqrt{\frac{b^2 + c^2 + d^2}{3}}, \quad x^2 \leq a^2.$$

Since

$$x^2 = \frac{b^2 + c^2 + d^2}{3} \geq \left(\frac{b + c + d}{3}\right)^2 = \frac{a^2}{9},$$

we have

$$\frac{a^2}{9} \leq x^2 \leq a^2.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} b^4 + c^4 + d^4 &= (b^2 + c^2 + d^2)^2 - 2(b^2c^2 + c^2d^2 + d^2b^2) \\ &= 9x^4 - 2(b^2c^2 + c^2d^2 + d^2b^2) \\ &\leq 9x^4 - \frac{2}{3}(bc + cd + db)^2 \\ &= 9x^4 - \frac{1}{6}[(b + c + d)^2 - b^2 - c^2 - d^2]^2 \\ &= 9x^4 - \frac{1}{6}(a^2 - 3x^2)^2 = \frac{45x^4 + 6a^2x^2 - a^4}{6}, \end{aligned}$$

hence

$$a^4 + b^4 + c^4 + d^4 \leq \frac{45x^4 + 6a^2x^2 + 5a^4}{6}.$$

Therefore, it suffices to prove that

$$2(45x^4 + 6a^2x^2 + 5a^4) \leq 7(a^2 + 3x^2)^2,$$

which is equivalent to the obvious inequality

$$(x^2 - a^2)(9x^2 - a^2) \leq 0.$$

The equality holds for  $-a/3 = b = c = d$  (or any cyclic permutation).

**Remark.** Similarly, we can prove the following generalization.

- If  $a_1, a_2, \dots, a_n$  are real numbers such that

$$a_1 + a_2 + \dots + a_n = 0,$$

then

$$\frac{(a_1^2 + a_2^2 + \dots + a_n^2)^2}{a_1^4 + a_2^4 + \dots + a_n^4} \geq \frac{n(n-1)}{n^2 - 3n + 3},$$

with equality for  $-a_1/(n-1) = a_2 = \dots = a_n$  (or any cyclic permutation).

□

**P 2.114.** Let  $a, b, c, d$  be real numbers such that

$$a + b + c + d = 0.$$

Prove that

$$(a^2 + b^2 + c^2 + d^2)^3 \geq 3(a^3 + b^3 + c^3 + d^3)^2.$$

(Vasile Cîrtoaje, 2011)

**Solution.** Applying the AM-GM inequality and the identity

$$(a + b + c)^3 = a^3 + b^3 + c^3 + 3(a + b)(b + c)(c + a),$$

we have

$$\begin{aligned} (a^2 + b^2 + c^2 + d^2)^3 &= [a^2 + b^2 + c^2 + (a + b + c)^2]^3 \\ &= [(a + b)^2 + (b + c)^2 + (c + a)^2]^3 \\ &\geq 27(a + b)^2(b + c)^2(c + a)^2 \\ &= 3[(a + b + c)^3 - a^3 - b^3 - c^3]^2 \\ &= 3(a^3 + b^3 + c^3 + d^3)^2. \end{aligned}$$



The equality holds for  $a = b = c = -d/3$  (or any cyclic permutation).

**Remark.** The following generalization holds (*Vasile Cîrtoaje*, 2011).

- If  $a_1, a_2, \dots, a_n$  are real numbers such that

$$a_1 + a_2 + \dots + a_n = 0,$$

then

$$(a_1^2 + a_2^2 + \dots + a_n^2)^3 \geq \frac{n(n-1)}{(n-2)^2} (a_1^3 + a_2^3 + \dots + a_n^3)^2.$$

Moreover,

- If  $k \geq 3$  is an odd number, and  $a_1, a_2, \dots, a_n$  are real numbers such that

$$a_1 + a_2 + \dots + a_n = 0,$$

then

$$(a_1^2 + a_2^2 + \dots + a_n^2)^k \geq \frac{n^k(n-1)^{k-2}}{[(n-1)^{k-1} - 1]^2} (a_1^k + a_2^k + \dots + a_n^k)^2,$$

with equality for  $a_1 = \dots = a_{n-1} = -a_n/(n-1)$  (or any cyclic permutation). □

**P 2.115.** If  $a, b, c, d$  are real numbers such that

$$a + b + c + d = 0,$$

then

$$a^4 + b^4 + c^4 + d^4 + 28abcd \geq 0.$$

(*Adrian Zahariuc*, 2015)

**First Solution.** Assume that  $a, b, c, d$  are nonzero and  $a \geq b \geq c \geq d$ . Since the statement remains unchanged by changing the sign of all numbers, it suffices to consider the cases

$$a \geq b \geq c \geq d > 0,$$

$$a \geq b \geq c > 0 > d,$$

$$a \geq b > 0 > c \geq d.$$

Clearly, the first and the third case are trivial. For

$$a \geq b \geq c > 0 > d,$$

we rewrite the inequality as

$$a^4 + b^4 + c^4 + (a + b + c)^4 \geq 28abc(a + b + c).$$

By the AM-GM inequality, we have

$$(a + b + c)^4 \geq 27abc(a + b + c).$$

Thus, it suffices to show that

$$a^4 + b^4 + c^4 \geq abc(a + b + c).$$

This inequality follows from

$$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c).$$

The equality holds for  $a = b = c = -d/3$  (or any cyclic permutation).

**Second Solution** Write the inequality as  $f_4(a, b, c) \geq 0$ , where

$$f_4(a, b, c) = a^4 + b^4 + c^4 + (a + b + c)^4 - 28abc(a + b + c).$$

According to P 2.60, it suffices to show that  $f_4(a, 1, 1) \geq 0$ . Indeed, we have

$$f_4(a, 1, 1) = 2(a - 1)^2(a + 3)^2 \geq 0.$$

□

**P 2.116.** If  $a, b, c, d$  are real numbers such that

$$abcd = 1,$$

then

$$(1 + a^2)(1 + b^2)(1 + c^2)(1 + d^2) \geq (a + b + c + d)^2.$$

(Pham Kim Hung, 2006)

**Solution.** Substituting  $a, b, c, d$  by  $|a|, |b|, |c|, |d|$ , respectively, the left side of the inequality remains unchanged, while the right side either remains unchanged or increases. Therefore, it suffices to prove the inequality only for  $a, b, c, d \geq 0$ . Among  $a, b, c, d$  there are two numbers less than or equal to 1, or larger than or equal to 1. Let  $b$  and  $d$  be these numbers; that is,

$$(1 - b)(1 - d) \geq 0.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} (1 + a^2)(1 + b^2)(1 + c^2)(1 + d^2) &= (1 + a^2 + b^2 + a^2b^2)(c^2 + 1 + d^2 + c^2d^2) \\ &\geq (c + a + bd + abcd)^2 = (c + a + bd + 1)^2. \end{aligned}$$

So, it suffices to show that

$$c + a + bd + 1 \geq a + b + c + d,$$

which is equivalent to  $(1 - b)(1 - d) \geq 0$ . The equality holds for  $a = b = c = d = 1$ .

□

**P 2.117.** Let  $a, b, c, d$  be real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

Prove that

$$(abc)^3 + (bcd)^3 + (cda)^3 + (dab)^3 \leq 4.$$

(Vasile Cîrtoaje, 2004)

**Solution.** Substituting  $a, b, c, d$  by  $|a|, |b|, |c|, |d|$ , respectively, the hypothesis and the right side of the inequality remains unchanged, while the left side either remains unchanged or decreases. Therefore, it suffices to prove the inequality only for  $a, b, c, d \geq 0$ . Setting

$$x = a^2, \quad y = b^2, \quad z = c^2, \quad t = d^2,$$

we need to prove that

$$(xyz)^{3/2} + (yzt)^{3/2} + (ztx)^{3/2} + (txy)^{3/2} \leq 4$$

for

$$x + y + z + t = 4.$$

By the AM-GM inequality, we have

$$4\sqrt[4]{xyz} \leq 1 + x + y + z = 5 - t,$$

$$(xyz)^{3/2} = xyz\sqrt{xyz} \leq \frac{xyz(5-t)^2}{16};$$

analogously,

$$(yzt)^{3/2} \leq \frac{yzt(5-x)^2}{16}, \quad (ztx)^{3/2} \leq \frac{ztx(5-y)^2}{16}, \quad (txy)^{3/2} \leq \frac{txy(5-z)^2}{16}.$$

Therefore, it suffices to show that

$$xyz(5-t)^2 + yzt(5-x)^2 + ztx(5-y)^2 + txy(5-z)^2 \leq 64,$$

which is equivalent to  $E(x, y, z, t) \geq 0$ , where

$$E(x, y, z, t) = 36xyzt - 25(xyz + yzt + ztx + txy) + 64.$$

To prove this inequality, we use the mixing variables method. Without loss of generality, assume that  $x \geq y \geq z \geq t \geq 0$ . Setting

$$u = \frac{x + y + z}{3},$$

we have

$$3u + t = 4, \quad t \leq u \leq \frac{4}{3}, \quad u^3 \geq xyz.$$

We will show that

$$E(x, y, z, t) \geq E(u, u, u, t) \geq 0.$$

The left inequality is equivalent to

$$25[(u^3 - xyz) + t(3u^2 - xy - yz - zx)] \geq 36t(u^3 - xyz),$$

$$(25 - 36t)(u^3 - xyz) + 25t(3u^2 - xy - yz - zx) \geq 0.$$

By Schur's inequality

$$(x + y + z)^3 + 9xyz \geq 4(x + y + z)(xy + yz + zx),$$

we get

$$9u^3 + 3xyz \geq 4u(xy + yz + zx),$$

hence

$$3u^2 - xy - yz - zx \geq 3u^2 - \frac{9u^3 + 3xyz}{4u} = \frac{3(u^3 - xyz)}{4u}.$$

Therefore, it suffices to prove that

$$25 - 36t + \frac{75t}{4u} \geq 0.$$

Write this inequality as

$$25(4u + 3t) \geq 144ut,$$

then in the homogeneous form

$$25(4u + 3t)(3u + t) \geq 576ut,$$

or, equivalently,

$$75(4u^2 + t^2) \geq 251ut.$$

This inequality is true, since

$$75(4u^2 + t^2) - 251ut \geq 75(4u^2 + t^2 - 4ut) = 75(2u - t)^2 \geq 0.$$

The right inequality  $E(u, u, u, t) \geq 0$  holds also, since

$$\begin{aligned} E(u, u, u, t) &= (36u^3 - 75u^2)t - 25u^3 + 64 \\ &= (36u^3 - 75u^2)(4 - 3u) - 25u^3 + 64 \\ &= 4(16 - 75u^2 + 86u^3 - 27u^4) \\ &= 4(1 - u)^2(16 + 32u - 27u^2) \\ &= 4(1 - u)^2[4(4 - u) + 9u(4 - 3u)] \geq 0. \end{aligned}$$

This completes the proof. The equality holds for  $a = b = c = d = 1$ .

□

**P 2.118.** Let  $a, b, c, d$  be real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 1.$$

Prove that

$$(1-a)^4 + (1-b)^4 + (1-c)^4 + (1-d)^4 \geq a^4 + b^4 + c^4 + d^4.$$

(Vasile Cîrtoaje, 2007)

**Solution.** The desired inequality follows by summing the inequalities

$$(1-a)^4 + (1-b)^4 \geq c^4 + d^4,$$

$$(1-c)^4 + (1-d)^4 \geq a^4 + b^4.$$

Since

$$(1-a)^4 + (1-b)^4 \geq 2(1-a)^2(1-b)^2$$

and

$$c^4 + d^4 \geq \frac{1}{2}(c^2 + d^2)^2,$$

the former inequality holds if

$$2(1-a)(1-b) \geq c^2 + d^2.$$

Indeed,

$$2(1-a)(1-b) - c^2 - d^2 = 2(1-a)(1-b) + a^2 + b^2 - 1 = (a+b-1)^2 \geq 0.$$

The equality holds for  $a = b = c = d = \frac{1}{2}$ .

□

**P 2.119.** If  $a, b, c, d \geq \frac{-1}{2}$  such that

$$a + b + c + d = 4,$$

then

$$\frac{1-a}{1-a+a^2} + \frac{1-b}{1-b+b^2} + \frac{1-c}{1-c+c^2} + \frac{1-d}{1-d+d^2} \geq 0.$$

(Vasile Cîrtoaje, 2014)

**Solution** (by Nguyen Van Quy). Assume that  $a \leq b \leq c \leq d$  and consider two cases:  $a > 0$  and  $a \leq 0$ .

Case 1:  $a > 0$ . Write the inequality as

$$\frac{a^2}{1-a+a^2} + \frac{b^2}{1-b+b^2} + \frac{c^2}{1-c+c^2} + \frac{d^2}{1-d+d^2} \leq 4.$$

We have

$$\frac{a^2}{1-a+a^2} + \frac{b^2}{1-b+b^2} + \frac{c^2}{1-c+c^2} + \frac{d^2}{1-d+d^2} \leq \frac{a^2}{a} + \frac{b^2}{b} + \frac{c^2}{c} + \frac{d^2}{d} = 4.$$

*Case 2:*  $-1/2 \leq a \leq 0$ . We can check that the equality holds for  $a = -1/2$  and  $b = c = d = 3/2$  (or any cyclic permutation). Define the function

$$f(x) = \frac{1-x}{1-x+x^2} + k_1x + k_2, \quad x \geq \frac{-1}{2},$$

such that

$$f(3/2) = f'(3/2) = 0.$$

We get

$$k_1 = \frac{12}{49}, \quad k_2 = \frac{-4}{49},$$

when

$$f(x) = \frac{1-x}{1-x+x^2} + \frac{12x-4}{49} = \frac{(2x-3)^2(3x+5)}{49(1-x+x^2)}.$$

Since  $f(x) \geq 0$  for  $x \geq -1/2$ , we have

$$\frac{1-x}{1-x+x^2} \geq \frac{4-12x}{49}.$$

Therefore,

$$\frac{1-b}{1-b+b^2} + \frac{1-c}{1-c+c^2} + \frac{1-d}{1-d+d^2} \geq \frac{12-12(b+c+d)}{49} = \frac{12(a-3)}{49}.$$

Thus, it suffices to show that

$$\frac{1-a}{1-a+a^2} + \frac{12(a-3)}{49} \geq 0.$$

Indeed,

$$\frac{1-a}{1-a+a^2} + \frac{12(a-3)}{49} = \frac{(2a+1)(6a^2-27a+13)}{49(1-a+a^2)} \geq 0.$$

The proof is completed. The equality holds for  $a = b = c = d = 1$ , and also for  $a = -1/2$  and  $b = c = d = 3/2$  (or any cyclic permutation).

□

**P 2.120.** If  $a, b, c, d$  are real numbers such that  $a \geq b \geq c \geq d$ ,  $b+c \geq 0$  and

$$a^2 + b^2 + c^2 + d^2 = 4,$$

then

$$a^2c^2 + b^2d^2 \leq 2.$$

(Vasile Cîrtoaje, 2020)

**Solution.** We have

$$\begin{aligned}
 2 - a^2c^2 - b^2d^2 &= 2 + (a^2 - b^2)(b^2 - c^2) - b^2(a^2 + c^2 + d^2 - b^2) \\
 &= 2 + (a^2 - b^2)(b^2 - c^2) - b^2(4 - 2b^2) \\
 &= (a^2 - b^2)(b^2 - c^2) + 2(b^2 - 1)^2 \\
 &= (a - b)(b - c)(a + b)(b + c) + 2(b^2 - 1)^2 \geq 0.
 \end{aligned}$$

The equality holds for  $a = b = 1 \geq c \geq d$  and  $c^2 + d^2 = 2$ , and for  $a \geq b = c = 1 \geq d$  and  $a^2 + d^2 = 2$ .

□

**P 2.121.** Let  $a, b, c, d$  be real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

If  $a \geq b \geq c \geq d$ , then

$$1 - abcd \leq (a - d)^2.$$

(Vasile Cîrtoaje, 2019)

**Solution.** There are five cases to consider.

*Case 1:*  $a \geq b \geq c \geq d \geq 0$ . For  $a = d$ , the inequality is a trivial equality. Consider next that  $a > d$ , give up the condition  $b \geq c$  (consider only that  $b, c \in [d, a]$ ) and write the inequality in the homogeneous form  $f \geq 0$ , where

$$f = 16abcd + 4(a - d)^2(a^2 + b^2 + c^2 + d^2) - (a^2 + b^2 + c^2 + d^2)^2.$$

For fixed  $a, c$  and  $d$ ,  $f$  is a function of  $b$ ,  $b \in [d, a]$ . We have

$$f'(b) = 16acd + 8(a - d)^2b - 4b(a^2 + b^2 + c^2 + d^2) = 4bh(b),$$

where

$$h(b) = \frac{4acd}{b} + 2(a - d)^2 - (a^2 + b^2 + c^2 + d^2).$$

Since  $h(b)$  is a decreasing function, there are three possible cases: (1)  $h(b) \geq 0$  for  $b \in [d, a]$ , hence  $f(b)$  is increasing on  $[d, a]$ ; (2)  $h(b) \geq 0$  for  $b \in [d, d_1]$  and  $h(b) \leq 0$  for  $b \in [d_1, a]$ , hence  $f(b)$  is increasing on  $[d, d_1]$  and decreasing on  $[d_1, a]$ ; (3)  $h(b) \leq 0$  for  $b \in [d, a]$ , hence  $f(b)$  is decreasing on  $[d, a]$ . In all these cases  $f(b)$  has the minimum value when  $b \in \{a, d\}$ . As a consequence, we only need to prove the required inequality for  $b \in \{a, d\}$ . Similarly, we only need to prove the required inequality for  $c \in \{a, d\}$ . So, we need to show that

$$16a^k d^{4-k} + 4(a - d)^2[ka^2 + (4 - k)d^2] - [ka^2 + (4 - k)d^2]^2 \geq 0,$$

where

$$k \in \{1, 2, 3\}.$$

For  $d = 0$ , the inequality reduces to

$$k(4 - k)a^4 \geq 0,$$

which is true for  $k \in \{1, 2, 3\}$ . Next, due to homogeneity, we may set  $d = 1$  (which involves  $a > 1$ ). The required inequality becomes

$$4(a - 1)^2(ka^2 + 4 - k) \geq (ka^2 + 4 - k)^2 - 16a^k.$$

For  $k = 1$ , we need to show that

$$4(a - 1)^2(a^2 + 3) \geq (a^2 + 3)^2 - 16a,$$

that is

$$\begin{aligned} 4(a - 1)^2(a^2 + 3) &\geq (a - 1)^2(a^2 + 2a + 9), \\ (a - 1)^2(3a^2 - 2a + 3) &\geq 0. \end{aligned}$$

The last inequality is clearly true.

For  $k = 2$ , we need to show that

$$2(a - 1)^2(a^2 + 1) \geq (a^2 + 1)^2 - 4a^2,$$

that is

$$\begin{aligned} 2(a - 1)^2(a^2 + 1) &\geq (a^2 - 1)^2, \\ (a - 1)^4 &\geq 0. \end{aligned}$$

For  $k = 3$ , we need to show that

$$4(a - 1)^2(3a^2 + 1) \geq (3a^2 + 1)^2 - 16a^3,$$

that is

$$\begin{aligned} 4(a - 1)^2(3a^2 + 1) &\geq (a - 1)^2(9a^2 + 2a + 1), \\ (a - 1)^2(3a^2 - 2a + 3) &\geq 0. \end{aligned}$$

*Case 2:*  $a \geq b \geq c \geq 0 \geq d$ . Replacing  $d$  by  $-d$ , we need to show that

$$(a + d)^2 \geq 1 + abcd$$

for  $a \geq b \geq c \geq 0$  and  $d \geq 0$  satisfying  $a^2 + b^2 + c^2 + d^2 = 4$ . It is enough to show that

$$2(a + d)^2 \geq 2 + ad(b^2 + c^2),$$

that is

$$\begin{aligned} 2(a + d)^2 &\geq 2 + ad(4 - a^2 - d^2), \\ (a^2 + d^2)(2 + ad) &\geq 2. \end{aligned}$$



Since

$$4 = a^2 + b^2 + c^2 + d^2 \leq 3a^2 + d^2,$$

we have

$$(a^2 + d^2)(2 + ad) \geq 2(a^2 + d^2) = \frac{4a^2 + 4d^2}{2} \geq \frac{3a^2 + d^2}{2} \geq 2.$$

*Case 3:*  $a \geq b \geq 0 \geq c \geq d$ . Replacing  $c$  by  $-c$  and  $d$  by  $-d$ , we need to show that

$$abcd + (a + d)^2 \geq 1$$

for  $a \geq b \geq 0$  and  $d \geq c \geq 0$  satisfying  $a^2 + b^2 + c^2 + d^2 = 4$ . It is enough to show that

$$(a + d)^2 \geq 1.$$

Since

$$4 = a^2 + b^2 + c^2 + d^2 \leq 2a^2 + 2d^2,$$

we have

$$(a + d)^2 \geq a^2 + d^2 \geq 2.$$

*Case 4:*  $a \geq 0 \geq b \geq c \geq d$ . Replacing  $a, b, c, d$  with  $-d, -c, -b, -a$ , respectively, this case reduces to the case 1.

*Case 5:*  $0 \geq a \geq b \geq c \geq d$ . Replacing  $a, b, c, d$  with  $-d, -c, -b, -a$ , respectively, this case reduces to the case  $a \geq b \geq c \geq d \geq 0$ .

The proof is completed. The equality occurs for  $a = b = c = d = 1$ .

□

**P 2.122.** If  $a, b, c, d, e \geq -3$  such that

$$a + b + c + d + e = 5,$$

then

$$\frac{1-a}{1+a+a^2} + \frac{1-b}{1+b+b^2} + \frac{1-c}{1+c+c^2} + \frac{1-d}{1+d+d^2} + \frac{1-e}{1+e+e^2} \geq 0.$$

(Vasile Cîrtoaje, 2014)

**Solution.** Assume that  $a \leq b \leq c \leq d \leq e$  and consider two cases:  $a \geq 0$  and  $a \leq 0$ .

*Case 1:*  $a \geq 0$ . For any  $x \geq 0$ , we have

$$\frac{1-x}{1+x+x^2} - \frac{1-x}{3} = \frac{(x-1)^2(x+2)}{3(1+x+x^2)} \geq 0.$$

Therefore, it suffices to show that

$$\frac{1-a}{3} + \frac{1-b}{3} + \frac{1-c}{3} + \frac{1-d}{3} + \frac{1-e}{3} \geq 0,$$

which is an identity.

*Case 2:*  $-3 \leq a \leq 0$ . We can check that the equality holds for  $a = -3$  and  $b = c = d = e = 2$ . Based on this, define the function

$$f(x) = \frac{1-x}{1+x+x^2} + k_1x + k_2, \quad x \geq -3,$$

such that

$$f(2) = f'(2) = 0.$$

We get

$$k_1 = \frac{2}{49}, \quad k_2 = \frac{3}{49},$$

when

$$f(x) = \frac{1-x}{1+x+x^2} + \frac{2x+3}{49} = \frac{(x-2)^2(2x+13)}{49(1+x+x^2)}.$$

Since  $f(x) \geq 0$  for  $x \geq -3$ , we have

$$\frac{1-x}{1+x+x^2} \geq \frac{-2x-3}{49}.$$

Thus, it suffices to show that

$$\frac{1-a}{1+a+a^2} - \frac{2b+3}{49} - \frac{2c+3}{49} - \frac{2d+3}{49} - \frac{2e+3}{49} \geq 0,$$

which is equivalent to

$$\begin{aligned} \frac{1-a}{1+a+a^2} - \frac{2(b+c+d+e)+12}{49} &\geq 0, \\ \frac{1-a}{1+a+a^2} - \frac{2(5-a)+12}{49} &\geq 0, \\ \frac{(a+3)(2a^2-26a+9)}{49(1+a+a^2)} &\geq 0. \end{aligned}$$

Clearly, the last inequality is true for  $-3 \leq a \leq 0$ .

The proof is completed. The equality holds for  $a = b = c = d = e = 1$ , and also for  $a = -3$  and  $b = c = d = e = 2$  (or any cyclic permutation). □

**P 2.123.** Let  $a, b, c, d, e$  be real numbers such that

$$a + b + c + d + e = 0.$$

Prove that

$$30(a^4 + b^4 + c^4 + d^4 + e^4) \geq 7(a^2 + b^2 + c^2 + d^2 + e^2)^2.$$

(Vasile Cîrtoaje, 2010)

**Solution.** Write the inequality as  $E(a, b, c, d, e) \geq 0$ , where

$$E(a, b, c, d, e) = 30(a^4 + b^4 + c^4 + d^4 + e^4) - 7(a^2 + b^2 + c^2 + d^2 + e^2)^2.$$

Among the numbers  $a, b, c, d, e$  there exist three with the same sign. Let  $a, b, c$  be these numbers. We will show that

$$E(a, b, c, d, e) \geq E(a, b, c, x, x) \geq 0,$$

where

$$x = \frac{d+e}{2} = \frac{-(a+b+c)}{2}.$$

The inequality  $E(a, b, c, d, e) \geq E(a, b, c, x, x)$  is equivalent to

$$30(d^4 + e^4 - 2x^4) \geq 7(d^2 + e^2 - 2x^2)(2a^2 + 2b^2 + 2c^2 + d^2 + e^2 + 2x^2).$$

Since

$$d^4 + e^4 - 2x^4 = \frac{(d-e)^2(7d^2 + 10de + 7e^2)}{8}$$

and

$$d^2 + e^2 - 2x^2 = \frac{(d-e)^2}{2},$$

we need to show that

$$15(7d^2 + 10de + 7e^2) \geq 14(2a^2 + 2b^2 + 2c^2 + d^2 + e^2 + 2x^2),$$

which reduces to

$$21(d^2 + e^2) + 34de \geq 7(a^2 + b^2 + c^2).$$

Since  $a, b, c$  have the same sign, we have

$$a^2 + b^2 + c^2 \leq (a+b+c)^2 = (d+e)^2.$$

Thus, it suffices to prove that

$$21(d^2 + e^2) + 34de \geq 7(d+e)^2,$$

which is equivalent to the obvious inequality

$$4(d^2 + e^2) + 10(d+e)^2 \geq 0.$$

The inequality  $E(a, b, c, x, x) \geq 0$ , where  $x = -(a+b+c)/2$ , can be written as

$$23 \sum a^4 + 2 \left( \sum a \right)^4 \geq 7 \left( \sum a^2 \right) \left( \sum a \right)^2 + 14 \sum a^2 b^2.$$

Since

$$\left( \sum a \right)^2 - 3 \sum ab = \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2] \geq 0,$$

it suffices to prove that

$$23 \sum a^4 + 6 \left( \sum ab \right) \left( \sum a \right)^2 \geq 7 \left( \sum a^2 \right) \left( \sum a \right)^2 + 14 \sum a^2 b^2.$$

This is equivalent to

$$\sum a^4 + abc \sum a \geq \frac{1}{2} \sum ab(a^2 + b^2) + \sum a^2 b^2,$$

which follows by summing Schur's inequality of degree four

$$\sum a^4 + abc \sum a \geq \sum ab(a^2 + b^2)$$

and the obvious inequality

$$\frac{1}{2} \sum ab(a^2 + b^2) \geq \sum a^2 b^2.$$

This completes the proof. The equality holds for  $a = b = c = 2$  and  $d = e = -3$  (or any permutation thereof).

**Remark.** Notice that the following generalization holds (Vasile Cîrtoaje, 2010).

- If  $n$  is an odd positive integer and  $a_1, a_2, \dots, a_n$  are real numbers such that

$$a_1 + a_2 + \dots + a_n = 0,$$

then

$$\frac{(a_1^2 + a_2^2 + \dots + a_n^2)^2}{a_1^4 + a_2^4 + \dots + a_n^4} \leq \frac{n(n^2 - 1)}{n^2 + 3},$$

with equality when  $(n+1)/2$  of  $a_1, a_2, \dots, a_n$  are equal to  $(n-1)/2$  and the other  $(n-1)/2$  numbers are equal to  $-(n+1)/2$ .

□

**P 2.124.** If  $a, b, c, d, e$  are real numbers such that  $a + b + c + d + e = 5$ , then

$$(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1)(e^2 - e + 1) \geq 1.$$

(Vasile Cîrtoaje, 2015)

**Solution** (by KaiRain). Without loss of generality, consider

$$e = \min\{a, b, c, d, e\}, \quad e \leq 1.$$

Write the inequality as

$$A(e^2 - e + 1) \geq 4,$$

where

$$A = 4(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1),$$

and use the inequality

$$2(a^2 - a + 1)(b^2 - b + 1) \geq (a + b - 1)^2 + 1,$$

which is equivalent to

$$(2ab - a - b)^2 + (a - b)^2 \geq 0.$$

According to this result and the Cauchy-Schwarz inequality, we have

$$A \geq [(a + b - 1)^2 + 1][1 + (c + d - 1)^2] \geq (a + b - 1 + c + d - 1)^2 = (3 - e)^2.$$

Thus, it suffices to show that

$$(3 - e)^2(e^2 - e + 1) \geq 4,$$

which is equivalent to

$$(e - 1)^2(e^2 - 5e + 5) \geq 0.$$

Since

$$e^2 - 5e + 5 \geq 5(1 - e) \geq 0,$$

the proof is completed. The equality holds for  $a = b = c = d = e = 1$ .

□

**P 2.125.** If  $a, b, c, d, e$  are real numbers, then

$$4(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1)(e^2 - e + 1) \geq (abcde - 1)^2.$$

(Vasile Cîrtoaje, 2015)

**Solution.** Firstly, we consider the cases  $abcde = 1$ ,  $a = 0$  and  $bcde = 2$ . For  $abcde = 1$ , the inequality is trivial. For  $a = 0$ , we need to show that

$$4(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1)(e^2 - e + 1) \geq 1.$$

Indeed, we have

$$4(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1)(e^2 - e + 1) \geq 4 \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} > 1.$$

For  $bcde = 2$ , the inequality becomes

$$4(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1)(e^2 - e + 1) \geq (2a - 1)^2.$$

Since

$$(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1)(e^2 - e + 1) \geq |b| \cdot |c| \cdot |d| \cdot |e| = 2,$$

it suffices to show that

$$8(a^2 - a + 1) \geq (2a - 1)^2,$$

which is clearly true. Otherwise, we write the inequality as  $F(a) \geq 1$ , where

$$F(a) = 4f(a)(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1)(e^2 - e + 1),$$

$$f(a) = \frac{a^2 - a + 1}{(ga - 1)^2}, \quad g = bcde,$$

$$f'(a) = \frac{(g - 2)a - 2g + 1}{(ag - 1)^3}.$$

If  $g \neq 2$ , we have  $f'(a_1) = 0$  for

$$a_1 = \frac{2g - 1}{g - 2}.$$

*Case 1:*  $g \in (-\infty, 0) \cup (2, \infty)$ .  $f(a)$  is increasing on  $\left(-\infty, \frac{1}{g}\right) \cup [a_1, \infty)$  and decreasing on  $\left(\frac{1}{g}, a_1\right]$ . Thus, it suffices to consider  $a = -\infty$  and  $a = a_1$ . For  $a = -\infty$ , we have

$$F(a) = \frac{4}{g^2}(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1)(e^2 - e + 1) \geq \frac{4}{g^2} \cdot \frac{3b^2}{4} \cdot \frac{3c^2}{4} \cdot \frac{3d^2}{4} \cdot \frac{3e^2}{4} > 1.$$

It remains the case  $a = a_1$ .

*Case 2:*  $0 < g < 2$ .  $f(a)$  is increasing on  $\left[a_1, \frac{1}{g}\right)$  and decreasing on  $(-\infty, a_1] \cup \left(\frac{1}{g}, \infty\right)$ . Thus, it suffices to consider  $a = \infty$  and  $a = a_1$  and . For  $a = \infty$ , the inequality holds. It remains the case  $a = a_1$ .

For  $a = a_1$ , we have

$$abcde = \frac{a(2a - 1)}{a - 2}.$$

Due to symmetry, it suffices to consider the case  $a = a_1$ ,  $b = b_1$ ,  $c = c_1$ ,  $d = d_1$ ,  $e = e_1$ , when

$$abcde = \frac{a(2a - 1)}{a - 2} = \frac{b(2b - 1)}{b - 2} = \frac{c(2c - 1)}{c - 2} = \frac{d(2d - 1)}{d - 2} = \frac{e(2e - 1)}{e - 2}. \quad (*)$$

From

$$\frac{a(2a - 1)}{a - 2} = \frac{b(2b - 1)}{b - 2},$$

we get

$$(a - b)(ab - 2a - 2b + 1) = 0.$$

The case  $ab - 2a - 2b + 1 = 0$  implies

$$b = \frac{2a - 1}{a - 2},$$

and from

$$abcde = \frac{a(2a - 1)}{a - 2},$$

we get  $abcde = ab$ , hence  $cde = 1$ . Since

$$(c^2 - c + 1)(d^2 - d + 1)(e^2 - e + 1) \geq |c| \cdot |d| \cdot |e| = 1$$

and

$$4(a^2 - a + 1)(b^2 - b + 1) - 3(ab - 1)^2 = (ab - 2a - 2b + 1)^2 \geq 0,$$

we have

$$F(a) \geq \frac{4(a^2 - a + 1)(b^2 - b + 1)}{(ab - 1)^2} > 1.$$

As a consequence, it suffices to consider the case  $a = b = c = d = e = f$ , when (\*) implies

$$a = b = c = d = e = \frac{1}{4} \left( 3 + \sqrt{5} \pm \sqrt{6\sqrt{5} - 2} \right)$$

(from  $a + \frac{1}{a} = \frac{3 + \sqrt{5}}{2}$ ). In this case, we have

$$F(a) = \frac{4(a^2 - a + 1)^5}{(a^5 - 1)^2} = 1.$$

Note that

$$4(a^2 - a + 1)^5 - (a^5 - 1)^2 = (a^4 - 3a^3 + 3a^2 - 3a + 1)^2(3a^2 - 2a + 3).$$

The equality occurs for

$$a = b = c = d = e = \frac{1}{4} \left( 3 + \sqrt{5} \pm \sqrt{6\sqrt{5} - 2} \right).$$

□

**P 2.126.** If  $a_1, a_2, \dots, a_5$  are real numbers such that

$$a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3 = 0,$$

then

$$\sum_{i < j} a_i a_j \leq 0.$$

(Vasile Cîrtoaje, 2019)

**Solution.** Since the statement remain unchanged by replacing all  $a_i$  with  $-a_i$ , there are two cases to consider:  $a_1 \leq 0, a_2, a_3, a_4, a_5 \geq 0$ , and  $a_1, a_2 \leq 0, a_3, a_4, a_5 \geq 0$ .

In the first case, we need to show that if  $x, a, b, c, d \geq 0$  such that

$$x^3 = a^3 + b^3 + c^3 + d^3,$$

then

$$(a + b + c + d)x \geq ab + ac + ad + bc + bd + cd.$$

Write the inequality as

$$2(a + b + c + d)x + a^2 + b^2 + c^2 + d^2 \geq (a + b + c + d)^2.$$

Denoting

$$s = \frac{a + b + c + d}{4},$$

the inequality becomes

$$8sx + a^2 + b^2 + c^2 + d^2 \geq 16s^2.$$

Since

$$a^2 + b^2 + c^2 + d^2 \geq 4s^2$$

and

$$x^3 \geq 4s^3 \geq \frac{27}{8}s^3 \quad x \geq \frac{3}{2}s,$$

we have

$$8sx + a^2 + b^2 + c^2 + d^2 - 16s^2 \geq 12s^2 + 4s^2 - 16s^2 = 0.$$

In the second case, we need to show that if  $x, y, a, b, c \geq 0$  such that

$$x^3 + y^3 = a^3 + b^3 + c^3,$$

then

$$(a + b + c)(x + y) \geq xy + ab + bc + ca.$$

From the known inequalities

$$(x + y)^3 \geq x^3 + y^3, \quad (x + y)^3 \leq 4(x^3 + y^3),$$

we get

$$\begin{aligned} x + y &\geq \sqrt[3]{x^3 + y^3} = \sqrt[3]{a^3 + b^3 + c^3}, \\ x + y &\leq \sqrt[3]{4x^3 + 4y^3} = \sqrt[3]{4(a^3 + b^3 + c^3)}. \end{aligned}$$

Denoting  $t = x + y$ , we have

$$t_1 \leq t \leq t_2,$$

where

$$t_1 = \sqrt[3]{a^3 + b^3 + c^3}, \quad t_2 = \sqrt[3]{4(a^3 + b^3 + c^3)}.$$

On the other hand, it is enough to prove the inequality

$$(a + b + c)(x + y) \geq \frac{(x + y)^2}{4} + \frac{(a + b + c)^2}{3},$$

or, better, the inequality

$$(a + b + c)(x + y) \geq \frac{5(x + y)^2}{12} + \frac{(a + b + c)^2}{3},$$



which is equivalent to

$$5t^2 - 12(a+b+c)t + 4(a+b+c)^2 \leq 0,$$

$$[t - 2(a+b+c)][5t - 2(a+b+c)] \leq 0.$$

This is true if  $t_1 \geq \frac{2}{5}(a+b+c)$  and  $t_2 \leq 2(a+b+c)$ . Thus, we need to show that

$$a^3 + b^3 + c^3 \geq \frac{8}{125}(a+b+c)^3$$

and

$$a^3 + b^3 + c^3 \leq 2(a+b+c)^3.$$

Indeed, we have

$$a^3 + b^3 + c^3 \geq \frac{1}{9}(a+b+c)^3 \geq \frac{8}{125}(a+b+c)^3$$

and

$$a^3 + b^3 + c^3 \leq (a+b+c)^3 \leq 2(a+b+c)^3.$$

Thus, the proof is completed. The equality holds for  $a_1 = a_2 = a_3 = a_4 = a_5 = 0$ .

□

**P 2.127.** Let  $a_1, a_2, \dots, a_n \geq -1$  such that

$$a_1 + a_2 + \dots + a_n = 0.$$

Prove that

$$(n-2)(a_1^2 + a_2^2 + \dots + a_n^2) \geq a_1^3 + a_2^3 + \dots + a_n^3.$$

(Vasile Cîrtoaje, 2005)

**Solution.** Without loss of generality, assume that

$$a_1 \geq a_2 \geq \dots \geq a_n.$$

Write the inequality as

$$\sum_{i=1}^n a_i f(a_i) \geq 0,$$

where

$$f(x) = (n-2)x - x^2.$$

Since

$$\begin{aligned} f(a_i) - f(a_{i+1}) &= (a_i - a_{i+1})(n-2 - a_i - a_{i+1}) \\ &\geq (a_i - a_{i+1})(n-2 - a_1 - a_2) \\ &= (a_i - a_{i+1})(n-2 + a_3 + \dots + a_n) \\ &= (a_i - a_{i+1})[(1 + a_3) + \dots + (1 + a_n)] \geq 0, \end{aligned}$$

we have  $a_1 \geq a_2 \geq \cdots \geq a_n$  and

$$f(a_1) \geq f(a_2) \geq \cdots \geq f(a_n).$$

Therefore, by Chebyshev's inequality, we get

$$n \sum_{i=1}^n a_i f(a_i) \geq (a_1 + a_2 + \cdots + a_n)[f(a_1) + f(a_2) + \cdots + f(a_n)] = 0.$$

The equality holds for  $a_1 = a_2 = \cdots = a_n = 0$ , and for  $a_1 = n - 1$  and  $a_2 = \cdots = a_n = -1$  (or any cyclic permutation). □

**P 2.128.** Let  $a_1, a_2, \dots, a_n \geq -1$  such that

$$a_1 + a_2 + \cdots + a_n = 0.$$

Prove that

$$(n - 2)(a_1^2 + a_2^2 + \cdots + a_n^2) + (n - 1)(a_1^3 + a_2^3 + \cdots + a_n^3) \geq 0.$$

(Vasile Cîrtoaje, 2005)

**Solution.** For the nontrivial case  $a_1^2 + a_2^2 + \cdots + a_n^2 \neq 0$ , write the inequality as

$$(n - 1) \frac{S_3}{S_2} + n - 2 \geq 0,$$

where

$$S_2 = a_1^2 + a_2^2 + \cdots + a_n^2, \quad S_3 = a_1^3 + a_2^3 + \cdots + a_n^3.$$

Without loss of generality, assume that

$$a_1 \leq a_2 \leq \cdots \leq a_n, \quad a_1 < 0.$$

For any  $p > 0$  such that  $a_1 + p \geq 0$ , by the Cauchy-Schwarz inequality, we have

$$\sum_{i=1}^n (a_i + 1)^2 (a_i + p) \geq \frac{[\sum_{i=1}^n (a_i + 1)(a_i + p)]^2}{\sum_{i=1}^n (a_i + p)},$$

which is equivalent to

$$\sum_{i=1}^n a_i^3 + (p + 2) \sum_{i=1}^n a_i^2 + np \geq \frac{(\sum_{i=1}^n a_i^2 + np)^2}{np},$$

$$\frac{S_3}{S_2} \geq \frac{S_2}{np} - p.$$

Thus, it suffices to show that

$$S_2 + \frac{n(n-2)}{n-1}p \geq np^2.$$

Case 1:  $S_2 \geq \frac{n}{n-1}$ . Choosing  $p = 1$ , we have

$$a_1 + p \geq 0$$

and

$$S_2 + \frac{n(n-2)}{n-1}p - np^2 = S_2 - \frac{n}{n-1} \geq 0.$$

Case 2:  $0 < S_2 \leq \frac{n}{n-1}$ . We set

$$p = \sqrt{\frac{n-1}{n}S_2}.$$

From

$$\begin{aligned} p^2 - a_1^2 &= \frac{n-1}{n}S_2 - a_1^2 \\ &= \frac{n-1}{n}(a_2^2 + \cdots + a_n^2) - \frac{1}{n}a_1^2 \\ &\geq \frac{(a_2 + \cdots + a_n)^2}{n} - \frac{a_1^2}{n} = 0, \end{aligned}$$

we get  $a_1 + p \geq 0$ . In addition,

$$\begin{aligned} S_2 + \frac{n(n-2)}{n-1}p - np^2 &= S_2 + (n-2)\sqrt{\frac{n}{n-1}S_2} - (n-1)S_2 \\ &= (n-2)\sqrt{S_2} \left( \sqrt{\frac{n}{n-1}} - \sqrt{S_2} \right) \geq 0. \end{aligned}$$

The equality holds for  $a_1 = a_2 = \cdots = a_n = 0$ , and also for  $a_1 = -1$  and  $a_2 = \cdots = a_n = 1/(n-1)$  (or any cyclic permutation). □

**P 2.129.** Let  $a_1, a_2, \dots, a_n \geq n-1 - \sqrt{n^2 - n + 1}$  be nonzero real numbers such that

$$a_1 + a_2 + \cdots + a_n = n.$$

Prove that

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} \geq n.$$

(Vasile Cîrtoaje, 2010)

**Solution.** Without loss of generality, assume that

$$a_1 \leq a_2 \leq \cdots \leq a_n.$$

There are two cases to consider:  $a_1 > 0$  and  $a_1 < 0$ .

*Case 1:*  $a_1 > 0$ . By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} &\geq \frac{1}{n} \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right)^2 \\ &\geq \frac{1}{n} \left( \frac{n^2}{a_1 + a_2 + \cdots + a_n} \right)^2 = n. \end{aligned}$$

*Case 2:*  $a_1 < 0$ . There exists  $k$ ,  $1 \leq k \leq n-1$ , such that

$$a_1 \leq \cdots \leq a_k < 0 < a_{k+1} \leq \cdots \leq a_n.$$

Let us denote

$$x = \frac{a_1 + \cdots + a_k}{k}, \quad y = \frac{a_{k+1} + \cdots + a_n}{n-k}.$$

We have

$$-1 < n-1 - \sqrt{n^2 - n + 1} \leq x < 0, \quad y > 1, \quad kx + (n-k)y = n.$$

From  $k \geq 1$  and  $k(y-x) = n(y-1) > 0$ , we get

$$y - x \leq n(y-1),$$

hence

$$y \geq \frac{n-x}{n-1}.$$

In addition, from  $n-1 - \sqrt{n^2 - n + 1} \leq x$ , we get

$$n + 2(n-1)x - x^2 \geq 0.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{1}{a_1^2} + \cdots + \frac{1}{a_k^2} \geq \frac{1}{k} \left( \frac{1}{-a_1} + \cdots + \frac{1}{-a_k} \right)^2 \geq \frac{1}{k} \left( \frac{k^2}{-a_1 - \cdots - a_k} \right)^2 = \frac{k}{x^2}$$

and

$$\frac{1}{a_{k+1}^2} + \cdots + \frac{1}{a_n^2} \geq \frac{1}{n-k} \left( \frac{1}{a_{k+1}} + \cdots + \frac{1}{a_n} \right)^2 \geq \frac{1}{n-k} \left[ \frac{(n-k)^2}{a_{k+1} + \cdots + a_n} \right]^2 = \frac{n-k}{y^2}.$$

Then, it suffices to prove that

$$\frac{k}{x^2} + \frac{n-k}{y^2} \geq n,$$

which is equivalent to

$$k(y-x)(y+x) \geq nx^2(y-1)(y+1).$$

Since  $k(y-x) = n(y-1) > 0$ , we need to show that  $y+x \geq x^2(y+1)$ , which is equivalent to

$$(1-x)(x+y+xy) \geq 0.$$

This is true, since  $1-x > 0$  and

$$x+y+xy \geq x + \frac{(1+x)(n-x)}{n-1} = \frac{n+2(n-1)x-x^2}{n-1} \geq 0.$$

The equality holds when  $a_1 = a_2 = \dots = a_n = 1$ , as well as when one of  $a_1, a_2, \dots, a_n$  is  $n-1-\sqrt{n^2-n+1}$  and the others are  $\frac{1+\sqrt{n^2-n+1}}{n-1}$ . □

**P 2.130.** Let  $a_1, a_2, \dots, a_n \leq \frac{n}{n-2}$  be real numbers such that

$$a_1 + a_2 + \dots + a_n = n.$$

If  $k$  is a positive integer,  $k \geq 2$ , then

$$a_1^k + a_2^k + \dots + a_n^k \geq n.$$

(Vasile Cîrtoaje, 2012)

**Solution.** First we show that at most one of  $a_i$  is negative. Assume, for the sake of contradiction, that  $a_{n-1} < 0$  and  $a_n < 0$ . Then,

$$a_{n-1} + a_n = n - (a_1 + \dots + a_{n-2}) \geq n - (n-2) \cdot \frac{n}{n-2} = 0,$$

which is a contradiction. There are two cases to consider.

*Case 1:*  $a_1, a_2, \dots, a_n \geq 0$ . The desired inequality is just Jensen's inequality applied to the convex function  $f(x) = x^k$ .

*Case 2:*  $a_1, a_2, \dots, a_{n-1} \geq 0$  and  $a_n < 0$ . Let us denote

$$x = \frac{a_1 + a_2 + \dots + a_{n-1}}{n-1}, \quad y = -a_n.$$

We have

$$(n-1)x - y = n, \quad x \geq y > 0.$$

The condition  $x \geq y$  follows from

$$x - y = n - (n-2)x \geq 0.$$

By Jensen's inequality, we have

$$a_1^k + a_2^k + \cdots + a_{n-1}^k \geq (n-1)x^k.$$

In addition,  $a_n^k \geq -y^k$ . Thus, it suffices to show that

$$(n-1)x^k - y^k \geq n.$$

We will use the inequality

$$x^k - y^k > (x-y)^k,$$

which is equivalent to

$$1 - \left(\frac{y}{x}\right)^k > \left(1 - \frac{y}{x}\right)^k.$$

Indeed,

$$1 - \left(\frac{y}{x}\right)^k - \left(1 - \frac{y}{x}\right)^k > 1 - \left(\frac{y}{x}\right) - \left(1 - \frac{y}{x}\right) = 0.$$

Therefore, it suffices to show that

$$(n-2)x^k + (x-y)^k \geq n.$$

By Jensen's inequality, we have

$$(n-2)x^k + (x-y)^k \geq [(n-2)+1] \left[ \frac{(n-2)x + (x-y)}{(n-2)+1} \right]^k = n \left( \frac{n}{n-1} \right)^{k-1} > n.$$

This completes the proof. The equality holds for  $a_1 = a_2 = \cdots = a_n = 1$ .

□

**P 2.131.** If  $a_1, a_2, \dots, a_n \geq \frac{-n}{n-2}$ ,  $n \geq 3$ , then

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} \geq \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}.$$

(Vasile Cîrtoaje, 2012)

**Solution.** Let

$$f(x) = \frac{1}{x^2} - \frac{1}{x}, \quad x \geq \frac{-n}{n-2}, \quad x \neq 0.$$

We need to show that

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq 0.$$

Consider

$$a_1, \dots, a_k < 0, \quad a_{k+1}, \dots, a_n > 0, \quad k = 0, 1, \dots, n-1.$$

Since  $a_1 + \cdots + a_k \geq \frac{-kn}{n-2}$ , it follows that

$$a_{k+1} + \cdots + a_n = n - (a_1 + \cdots + a_k) \leq \frac{n(n+k-2)}{n-2}.$$

Denote

$$x = \frac{1}{a_{k+1}} + \cdots + \frac{1}{a_n}.$$

Since

$$f(a_i) \geq \frac{2(n-2)(n-1)}{n^2}, \quad i = 1, 2, \dots, k,$$

we get

$$\begin{aligned} f(a_1) + f(a_2) + \cdots + f(a_n) &\geq \frac{2k(n-2)(n-1)}{n^2} + f(a_{k+1}) + \cdots + f(a_n) \\ &= \frac{2k(n-2)(n-1)}{n^2} + \left( \frac{1}{a_{k+1}^2} + \cdots + \frac{1}{a_n^2} \right) - x \geq \frac{2k(n-2)(n-1)}{n^2} + \frac{x^2}{n-k} - x. \end{aligned}$$

Thus, we need to show that

$$nx^2 - n(n-k)x + \frac{2k(n-k)(n-2)(n-1)}{n} \geq 0.$$

For  $9k \geq \frac{n^3}{(3n-4)^2}$ , we have

$$\begin{aligned} nx^2 + \frac{2k(n-k)(n-2)(n-1)}{n} - n(n-k)x &\geq \\ &\geq [2\sqrt{2k(n-k)(n-2)(n-1)} - n(n-k)]x \\ &= \sqrt{n-k}[2\sqrt{2k(n-2)(n-1)} - n\sqrt{n-k}]x \geq 0. \end{aligned}$$

So, it remains to study the case  $9k \leq \frac{n^3}{(3n-4)^2}$ , when it suffices to show that

$$x \geq \frac{n(n-k) + \sqrt{(n-k)[n^3 - k(3n-4)^2]}}{2n}.$$

Because

$$x \geq \frac{(n-k)^2}{a_{k+1} + \cdots + a_n} \geq \frac{(n-k)^2(n-2)}{n(n+k-2)},$$

it suffices to show that

$$\frac{(n-k)^2(n-2)}{n+k-2} \geq \frac{n(n-k) + \sqrt{(n-k)[n^3 - k(3n-4)^2]}}{2},$$

that is equivalent to

$$\begin{aligned}\frac{(n-k)[n^2-2n-(3n-4)k]}{n+k-2} &\geq \sqrt{(n-k)[n^3-k(3n-4)^2]}, \\ (n-k)[n^2-2n-(3n-4)k]^2 &\geq (k+n-2)^2[n^3-k(3n-4)^2], \\ 32k(k-1)(n-1)^2(n-2) &\geq 0.\end{aligned}$$

The equality occurs for  $k=0$ , that means for  $a_1=a_2=\dots=a_n=1$ , and for  $k=1$ , that means  $a_1=\frac{-n}{n+2}$  and  $a_2=\dots=a_n=\frac{n}{n-2}$  (or any cyclic permutation).  $\square$

**P 2.132.** If  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) are real numbers such that

$$a_1, a_2, \dots, a_n \geq \frac{-(3n-2)}{n-2}, \quad a_1 + a_2 + \dots + a_n = n,$$

then

$$\frac{1-a_1}{(1+a_1)^2} + \frac{1-a_2}{(1+a_2)^2} + \dots + \frac{1-a_n}{(1+a_n)^2} \geq 0.$$

(Vasile Cîrtoaje, 2014)

**Solution.** Since the inequality holds for  $n=3$  (see P 2.25), consider further that  $n \geq 4$ . Assume that  $a_1 \leq a_2 \leq \dots \leq a_n$  and consider two cases:  $a_1 \geq 0$  and  $a_1 \leq 0$ .

*Case 1:*  $a_1 \geq 0$ . For  $x \geq 0$ , from

$$\frac{1-x}{(1+x)^2} + \frac{x-1}{4} = \frac{(x-1)^2(x+3)}{4(1+x)^2},$$

we get

$$\frac{1-x}{(1+x)^2} \geq \frac{1-x}{4}.$$

Therefore, it suffices to show that

$$\frac{1-a_1}{4} + \frac{1-a_2}{4} + \dots + \frac{1-a_n}{4} \geq 0,$$

which is an identity.

*Case 2:*  $-(3n-2)/(n-2) \leq a_1 \leq 0$ . We can check that the equality holds for

$$a_1 = \frac{-(3n-2)}{n-2}, \quad a_2 = a_3 = \dots = a_n = \frac{n+2}{n-2}.$$

Based on this, define the function

$$f(x) = \frac{1-x}{(1+x)^2} + k_1x + k_2, \quad x \geq \frac{-(3n-2)}{n-2}, \quad x \neq -1,$$



such that

$$f\left(\frac{n+2}{n-2}\right) = f'\left(\frac{n+2}{n-2}\right) = 0.$$

We get

$$\begin{aligned} k_1 &= \frac{(n-4)(n-2)^2}{4n^3}, \\ k_2 &= \frac{(n-2)(-n^2+6n+8)}{4n^3}, \\ f(x) &= \frac{[(n-2)x - n - 2]^2[(n-4)x + 3n - 4]}{4n^3(1+x)^2}. \end{aligned}$$

Since  $f(x) \geq 0$  for  $n \geq 4$  and  $x \geq -(3n-2)/(n-2)$ ,  $x \neq -1$ , we have

$$\frac{1-x}{(1+x)^2} \geq -k_1x - k_2.$$

Based on this result, we get

$$\frac{1-a_2}{(1+a_2)^2} + \cdots + \frac{1-a_n}{(1+a_n)^2} \geq -k_1(a_2 + a_3 + \cdots + a_n) - (n-1)k_2.$$

Thus, it suffices to show that

$$\frac{1-a_1}{(1+a_1)^2} - k_1(a_2 + a_3 + \cdots + a_n) - (n-1)k_2 \geq 0,$$

which is equivalent to

$$\frac{1-a_1}{(1+a_1)^2} - k_1(n-a_1) - (n-1)k_2 \geq 0,$$

$$[(n-2)a_1 + 3n - 2][(n-4)(n-2)a_1^2 - 2(n^2 + 4n - 8)a_1 + n^2 - 2n + 8] \geq 0.$$

The last inequality is clearly true for  $n \geq 4$  and  $-(3n-2)/(n-2) \leq a_1 \leq 0$ . This completes the proof. The equality holds for  $a_1 = a_2 = \cdots = a_n = 1$ , and also for  $a_1 = \frac{-(3n-2)}{n-2}$  and  $a_2 = \cdots = a_n = \frac{n+2}{n-2}$  (or any cyclic permutation).

□

**P 2.133.** Let  $a_1, a_2, \dots, a_n$  be real numbers.

(a) If  $k \geq n$ , then

$$\frac{(a_1 + a_2 + \cdots + a_n + k - n)^2}{(a_1^2 + k - 1)(a_2^2 + k - 1) \cdots (a_n^2 + k - 1)} \leq \frac{1}{k^{n-2}};$$

(b) If  $k \geq \frac{n}{2}$ , then

$$\frac{a_1 + a_2 + \cdots + a_n + k - n}{(a_1^2 + 2k - 1)(a_2^2 + 2k - 1) \cdots (a_n^2 + 2k - 1)} \leq \frac{1}{2(2k)^{n-1}};$$

$$(c) \quad \frac{(a_1 + a_2 + \cdots + a_n)^2}{(a_1^2 + n - 1)(a_2^2 + n - 1) \cdots (a_n^2 + n - 1)} \leq \frac{1}{n^{n-2}};$$

$$(d) \quad \frac{a_1 + a_2 + \cdots + a_n}{(a_1^2 + 2n - 1)(a_2^2 + 2n - 1) \cdots (a_n^2 + 2n - 1)} \leq \frac{1}{2(2n)^{n-1}}.$$

(Vasile Cîrtoaje, 1994)

**Solution** (by Gabriel Dospinescu). a) Assume that

$$a_1^2 \leq \cdots \leq a_j^2 \leq 1 \leq a_{j+1}^2 \leq \cdots \leq a_n^2,$$

where  $0 \leq j \leq n$ . By Bernoulli's inequality and Cauchy-Schwarz inequality, we have

$$\begin{aligned} \frac{1}{k^n} \prod_{i=1}^n (a_i^2 + k - 1) &= \prod_{i=1}^n \left( 1 + \frac{a_i^2 - 1}{k} \right) \\ &= \prod_{i=1}^j \left( 1 + \frac{a_i^2 - 1}{k} \right) \prod_{i=j+1}^n \left( 1 + \frac{a_i^2 - 1}{k} \right) \\ &\geq \left( 1 + \sum_{i=1}^j \frac{a_i^2 - 1}{k} \right) \left( 1 + \sum_{i=j+1}^n \frac{a_i^2 - 1}{k} \right) \\ &= \frac{1}{k^2} [(a_1^2 + \cdots + a_j^2) + (n - j) + (k - n)] [j + (a_{j+1}^2 + \cdots + a_n^2) + (k - n)] \\ &\geq \frac{1}{k^2} [(a_1 + \cdots + a_j) + (a_{j+1} + \cdots + a_n) + (k - n)]^2 \\ &= \frac{1}{k^2} (a_1 + a_2 + \cdots + a_n + k - n)^2. \end{aligned}$$

Thus, the proof is completed. The equality occurs for  $a_1 = a_2 = \cdots = a_n = 1$ .

(b) Replacing  $k$  by  $2k$  ( $k \geq n/2$ ), the inequality in (a) becomes

$$\frac{(a_1 + a_2 + \cdots + a_n + 2k - n)^2}{(a_1^2 + 2k - 1)(a_2^2 + 2k - 1) \cdots (a_n^2 + 2k - 1)} \leq \frac{1}{(2k)^{n-2}}.$$

Thus, we only need to show that

$$4k(a_1 + a_2 + \cdots + a_n + k - n) \leq (a_1 + a_2 + \cdots + a_n + 2k - n)^2,$$

which is equivalent to

$$(a_1 + a_2 + \cdots + a_n - n)^2 \geq 0.$$

The equality occurs for  $a_1 = a_2 = \cdots = a_n = 1$ .

(c) The inequality follows from (a) for  $k = n$ .

(d) The inequality follows from (b) for  $k = n$ .

□

**P 2.134.** Let  $a_1, a_2, \dots, a_n$  be real numbers. If  $k \geq \frac{n}{4}$ , then

$$(a) \quad \frac{(a_1 + a_2 + \cdots + a_n + 2k - n)^2}{(a_1^2 - a_1 + k)(a_2^2 - a_2 + k) \cdots (a_n^2 - a_n + k)} \leq \frac{4}{k^{n-2}};$$

$$(b) \quad \frac{(a_1 + a_2 + \cdots + a_n)^2}{(a_1^2 - a_1 + \frac{n}{2})(a_2^2 - a_2 + \frac{n}{2}) \cdots (a_n^2 - a_n + \frac{n}{2})} \leq \frac{2^n}{n^{n-2}}.$$

(Vasile Cîrtoaje, 1994)

**Solution.** (a) The inequality follows from the inequality (a) in P 2.133 by replacing all  $a_i$  with  $2a_i - 1$  and  $k$  with  $4k$ . The equality occurs for  $a_1 = a_2 = \cdots = a_n = 1$ .

(b) The inequality follows from (a) for  $k = \frac{n}{2}$ .

□

**P 2.135.** Let  $a_1, a_2, \dots, a_n$  be real numbers. If  $k \geq n$ , then

$$(a) \quad \frac{(a_1 + a_2 + \cdots + a_n)^2 + n(k - n)}{(a_1^2 + k - 1)(a_2^2 + k - 1) \cdots (a_n^2 + k - 1)} \leq \frac{n}{k^{n-1}};$$

$$(b) \quad \frac{(a_1 + a_2 + \cdots + a_n)^2 + n^2}{(a_1^2 + 2n - 1)(a_2^2 + 2n - 1) \cdots (a_n^2 + 2n - 1)} \leq \frac{n}{(2n)^{n-1}}.$$

(Vasile Cîrtoaje, 2018)

**Solution.** (a) We will use the induction method. For  $n = 1$ , the inequality is an identity. Assume that the inequality is true for  $n - 1$  variables. Since  $k \geq n > n - 1$ , we have

$$\frac{(n - 1)x^2 + k - n + 1}{(a_1^2 + k - 1)(a_2^2 + k - 1) \cdots (a_{n-1}^2 + k - 1)} \leq \frac{1}{k^{n-2}},$$

where

$$x = \frac{a_1 + a_2 + \cdots + a_{n-1}}{n - 1}.$$

We need to show that

$$\frac{[(n - 1)x + a_n]^2 + n(k - n)}{(a_1^2 + k - 1)(a_2^2 + k - 1) \cdots (a_n^2 + k - 1)} \leq \frac{n}{k^{n-1}}$$

for  $k \geq n$ . Using the induction hypothesis, it suffices to prove that

$$\frac{[(n-1)x + a_n]^2 + n(k-n)}{n(a_n^2 + k-1)} \leq \frac{(n-1)x^2 + k - n + 1}{k},$$

which is equivalent to

$$nx^2a_n^2 + (k-n)(x^2 + a_n^2) - 2kxa_n + n \geq 0,$$

$$n(xa_n - 1)^2 + (k-n)(x - a_n)^2 \geq 0.$$

The equality occurs for  $a_1 = a_2 = \dots = a_n = 1$ .

(b) The inequality follows from (a) for  $k = 2n$ .

□

**P 2.136.** If  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) are real numbers such that

$$a_1 \leq a_2 \leq \dots \leq a_n, \quad a_1 + a_2 + \dots + a_n = 0,$$

then

$$a_1^2 + a_2^2 + \dots + a_n^2 + na_1a_n \leq 0.$$

(Vasile Cîrtoaje, 2009)

**Solution.** For the nontrivial case  $a_1^2 + a_2^2 + \dots + a_n^2 \neq 0$ , let  $a_1 = a < 0$  and  $a_n = b > 0$  be fixed. We claim that for

$$a \leq a_2 \leq \dots \leq a_{n-1} \leq b, \quad a_2 + \dots + a_{n-1} = -a - b,$$

the sum  $S = a_2^2 + \dots + a_{n-1}^2$  has the maximum value when at least  $n-3$  of the numbers  $a_2, \dots, a_{n-1}$  are equal to  $a$  or  $b$ . In the contrary case, if  $a < a_i \leq a_j < b$ , then

$$a_i^2 + a_j^2 < c_i^2 + c_j^2$$

for all  $c_i$  and  $c_j$  such that

$$a \leq c_i < a_i \leq a_j < c_j \leq b, \quad c_i + c_j = a_i + a_j;$$

indeed,

$$a_i^2 + a_j^2 - c_i^2 - c_j^2 = (a_i - c_i)(a_i + c_i) + (a_j - c_j)(a_j + c_j) = (a_i - c_i)(a_i + c_i - a_j - c_j) < 0.$$

This result confirms our claim. Therefore, it suffices to consider the case where at least  $n-3$  of the numbers  $a_2, \dots, a_{n-1}$  are equal to  $a$  or  $b$ . More precisely, assume that  $k$  of  $a_2, \dots, a_{n-1}$  are equal to  $a$  and  $m$  of  $a_2, \dots, a_{n-1}$  are equal to  $b$ , where

$$k + m = n - 3, \quad k, m \geq 0.$$

Therefore, it suffices to show that

$$(k+1)a^2 + c^2 + (m+1)b^2 + (k+m+3)ab \leq 0,$$

where

$$a \leq c \leq b, \quad (k+1)a + c + (m+1)b = 0.$$

We have

$$\begin{aligned} (k+1)a^2 + c^2 + (m+1)b^2 + (k+m+3)ab &= c^2 + (a+b)[(k+1)a + (m+1)b] + ab \\ &= c^2 - (a+b)c + ab = (c-a)(c-b) \leq 0. \end{aligned}$$

The equality holds if and only if

$$a_1, a_2, \dots, a_n \in \{a_1, a_n\}, \quad a_1 + a_2 + \dots + a_n = 0.$$

□



## Chapter 3

# Symmetric Polynomial Inequalities in Nonnegative Variables

### 3.1 Applications

**3.1.** If  $a, b, c$  are positive real numbers, then

$$a^2 + b^2 + c^2 + 2abc + 1 \geq 2(ab + bc + ca).$$

**3.2.** Let  $a, b, c$  be nonnegative real numbers. If  $0 \leq k \leq \sqrt{2}$ , then

$$a^2 + b^2 + c^2 + kabc + 2k + 3 \geq (k + 2)(a + b + c).$$

**3.3.** If  $a, b, c$  are positive real numbers, then

$$abc(a + b + c) + 2(a^2 + b^2 + c^2) + 3 \geq 4(ab + bc + ca).$$

**3.4.** If  $a, b, c$  are positive real numbers, then

$$a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2) + 3 \geq 3(ab + bc + ca).$$

**3.5.** If  $a, b, c$  are positive real numbers, then

$$\left( \frac{a^2 + b^2 + c^2}{3} \right)^3 \geq a^2 b^2 c^2 + (a - b)^2 (b - c)^2 (c - a)^2.$$

**3.6.** If  $a, b, c$  are positive real numbers, then

$$(a + b + c - 3)(ab + bc + ca - 3) \geq 3(abc - 1)(a + b + c - ab - bc - ca).$$

**3.7.** If  $a, b, c$  are positive real numbers, then

$$(a) \quad a^3 + b^3 + c^3 + ab + bc + ca + 9 \geq 5(a + b + c); \quad 7 \quad (b) \quad a^3 + b^3 + c^3 + 4(ab + bc + ca) + 18 \geq 11(a + b + c).$$

**3.8.** If  $a, b, c$  are positive real numbers, then

$$(a) \quad a^3 + b^3 + c^3 + abc + 8 \geq 4(a + b + c);$$

$$(b) \quad 4(a^3 + b^3 + c^3) + 15abc + 54 \geq 27(a + b + c).$$

**3.9.** Let  $a, b, c$  be nonnegative real numbers such that

$$a + b + c = a^2 + b^2 + c^2.$$

Prove that

$$ab + bc + ca \geq a^2b^2 + b^2c^2 + c^2a^2.$$

**3.10.** If  $a, b, c$  are nonnegative real numbers, then

$$(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) \geq (ab + bc + ca)^3.$$

**3.11.** If  $a, b, c$  are nonnegative real numbers, then

$$(2a^2 + bc)(2b^2 + ca)(2c^2 + ab) \geq (ab + bc + ca)^3.$$

**3.12.** Let  $a, b, c$  be nonnegative real numbers such that

$$a + b + c = 2.$$

Prove that

$$(a) \quad (a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \leq (a + b)(b + c)(c + a);$$

$$(b) \quad (a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \leq 2.$$



**3.13.** Let  $a, b, c$  be nonnegative real numbers such that

$$a + b + c = 2.$$

Prove that

$$(a^3 + b^3)(b^3 + c^3)(c^3 + a^3) \leq 2.$$

**3.14.** Let  $a, b, c$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 = 2.$$

Prove that

$$(a^3 + b^3)(b^3 + c^3)(c^3 + a^3) \leq 2.$$

**3.15.** If  $a, b, c$  are nonnegative real numbers such that

$$a + b + c = 2,$$

then

$$(3a^2 - 2ab + 3b^2)(3b^2 - 2bc + 3c^2)(3c^2 - 2ca + 3a^2) \leq 36.$$

**3.16.** Let  $a, b, c$  be nonnegative real numbers such that

$$a + b + c = 3.$$

Prove that

$$(a^2 - 4ab + b^2)(b^2 - 4bc + c^2)(c^2 - 4ca + a^2) \leq 3.$$

**3.17.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = 3,$$

then

$$abc + \frac{12}{ab + bc + ca} \geq 5.$$

**3.18.** If  $a, b, c$  are positive real numbers such that

$$a^2 + b^2 + c^2 = 3,$$

then

$$5(a + b + c) + \frac{3}{abc} \geq 18.$$

**3.19.** If  $a, b, c$  are positive real numbers such that

$$a^2 + b^2 + c^2 = 3,$$

then

$$12 + 9abc \geq 7(ab + bc + ca).$$

**3.20.** If  $a, b, c$  are positive real numbers such that

$$a^2 + b^2 + c^2 = 3,$$

then

$$21 + 18abc \geq 13(ab + bc + ca).$$

**3.21.** If  $a, b, c$  are positive real numbers such that

$$a^2 + b^2 + c^2 = 3,$$

then

$$(2 - ab)(2 - bc)(2 - ca) \geq 1.$$

**3.22.** Let  $a, b, c$  be positive real numbers such that

$$abc = 1.$$

Prove that

$$\left(\frac{a+b+c}{3}\right)^5 \geq \frac{a^2+b^2+c^2}{3}.$$

**3.23.** If  $a, b, c$  are positive real numbers such that

$$abc = 1,$$

then

$$a^3 + b^3 + c^3 + a^{-3} + b^{-3} + c^{-3} + 21 \geq 3(a+b+c)(a^{-1} + b^{-1} + c^{-1}).$$

**3.24.** If  $a, b, c$  are positive real numbers such that

$$abc = 1,$$

then

$$a^2 + b^2 + c^2 - ab - bc - ca \geq \frac{9}{4}(a+b+c-3).$$

**3.25.** If  $a, b, c$  are positive real numbers such that

$$abc = 1,$$

then

$$a^2 + b^2 + c^2 + a + b + c \geq 2(ab + bc + ca).$$

**3.26.** If  $a, b, c$  are positive real numbers such that

$$abc = 1,$$

then

$$a^2 + b^2 + c^2 + 15(ab + bc + ca) \geq 16(a + b + c).$$

**3.27.** If  $a, b, c$  are positive real numbers such that

$$abc = 1,$$

then

$$\frac{2}{a + b + c} + \frac{1}{3} \geq \frac{3}{ab + bc + ca}.$$

**3.28.** Let  $a, b, c$  be the lengths of the sides of a triangle. If

$$a^2 + b^2 + c^2 = 3,$$

then

$$a + b + c \geq 2 + abc.$$

**3.29.** If  $a, b, c$  are positive real numbers such that

$$abc = 1,$$

then

$$\sqrt[3]{(1+a)(1+b)(1+c)} \geq \sqrt[4]{4(1+a+b+c)}.$$

**3.30.** If  $a, b, c$  are positive real numbers, then

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq 18(a^2 - bc)(b^2 - ca)(c^2 - ab).$$

**3.31.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = 3,$$

then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq a^2 + b^2 + c^2.$$

**3.32.** If  $a, b, c$  are positive real numbers such that

$$ab + bc + ca = 3,$$

then

$$a^3 + b^3 + c^3 + 7abc \geq 10.$$

**3.33.** If  $a, b, c$  are nonnegative real numbers such that

$$a^3 + b^3 + c^3 = 3,$$

then

$$a^4b^4 + b^4c^4 + c^4a^4 \leq 3.$$

**3.34.** If  $a, b, c$  are nonnegative real numbers, then

$$(a+1)^2(b+1)^2(c+1)^2 \geq 4(a+b+c)(ab+bc+ca) + 28abc.$$

**3.35.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = 3,$$

then

$$1 + 8abc \geq 9 \min\{a, b, c\}.$$

**3.36.** If  $a, b, c$  are positive real numbers such that

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 + 4abc \geq 5 \min\{a, b, c\}.$$

**3.37.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = abc,$$

then

$$(1 - a)(1 - b)(1 - c) + (\sqrt{3} - 1)^3 \geq 0.$$

**3.38.** If  $a, b, c$  are nonnegative real numbers such that

$$a + b + c = 2,$$

then

$$(a^2 + bc)(b^2 + ca)(c^2 + ab) \leq 1.$$

**3.39.** If  $a, b, c$  are nonnegative real numbers, then

$$(8a^2 + bc)(8b^2 + ca)(8c^2 + ab) \leq (a + b + c)^6.$$

**3.40.** If  $a, b, c$  are positive real numbers such that

$$a^2b^2 + b^2c^2 + c^2a^2 = 3,$$

then

$$a + b + c \geq abc + 2.$$

**3.41.** Let  $a, b, c$  be nonnegative real numbers such that

$$a + b + c = 5.$$

Prove that

$$(a^2 + 3)(b^2 + 3)(c^2 + 3) \geq 192.$$

**3.42.** If  $a, b, c$  are nonnegative real numbers, then

$$a^2 + b^2 + c^2 + abc + 2 \geq a + b + c + ab + bc + ca.$$

**3.43.** If  $a, b, c$  are nonnegative real numbers, then

$$\sum a^3(b + c)(a - b)(a - c) \geq 3(a - b)^2(b - c)^2(c - a)^2.$$

**3.44.** Prove that 2 is the greatest real number  $k$  such that

$$a + b + c + 4abc \geq k(ab + bc + ca)$$

for all  $a, b, c \in [0, 1]$ .

**3.45.** If  $a, b, c \geq \frac{2}{3}$  such that

$$a + b + c = 3,$$

then

$$a^2b^2 + b^2c^2 + c^2a^2 \geq ab + bc + ca.$$

**3.46.** If  $a, b, c$  are positive real numbers such that

$$a \leq 1 \leq b \leq c, \quad a + b + c = 3,$$

then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq a^2 + b^2 + c^2.$$

**3.47.** If  $a, b, c$  are positive real numbers such that

$$a \leq 1 \leq b \leq c, \quad a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c},$$

then

$$a^2 + b^2 + c^2 \leq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

**3.48.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c},$$

then

$$(abc - 1) \left( a^n + b^n + c^n - \frac{1}{a^n} - \frac{1}{b^n} - \frac{1}{c^n} \right) \leq 0$$

for any integer  $n \geq 2$ .

**3.49.** Let  $a, b, c$  be positive real numbers, and let

$$E(a, b, c) = a(a - b)(a - c) + b(b - c)(b - a) + c(c - a)(c - b).$$

Prove that

$$(a) \quad (a + b + c)E(a, b, c) \geq ab(a - b)^2 + bc(b - c)^2 + ca(c - a)^2;$$

$$(b) \quad 2 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) E(a, b, c) \geq (a - b)^2 + (b - c)^2 + (c - a)^2.$$

**3.50.** Let  $a \geq b \geq c$  be nonnegative real numbers. Schur's inequalities of third and fourth degree state that

$$(a) \quad a(a - b)(a - c) + b(b - c)(b - a) + c(c - a)(c - b) \geq 0;$$

$$(b) \quad a^2(a - b)(a - c) + b^2(b - c)(b - a) + c^2(c - a)(c - b) \geq 0.$$

Prove that (a) is sharper than (b) if

$$\sqrt{b} + \sqrt{c} \leq \sqrt{a},$$

and (b) is sharper than (a) if

$$\sqrt{b} + \sqrt{c} \geq \sqrt{a}.$$

**3.51.** If  $a, b, c$  are nonnegative real numbers such that

$$(a + b)(b + c)(c + a) = 8,$$

then

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca.$$

**3.52.** If

$$a, b, c \in [1, 4 + 3\sqrt{2}],$$

then

$$9(ab + bc + ca)(a^2 + b^2 + c^2) \geq (a + b + c)^4.$$

**3.53.** If  $a, b, c$  are nonnegative real numbers such that

$$a + b + c + abc = 4,$$

then

$$(a) \quad a^2 + b^2 + c^2 + 12 \geq 5(ab + bc + ca);$$

$$(b) \quad 3(a^2 + b^2 + c^2) + 13(ab + bc + ca) \geq 48.$$

**3.54.** Let  $a, b, c$  be the lengths of the sides of a triangle. If

$$a + b + c = 3,$$

then

$$a^2b^2 + b^2c^2 + c^2a^2 \geq ab + bc + ca.$$

**3.55.** Let  $a, b, c$  be the lengths of the sides of a triangle. If

$$a^2 + b^2 + c^2 = 3,$$

then

$$ab + bc + ca \geq 1 + 2abc.$$

**3.56.** Let  $a, b, c$  be the lengths of the sides of a triangle. If

$$a + b + c = 3,$$

then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{41}{6} \geq 3(a^2 + b^2 + c^2).$$

**3.57.** Let  $p$  and  $q$  be fixed real numbers such that there exist three nonnegative real numbers  $a, b, c$  satisfying

$$a \geq b \geq c \geq 0, \quad a + b + c = p, \quad ab + bc + ca = q.$$

Then,

(a) the product  $r = abc$  has the minimum value only when  $a = b$  or  $c = 0$ , and the maximum value only when  $b = c$ ;

(b) if  $a, b, c$  are the lengths of the sides of a triangle (non-degenerate or degenerate), then the product  $r = abc$  has the minimum value only when  $a = b \geq c$ , and the maximum value only when  $b = c \geq \frac{a}{2}$  or  $b + c = a$ .



**3.58.** Let  $a \geq b \geq c$  be positive real numbers such that

$$a + b + c = p, \quad abc = r > 0,$$

where  $p$  and  $r$  are fixed positive numbers satisfying  $p^3 \geq 27r$ . Prove that

$$q = ab + bc + ca$$

has the minimum value only when  $b = c$ , and has the maximum value only when  $a = b$ .

**3.59.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = 3,$$

then

$$\frac{9}{abc} + 16 \geq \frac{75}{ab + bc + ca}.$$

**3.60.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = 3,$$

then

$$8 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) + 9 \geq 10(a^2 + b^2 + c^2).$$

**3.61.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = 3,$$

then

$$7(a^2 + b^2 + c^2) + 8(a^2b^2 + b^2c^2 + c^2a^2) + 4a^2b^2c^2 \geq 49.$$

**3.62.** If  $a, b, c$  are nonnegative real numbers, then

$$(a^3 + b^3 + c^3 + abc)^2 \geq 2(a^2 + b^2)(b^2 + c^2)(c^2 + a^2).$$

**3.63.** If  $a, b, c$  are nonnegative real numbers, then

$$[ab(a + b) + bc(b + c) + ca(c + a)]^2 \geq 4(ab + bc + ca)(a^2b^2 + b^2c^2 + c^2a^2).$$

**3.64.** Let  $a, b, c$  be nonnegative real numbers such that

$$ab + bc + ca = 3.$$

Prove that

$$4(a^3 + b^3 + c^3) + 7abc + 125 \geq 48(a + b + c).$$

**3.65.** If  $a, b, c$  are nonnegative real numbers such that

$$a + b + c = 3,$$

then

$$4(a^4 + b^4 + c^4) + 45 \geq 19(a^2 + b^2 + c^2).$$

**3.66.** If

$$a, b, c \geq \frac{13 - 4\sqrt{10}}{3} \approx 0.117$$

such that  $a + b + c = 9$ , then

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq \sqrt{ab + bc + ca}.$$

**3.67.** If  $a, b, c$  are positive real numbers such that  $abc = 1$ , then

$$ab + bc + ca + \frac{6}{a + b + c} \geq 5.$$

**3.68.** Let  $f_n(a, b, c)$  be a symmetric homogeneous polynomial of degree  $n \leq 5$ . Prove that

(a) the inequality  $f_n(a, b, c) \geq 0$  holds for all nonnegative real numbers  $a, b, c$  if and only if  $f_n(a, 1, 1) \geq 0$  and  $f_n(0, b, c) \geq 0$  for all nonnegative real numbers  $a, b, c$ ;

(b) the inequality  $f_n(a, b, c) \geq 0$  holds for all the lengths  $a, b, c$  of the sides of a non-degenerate or degenerate triangle if and only if  $f_n(x, 1, 1) \geq 0$  for  $0 \leq x \leq 2$ , and  $f_n(y + z, y, z) \geq 0$  for all  $y, z \geq 0$ .

**3.69.** If  $a, b, c, d$  are nonnegative real numbers such that  $ab + ac + ad + bc + bd + cd \geq 6$ , then

$$(1 + a^2)(1 + b^2)(1 + c^2)(1 + d^2) \geq (a + b + c + d)^2.$$

**3.70.** Let  $a, b, c$  be nonnegative real numbers. If  $k \leq 2$ , then

$$\sum a(a-b)(a-c)(a-kb)(a-kc) \geq 0.$$

**3.71.** Let  $a, b, c$  be nonnegative real numbers. If  $k \in \mathbf{R}$ , then

$$\sum (b+c)(a-b)(a-c)(a-kb)(a-kc) \geq 0.$$

**3.72.** If  $a, b, c$  are nonnegative real numbers, then

$$\sum a(a-2b)(a-2c)(a-5b)(a-5c) \geq 0.$$

**3.73.** If  $a, b, c$  are the lengths of the sides of a triangle, then

$$a^4 + b^4 + c^4 + 9abc(a+b+c) \leq 10(a^2b^2 + b^2c^2 + c^2a^2).$$

**3.74.** If  $a, b, c$  are the lengths of the sides of a triangle, then

$$3(a^4 + b^4 + c^4) + 7abc(a+b+c) \leq 5 \sum ab(a^2 + b^2).$$

**3.75.** If  $a, b, c$  are the lengths of the sides of a triangle, then

$$\frac{b^2 + c^2 - 6bc}{a} + \frac{c^2 + a^2 - 6ca}{b} + \frac{a^2 + b^2 - 6ab}{c} + 4(a+b+c) \leq 0.$$

**3.76.** Let  $f_6(a, b, c)$  be a sixth degree symmetric homogeneous polynomial written in the form

$$f_6(a, b, c) = Ar^2 + B(p, q)r + C(p, q),$$

where  $A$  is the highest coefficient of  $f_6$  and

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

For  $A \leq 0$ , prove that:

(a) the inequality  $f_6(a, b, c) \geq 0$  holds for all nonnegative real numbers  $a, b, c$  if and only if  $f_6(a, 1, 1) \geq 0$  and  $f_6(0, b, c) \geq 0$  for all nonnegative real numbers  $a, b, c$ ;

(b) the inequality  $f_6(a, b, c) \geq 0$  holds for all lengths  $a, b, c$  of the sides of a non-degenerate or degenerate triangle if and only if  $f_6(x, 1, 1) \geq 0$  for  $0 \leq x \leq 2$ , and  $f_6(y + z, y, z) \geq 0$  for all  $y, z \geq 0$ .

**3.77.** If  $a, b, c$  are nonnegative real numbers, then

$$\sum a(b+c)(a-b)(a-c)(a-2b)(a-2c) \geq (a-b)^2(b-c)^2(c-a)^2.$$

**3.78.** Let  $a, b, c$  be nonnegative real numbers.

(a) If  $2 \leq k \leq 6$ , then

$$\sum a(a-b)(a-c)(a-kb)(a-kc) + \frac{4(k-2)(a-b)^2(b-c)^2(c-a)^2}{a+b+c} \geq 0.$$

(b) If  $k \geq 6$ , then

$$\sum a(a-b)(a-c)(a-kb)(a-kc) + \frac{(k+2)^2(a-b)^2(b-c)^2(c-a)^2}{4(a+b+c)} \geq 0.$$

**3.79.** If  $a, b, c$  are nonnegative real numbers, then

$$(3a^2 + 2ab + 3b^2)(3b^2 + 2bc + 3c^2)(3c^2 + 2ca + 3a^2) \geq 8(a^2 + 3bc)(b^2 + 3ca)(c^2 + 3ab).$$

**3.80.** Let  $a, b, c$  be nonnegative real numbers such that

$$a + b + c = 2.$$

If

$$\frac{-2}{3} \leq k \leq \frac{11}{8},$$

then

$$(a^2 + kab + b^2)(b^2 + kbc + c^2)(c^2 + kca + a^2) \leq k + 2.$$

**3.81.** Let  $a, b, c$  be nonnegative real numbers such that

$$a + b + c = 2.$$

Prove that

$$(2a^2 + bc)(2b^2 + ca)(2c^2 + ab) \leq 4.$$

**3.82.** Let  $a, b, c$  be nonnegative real numbers, no two of which are zero. Then,

$$\sum (a-b)(a-c)(a-2b)(a-2c) \geq \frac{5(a-b)^2(b-c)^2(c-a)^2}{ab+bc+ca}.$$

**3.83.** If  $a, b, c$  are positive real numbers such that

$$abc = 1,$$

then

$$ab + bc + ca + \frac{50}{a + b + c + 5} \geq \frac{37}{4}.$$

**3.84.** If  $a, b, c$  are positive real numbers, then

$$(a + b + c - 3) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 3 \right) + abc + \frac{1}{abc} \geq 2.$$

**3.85.** If  $a, b, c$  are positive real numbers such that

$$abc = 1,$$

then

$$(a) \quad \frac{3}{7} \left( ab + bc + ca - \frac{2}{3} \right) \geq \sqrt{\frac{2}{3}(a + b + c) - 1};$$

$$(b) \quad ab + bc + ca - 3 \geq \frac{46}{27}(\sqrt{a + b + c - 2} - 1).$$

**3.86.** Let  $a, b, c$  be positive real numbers.

(a) If  $abc = 2$ , then

$$(a + b + c - 3)^2 + 1 \geq \frac{a^2 + b^2 + c^2}{3}.$$

(b) If  $abc = \frac{1}{2}$ , then

$$a^2 + b^2 + c^2 + 3(3 - a - b - c)^2 \geq 3.$$

**3.87.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = 3,$$

then

$$4 \left( \frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \right) + 9abc \geq 21.$$

**3.88.** If  $a, b, c$  are nonnegative real numbers such that

$$ab + bc + ca = abc + 2,$$

then

$$a^2 + b^2 + c^2 + abc \geq 4.$$

**3.89.** If  $a, b, c$  are nonnegative real numbers such that  $a + b + c = 3$ , then

$$(a + b)(b + c)(c + a) \geq (a + bc)(b + ca)(c + ab).$$

**3.90.** Let  $a, b, c$  be positive numbers such that

$$a + b + c \leq 3\sqrt[4]{abc}.$$

Prove that

$$a^2 + b^2 + c^2 \leq 3.$$

**3.91.** If  $a, b, c$  are positive real numbers, then

$$\left(\frac{b+c}{a} - 2 - \sqrt{2}\right)^2 + \left(\frac{c+a}{b} - 2 - \sqrt{2}\right)^2 + \left(\frac{a+b}{c} - 2 - \sqrt{2}\right)^2 \geq 6.$$

**3.92.** If  $a, b, c$  are positive real numbers, then

$$2(a^3 + b^3 + c^3) + 9(ab + bc + ca) + 39 \geq 24(a + b + c).$$

**3.93.** If  $a, b, c$  are positive real numbers such that  $a^2 + b^2 + c^2 = 3$ , then

$$a^3 + b^3 + c^3 - 3 \geq |(a - b)(b - c)(c - a)|.$$

**3.94.** Let  $a, b, c$  be nonnegative real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove that

$$1 - abc \geq \frac{5}{3} \min\{(a - b)^2, (b - c)^2, (c - a)^2\}.$$

**3.95.** If  $a, b, c$  are nonnegative real numbers, then

$$a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 \geq 2|a^3b + b^3c + c^3a - ab^3 - bc^3 - ca^3|.$$

**3.96.** If  $a, b, c$  are nonnegative real numbers, then

$$a^4 + b^4 + c^4 - abc(a + b + c) \geq 2\sqrt{2} |a^3b + b^3c + c^3a - ab^3 - bc^3 - ca^3|.$$

**3.97.** If  $a, b, c$  are nonnegative real numbers such that  $a + b + c = 3$ , then

$$(a^3b + b^3c + c^3a - 3abc)(ab^3 + bc^3 + ca^3 - 3abc) \geq (a^2b^2 + b^2c^2 + c^2a^2 - 3abc)^2.$$

**3.98.** If  $a, b, c \geq -5$  such that

$$a + b + c = 3,$$

then

$$\frac{1-a}{1+a+a^2} + \frac{1-b}{1+b+b^2} + \frac{1-c}{1+c+c^2} \geq 0.$$

**3.99.** Let  $a, b, c \neq \frac{1}{k}$  be nonnegative real numbers such that

$$a + b + c = 3.$$

If  $k \geq \frac{4}{3}$ , then

$$\frac{1-a}{(1-ka)^2} + \frac{1-b}{(1-kb)^2} + \frac{1-c}{(1-kc)^2} \geq 0.$$

**3.100.** If  $a, b, c$  are positive real numbers such that

$$abc = 1,$$

then

$$3(2a^2 + 1)(2b^2 + 1)(2c^2 + 1) \leq (a + b + c)^4.$$

**3.101.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = \sqrt{3},$$

then

$$(3\sqrt{3} - 5) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq a^2 + b^2 + c^2.$$

**3.102.** If  $a, b, c \geq 1$  such that

$$a + b + c = 4,$$

then

$$12 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 5(a^2 + b^2 + c^2).$$

**3.103.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = 3, \quad c \leq \frac{15}{32},$$

then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq a^2 + b^2 + c^2.$$

**3.104.** If  $a \geq b \geq c \geq 0$  and  $ab + bc + ca = 3$ , then

$$(a) \quad b + c \leq 2;$$

$$(b) \quad b^2 + bc + c^2 \leq 3.$$

**3.105.** If  $a, b, c \in \left[0, 1 + \frac{1}{\sqrt{2}}\right]$  and  $a^2 + b^2 + c^2 = 3$ , then

$$a + b + c \geq abc + 2.$$

**3.106.** Let  $a, b, c \geq \frac{1}{6}$  be real numbers such that  $a^2 + b^2 + c^2 = 3$ . Then,

$$a + b + c \geq abc + 2.$$



**3.107.** If  $a, b, c$  are nonnegative real numbers such that

$$ab + bc + ca + 6abc = 9,$$

then

$$2(a + b + c) \geq ab + bc + ca + 3.$$

**3.108.** If  $a, b, c$  are nonnegative real numbers such that

$$ab + bc + ca + abc = 4,$$

then

$$4(a + b + c) + 3a^2b^2c^2 \geq 15.$$

**3.109.** If  $a, b, c \in \left[0, \frac{5}{3}\right]$  such that  $a + b + c = 3$ , then

$$(a + b)(b + c)(c + a) \geq 8\sqrt[3]{abc}.$$

**3.110.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$a + b + c + \left(1 - \frac{1}{\sqrt{3}}\right)(a - c)^2 \geq 3.$$

**3.111.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \leq \sqrt{\frac{2}{3}}(a - c).$$

**3.112.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \leq \frac{7}{10}(a - c)^2.$$

**3.113.** If  $a \geq b \geq c \geq \frac{1}{3}$  and  $a^2 + b^2 + c^2 = 3$ , then

$$1 - abc \leq \frac{11}{18} (a - c).$$

**3.114.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - \sqrt{abc} \leq \frac{2}{3} (a - c)^2.$$

**3.115.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \leq \frac{2}{3} a(a - c)^2.$$

**3.116.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \leq \frac{1}{9} (5a + c)(a - c)^2.$$

**3.117.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \geq \frac{2}{3} (b - c)^2.$$

**3.118.** Let  $a, b, c$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 = 3.$$

If  $a \geq b \geq c$ , then

$$1 - abc \geq \frac{2}{3} (1 + \sqrt{2})(a - b)(b - c).$$

**3.119.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

- (a)  $1 - abc \geq 2b(a - b)(b - c);$
- (b)  $1 - abc \geq (a - c)(a - b)(b - c);$
- (c)  $1 - abc \geq a(a - b)(b - c);$
- (d)  $1 - abc \geq (a + c)(a - b)(b - c).$

**3.120.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

- (a)  $1 - abc \geq \frac{2}{3} b(a - b)^2;$
- (b)  $1 - abc \geq \frac{2}{27} (2a + 7b)(a - b)^2.$

**3.121.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

- (a)  $1 - abc \geq \frac{1}{3} (b + c)(b - c)^2;$
- (b)  $1 - abc \geq \frac{2}{27} (7b + 2c)(b - c)^2.$

**3.122.** If  $a, b, c$  are nonnegative real numbers such that  $a^2 + b^2 + c^2 = 3$ , then

- (a)  $1 - \sqrt{abc} \geq (a - b)(b - c);$
- (b)  $1 - \sqrt[3]{a^2 b^2 c^2} \geq \frac{4}{3} (a - b)(b - c).$

**3.123.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \geq \frac{2}{3}b(\sqrt{a} - \sqrt{c})^2.$$

**3.124.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \geq \frac{8}{3}b(\sqrt{a} - \sqrt{b})^2.$$

**3.125.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \geq \frac{1}{3}(a + 3c)(\sqrt{a} - \sqrt{c})^2.$$

**3.126.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \geq \frac{2}{3}(a + 3c)(\sqrt{b} - \sqrt{c})^2.$$

**3.127.** If  $a \geq b \geq c > 0$ , then

$$(a) \quad a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq 12a^2c^2(b - c)^2;$$

$$(b) \quad a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq 10a^3c(b - c)^2.$$

**3.128.** If  $a \geq b \geq c > 0$ , then

$$a + b + c - 3\sqrt[3]{abc} \geq \frac{3(a - c)^2}{4(a + b + c)}.$$

**3.129.** If  $a \geq b \geq c > 0$ , then

$$\begin{aligned} \text{(a)} \quad & a + b + c - 3\sqrt[3]{abc} \geq \frac{3(b-c)^2}{4b+5c}; \\ \text{(b)} \quad & a + b + c - 3\sqrt[3]{abc} \geq \frac{25(b-c)^2}{7(3b+11c)}. \end{aligned}$$

**3.130.** If  $a \geq b \geq c > 0$ , then

$$\begin{aligned} \text{(a)} \quad & a + b + c - 3\sqrt[3]{abc} \geq \frac{3(a-b)^2}{5a+4b}; \\ \text{(b)} \quad & a + b + c - 3\sqrt[3]{abc} \geq \frac{64(a-b)^2}{7(11a+24b)}. \end{aligned}$$

**3.131.** If  $a \geq b \geq c \geq 0$ , then

$$\frac{(a-c)^2}{2(a+c)} \leq a + b + c - 3\sqrt[3]{abc} \leq \frac{2(a-c)^2}{a+5c}.$$

**3.132.** Let  $a, b, c$  be nonnegative real numbers such that  $a \geq b \geq c$ . Prove that

$$\begin{aligned} \text{(a)} \quad & a^3 + b^3 + c^3 - 3abc \geq 2(2b-a-c)^3; \\ \text{(b)} \quad & a^3 + b^3 + c^3 - 3abc \geq (a-2b+c)^3. \end{aligned}$$

**3.133.** Let  $a, b, c$  be nonnegative real numbers such that  $a \geq b \geq c$ . Prove that

$$\begin{aligned} \text{(a)} \quad & a^3 + b^3 + c^3 - 3abc \geq 3(a^2 - b^2)(b-c); \\ \text{(b)} \quad & a^3 + b^3 + c^3 - 3abc \geq \frac{9}{2}(a-b)(b^2 - c^2). \end{aligned}$$

**3.134.** Let  $a, b, c$  be nonnegative real numbers such that  $a = \max\{a, b, c\}$ . Prove that

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq 2(b^4 + c^4 + 4b^2c^2)(b-c)^2.$$

**3.135.** Let  $a, b, c$  be nonnegative real numbers such that  $a = \max\{a, b, c\}$ . Prove that

$$a^2 + b^2 + c^2 \geq \frac{9abc}{a+b+c} + \frac{5}{3}(b-c)^2.$$

**3.136.** Let  $a, b, c$  be nonnegative real numbers such that  $a + b > 0$ . Prove that

$$abc \geq (b + c - a)(c + a - b)(a + b - c) + \frac{ab(a - b)^2}{a + b}.$$

**3.137.** Let  $a, b, c$  be nonnegative real numbers such that  $a \geq b \geq c$ . Prove that

$$(a) \quad abc \geq (b + c - a)(c + a - b)(a + b - c) + \frac{2ab(a - b)^2}{a + b};$$

$$(b) \quad abc \geq (b + c - a)(c + a - b)(a + b - c) + \frac{27b(a - b)^4}{4a^2}.$$

**3.138.** Let  $a, b, c$  be nonnegative real numbers such that  $a + b > 0$ . Prove that

$$\sum a^2(a - b)(a - c) \geq a^2b^2 \left( \frac{a - b}{a + b} \right)^2.$$

**3.139.** Let  $a, b, c$  be nonnegative real numbers such that  $a \geq b \geq c$ . Prove that

$$(a) \quad \sum a^2(a - b)(a - c) \geq 4a^2b^2 \left( \frac{a - b}{a + b} \right)^2;$$

$$(b) \quad \sum a^2(a - b)(a - c) \geq \frac{27b(a - b)^4}{4a}.$$

**3.140.** If  $a, b, c$  are nonnegative real numbers such that  $ab + bc + ca = 3$ , then

$$(a) \quad a^2 + b^2 + c^2 - a - b - c \geq \frac{5}{8}(a - c)^2;$$

$$(b) \quad a^2 + b^2 + c^2 - a - b - c \geq \frac{5}{2} \min\{(a - b)^2, (b - c)^2, (c - a)^2\}.$$

**3.141.** If  $a, b, c$  are nonnegative real numbers such that  $ab + bc + ca = 3$ , then

$$\frac{a^3 + b^3 + c^3}{a + b + c} \geq 1 + \frac{5}{9}(a - c)^2.$$

**3.142.** If  $a, b, c$  are nonnegative real numbers such that

$$a \geq b \geq c, \quad ab + bc + ca = 3,$$

then

$$(a) \quad \frac{a^3 + b^3 + c^3}{a + b + c} \geq 1 + \frac{7}{9}(a - b)^2;$$

$$(b) \quad \frac{a^3 + b^3 + c^3}{a + b + c} \geq 1 + \frac{2}{3}(b - c)^2.$$

$$(c) \quad \frac{a^3 + b^3 + c^3}{a + b + c} \geq 1 + \frac{7}{3} \min\{(a - b)^2, (b - c)^2\}.$$

**3.143.** If  $a, b, c$  are nonnegative real numbers such that  $ab + bc + ca = 3$ , then

$$a^4 + b^4 + c^4 - a^2 - b^2 - c^2 \geq \frac{11}{4}(a - c)^2.$$

**3.144.** If  $a, b, c$  are nonnegative real numbers such that

$$a \geq b \geq c, \quad ab + bc + ca = 3,$$

then

$$(a) \quad a^4 + b^4 + c^4 - a^2 - b^2 - c^2 \geq \frac{11}{3}(a - b)^2;$$

$$(b) \quad a^4 + b^4 + c^4 - a^2 - b^2 - c^2 \geq \frac{10}{3}(b - c)^2.$$

**3.145.** Let

$$F(a, b, c) = 3(a^2 + b^2 + c^2) - (a + b + c)^2,$$

where  $a, b, c$  are positive real numbers such that  $a = \min\{a, b, c\}$  and  $a^2(b^2 + c^2) \geq 2$ . Then,

$$F(a, b, c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

**3.146.** Let

$$F(a, b, c) = a + b + c - 3\sqrt[3]{abc},$$

where  $a, b, c$  are positive real numbers. If  $\min\{a, b, c\} \geq \frac{1}{abc}$ , then

$$F(a, b, c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

**3.147.** Let

$$F(a, b, c) = a + b + c - 3\sqrt[3]{abc},$$

where  $a, b, c$  are positive real numbers such that  $a = \min\{a, b, c\}$  and  $a(b + c) \geq 2$ . Then,

$$F(a, b, c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

**3.148.** Let

$$F(a, b, c, d) = a + b + c + d - 4\sqrt[4]{abcd},$$

where  $a, b, c, d$  are positive real numbers. If

$$\min\{a^2, b^2, c^2, d^2\} \geq \frac{1}{abcd},$$

then

$$F(a, b, c, d) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\right).$$

**3.149.** Let

$$F(a, b, c, d) = a + b + c + d - 4\sqrt[4]{abcd},$$

where  $a, b, c, d$  are positive real numbers such that  $a(b + c + d) \geq 3$ . Then

$$F(a, b, c, d) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\right).$$

**3.150.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 1.$$

Prove that

$$(1 - a)(1 - b)(1 - c)(1 - d) \geq abcd.$$

**3.151.** Let  $a, b, c, d$  and  $x$  be positive real numbers such that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} = \frac{4}{x^2}.$$

If  $x \geq 2$ , then

$$(a - 1)(b - 1)(c - 1)(d - 1) \geq (x - 1)^4.$$



**3.152.** If  $a, b, c, d$  are positive real numbers, then

$$\frac{(1+a^3)(1+b^3)(1+c^3)(1+d^3)}{(1+a^2)(1+b^2)(1+c^2)(1+d^2)} \geq \frac{1+abcd}{2}.$$

**3.153.** Let  $a, b, c, d$  be positive real numbers such that

$$a + b + c + d = 4.$$

Prove that

$$\left(a + \frac{1}{a} - 1\right) \left(b + \frac{1}{b} - 1\right) \left(c + \frac{1}{c} - 1\right) \left(d + \frac{1}{d} - 1\right) + 3 \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

**3.154.** If  $a, b, c, d$  are nonnegative real numbers, then

$$4(a^3 + b^3 + c^3 + d^3) + 15(abc + bcd + cda + dab) \geq (a + b + c + d)^3.$$

**3.155.** Let  $a, b, c, d$  be positive real numbers such that

$$a + b + c + d = 4.$$

Prove that

$$1 + 2(abc + bcd + cda + dab) \geq 9 \min\{a, b, c, d\}.$$

**3.156.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a + b + c + d = 4.$$

Prove that

$$5(a^2 + b^2 + c^2 + d^2) \geq a^3 + b^3 + c^3 + d^3 + 16.$$

**3.157.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a + b + c + d = 4.$$

Prove that

$$3(a^2 + b^2 + c^2 + d^2) + 4abcd \geq 16.$$

**3.158.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a + b + c + d = 4.$$

Prove that

$$27(abc + cd + cda + dab) \leq 44abcd + 64.$$

**3.159.** Let  $a, b, c, d$  be positive real numbers such that

$$a + b + c + d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

Prove that

$$(1 - abcd) \left( a^2 + b^2 + c^2 + d^2 - \frac{1}{a^2} - \frac{1}{b^2} - \frac{1}{c^2} - \frac{1}{d^2} \right) \geq 0.$$

**3.160.** Let  $a, b, c, d$  be positive real numbers such that

$$a + b + c + d = 1.$$

Prove that

$$(1 - a)(1 - b)(1 - c)(1 - d) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \geq \frac{81}{16}.$$

**3.161.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a + b + c + d = a^3 + b^3 + c^3 + d^3 = 2.$$

Prove that

$$a^2 + b^2 + c^2 + d^2 \geq \frac{7}{4}.$$

**3.162.** Let  $a, b, c, d \in (0, 4]$  such that

$$abcd = 1.$$

Prove that

$$(1 + 2a)(1 + 2b)(1 + 2c)(1 + 2d) \geq (5 - 2a)(5 - 2b)(5 - 2c)(5 - 2d).$$

**3.163.** If  $a, b, c, d \in \left[0, 1 + \frac{1}{\sqrt{6}}\right]$  and  $a^2 + b^2 + c^2 + d^2 = 4$ , then

$$a + b + c + d \geq abcd + 3.$$

**3.164.** Let  $a, b, c, d$  be positive real numbers such that

$$(a + b + c + d) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \leq (1 + \sqrt{10})^2.$$

Prove that any three of  $a, b, c, d$  are the lengths of the sides of a triangle (non-degenerate or degenerate).

**3.165.** Let  $a, b, c, d$  be positive real numbers such that

$$(a + b + c + d) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \leq \frac{119}{6}.$$

Prove that there exist three of  $a, b, c, d$  which are the lengths of the sides of a triangle (non-degenerate or degenerate).

**3.166.** Let  $a, b, c, d$  be positive real numbers such that

$$3(a + b + c + d)^2 \geq 11(a^2 + b^2 + c^2 + d^2).$$

Prove that any three of  $a, b, c, d$  are the lengths of the sides of a triangle (non-degenerate or degenerate).

**3.167.** Let  $a, b, c, d$  be positive real numbers such that

$$15(a + b + c + d)^2 \geq 49(a^2 + b^2 + c^2 + d^2).$$

Prove that there exist three of  $a, b, c, d$  which are the lengths of the sides of a triangle (non-degenerate or degenerate).

**3.168.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

If  $a \geq b \geq c \geq d$ , then

$$a + b + c + d + (2 - \sqrt{2})(a - d)^2 \geq 4.$$

**3.169.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

If  $a \geq b \geq c \geq d$ , then

$$1 - abcd \leq \frac{\sqrt{3}}{2} (a - d).$$

**3.170.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

If  $a \geq b \geq c \geq d$ , then

$$1 - \sqrt{abcd} \leq \frac{3}{4}(a - d)^2.$$

**3.171.** If  $a, b, c, d$  are nonnegative real numbers such that  $a \geq b \geq c \geq d$  and

$$a^2 + b^2 + c^2 + d^2 = 4,$$

then

$$1 - (abcd)^{3/4} \leq \frac{3}{4}(a - d)^2.$$

**3.172.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

If  $a \geq b \geq c \geq d$ , then

$$(a) \quad 1 - \sqrt{abcd} \geq \frac{1}{2}(b - c)^2;$$

$$(b) \quad 1 - \sqrt{abcd} \geq \frac{1}{4}(a - d)^2.$$

**3.173.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

If  $a \geq b \geq c \geq d$ , then

$$1 - abcd \geq \frac{3}{4}(c - d)^2.$$

**3.174.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

If  $a \geq b \geq c \geq d$ , then

$$(a) \quad 1 - abcd \geq (a - b)(c - d);$$

$$(b) \quad 1 - abcd \geq \frac{1 + \sqrt{3}}{2} (a - b)(c - d).$$

**3.175.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

If  $a \geq b \geq c \geq d$ , then

$$1 - abcd \geq 3(a - b)(b - c)(c - d)(a - d).$$

**3.176.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

If  $a \geq b \geq c \geq d$ , then

$$(a) \quad 1 - \sqrt{abcd} \geq \frac{1}{3}(b - d)^2;$$

$$(b) \quad 1 - (abcd)^{3/4} \geq \frac{1}{2}(b - d)^2.$$

**3.177.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a^4 + b^4 + c^4 + d^4 = 4.$$

If  $a \geq b \geq c \geq d$ , then

$$(a) \quad 1 - \sqrt{abcd} \geq \frac{1}{2}(ac - bd)^2;$$

$$(b) \quad 1 - abcd \geq \frac{1}{\sqrt{2}} (ac - bd)^2.$$

**3.178.** If  $a, b, c, d$  are nonnegative real numbers such that  $a \geq b \geq c \geq d$  and

$$a^4 + b^4 + c^4 + d^4 = 4,$$

then

$$1 - abcd \geq \frac{3}{4}(ad - bc)^2.$$

**3.179.** If  $a, b, c, d$  are nonnegative real numbers such that  $a \geq b \geq c \geq d$  and

$$a + b + c + d = 4,$$

then

$$\begin{aligned} (a) \quad & \frac{a^4 + b^4 + c^4 + d^4}{4} - abcd \geq 2(b - c)^2, \\ (b) \quad & \frac{a^4 + b^4 + c^4 + d^4}{4} - abcd \geq \frac{3}{2}(a - b)^2; \\ (c) \quad & \frac{a^4 + b^4 + c^4 + d^4}{4} - abcd \geq \frac{4}{3}(a - c)^2; \\ (d) \quad & \frac{a^4 + b^4 + c^4 + d^4}{4} - abcd \geq \frac{4}{3}(c - d)^2. \end{aligned}$$

**3.180.** If  $a, b, c, d$  are nonnegative real numbers such that  $a \geq b \geq c \geq d$  and  $a + d \geq b + c$ , then

$$a + b + c + d - 4\sqrt[4]{abcd} \leq 2 \left( \sqrt{a} - \sqrt{d} \right)^2.$$

**3.181.** If  $a, b, c, d$  are nonnegative real numbers such that  $a \geq b \geq c \geq d$  and

$$a + kd \geq b + c, \quad k = (1 + \sqrt{2})^4 \approx 33.970,$$

then

$$a + b + c + d - 4\sqrt[4]{abcd} \leq 2 \left( \sqrt{a} - \sqrt{d} \right)^2.$$

**3.182.** If  $a, b, c, d$  are nonnegative real numbers such that  $a \geq b \geq c \geq d$  and

$$a + d \geq 2c,$$

then

$$a + b + c + d - 4\sqrt[4]{abcd} \leq \frac{5}{2} \left( \sqrt{a} - \sqrt{d} \right)^2.$$

**3.183.** If  $a, b, c, d$  are nonnegative real numbers such that  $a \geq b \geq c \geq d$  and

$$a + b + c + d = 4,$$

then

$$(a - d)^2 \leq \frac{a^4 + b^4 + c^4 + d^4}{4} - abcd \leq 4(a - d)^2.$$

**3.184.** If  $a \geq b \geq c \geq d \geq 0$ , then

- (a)  $a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{3}{2} \left( \sqrt{b} - 2\sqrt{c} + \sqrt{d} \right)^2;$
- (b)  $a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{2}{9} \left( 3\sqrt{b} - 2\sqrt{c} - \sqrt{d} \right)^2;$
- (c)  $a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{4}{19} \left( 3\sqrt{b} - \sqrt{c} - 2\sqrt{d} \right)^2;$
- (d)  $a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{3}{8} \left( \sqrt{b} - 3\sqrt{c} + 2\sqrt{d} \right)^2;$
- (e)  $a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{1}{2} \left( 2\sqrt{b} - 3\sqrt{c} + \sqrt{d} \right)^2;$
- (f)  $a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{1}{6} \left( 2\sqrt{b} + \sqrt{c} - 3\sqrt{d} \right)^2.$

**3.185.** If  $a \geq b \geq c \geq d \geq 0$ , then

- (a)  $a + b + c + d - 4\sqrt[4]{abcd} \geq \left( \sqrt{a} - \sqrt{d} \right)^2;$
- (b)  $a + b + c + d - 4\sqrt[4]{abcd} \geq 2 \left( \sqrt{b} - \sqrt{c} \right)^2;$
- (c)  $a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{4}{3} \left( \sqrt{b} - \sqrt{d} \right)^2;$
- (d)  $a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{3}{2} \left( \sqrt{c} - \sqrt{d} \right)^2.$

**3.186.** If  $a, b, c, d$  are positive real numbers such that

$$abcd \geq 1, \quad a \geq b \geq c \geq d, \quad ad \geq bc,$$

then

$$a + b + c + d \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

**3.187.** If  $a \geq b \geq c \geq d \geq 0$ , then

$$\frac{(a-d)^2}{2(a+d)} \leq a+b+c+d-4\sqrt[4]{abcd} \leq \frac{3(a-d)^2}{a+5d}.$$

**3.188.** If  $a \geq b \geq c \geq d \geq e \geq 0$ , then

$$a+b+c+d+e-5\sqrt[5]{abcde} \geq 2\left(\sqrt{b}-\sqrt{d}\right)^2.$$

**3.189.** Let  $a, b, c, d, e$  be nonnegative real numbers.

(a) If  $a+b+c=3(d+e)$ , then

$$4(a^4+b^4+c^4+d^4+e^4) \geq (a^2+b^2+c^2+d^2+e^2)^2.$$

(b) If  $a+b+c=d+e$ , then

$$12(a^4+b^4+c^4+d^4+e^4) \leq 7(a^2+b^2+c^2+d^2+e^2)^2.$$

**3.190.** Let  $a, b, c, d, e$  be nonnegative real numbers such that

$$a+b+c+d+e=5.$$

Prove that

$$a^4+b^4+c^4+d^4+e^4+150 \leq 31(a^2+b^2+c^2+d^2+e^2).$$

**3.191.** Let  $a, b, c, d, e$  be positive real numbers such that

$$a^2+b^2+c^2+d^2+e^2=5.$$

Prove that

$$abcde(a^4+b^4+c^4+d^4+e^4) \leq 5.$$

**3.192.** Let  $a, b, c, d, e$  be positive real numbers such that

$$a+b+c+d+e=5.$$

Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{20}{a^2+b^2+c^2+d^2+e^2} \geq 9.$$



**3.193.** If  $a, b, c, d, e \geq 1$ , then

$$\begin{aligned} & \left(a + \frac{1}{a}\right) \left(b + \frac{1}{b}\right) \left(c + \frac{1}{c}\right) \left(d + \frac{1}{d}\right) \left(e + \frac{1}{e}\right) + 68 \geq \\ & \geq 4(a + b + c + d + e) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right). \end{aligned}$$

**3.194.** Let  $a, b, c, d, e, f$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 = 6.$$

If  $a \geq b \geq c \geq d \geq e \geq f$ , then

$$1 - abcdef \leq \frac{3}{2}(a - f)^2.$$

**3.195.** If  $a \geq b \geq c \geq d \geq e \geq f \geq 0$ , then

$$a + b + c + d + e + f - 6\sqrt[6]{abcdef} \geq 2 \left(\sqrt{b} - \sqrt{e}\right)^2.$$

**3.196.** If  $a, b, c, d, e, f$  are nonnegative real numbers such that

$$a \geq b \geq c \geq d \geq e \geq f,$$

then

$$(a + b + c + d + e + f)^2 \geq 8(ac + bd + ce + df).$$

**3.197.** If  $a_1 \geq a_2 \geq \cdots \geq a_8 \geq 0$ , then

$$a_1 + a_2 + \cdots + a_8 - 8\sqrt[8]{a_1 a_2 \cdots a_8} \geq 3(\sqrt{a_6} - \sqrt{a_7})^2.$$

**3.198.** Let  $a, b, c$  and  $x, y, z$  be real numbers.

(a) If  $ab + bc + ca > 0$ , then

$$[(b + c)x + (c + a)y + (a + b)z]^2 \geq 4(ab + bc + ca)(xy + yz + zx);$$

(b) If  $a, b, c \geq 0$ , then

$$[(b + c)x + (c + a)y + (a + b)z]^2 \geq 4(a + b + c)(ayz + bzx + cxy).$$

**3.199.** If  $a, b, c$  and  $x, y, z$  are nonnegative real numbers such that

$$x^3 + y^3 + z^3 = a^3 + b^3 + c^3,$$

then

$$(a + b + c)(x + y + z) \geq ab + bc + ca + xy + yz + zx.$$

**3.200.** Let  $a, b, c$  and  $x, y, z$  be positive real numbers such that

$$(a + b + c)(x + y + z) = (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = 4.$$

Prove that

$$abcxyz < \frac{1}{36}.$$

**3.201.** Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be nonnegative real numbers such that

$$a_1^2 + a_2^2 + \dots + a_n^2 = b_1^2 + b_2^2 + \dots + b_n^2.$$

Then, for  $n = 3$  and  $n = 4$ , the following inequalities holds:

$$(n - 1)(a_1 + a_2 + \dots + a_n)(b_1 + b_2 + \dots + b_n) \geq n \left( \sum_{i < j} a_i a_j + \sum_{i < j} b_i b_j \right).$$

**3.202.** Let  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) be positive real numbers such that

$$a_1 + a_2 + \dots + a_n = a_1^2 + a_2^2 + \dots + a_n^2 = n - 1.$$

Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \geq \frac{n^2(2n - 3)}{2(n - 1)(n - 2)}.$$

**3.203.** Let  $a_1, a_2, \dots, a_n$  be positive real numbers such that

$$a_1 + a_2 + \dots + a_n = n.$$

Prove that

$$n^2 \left( \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} - n \right) \geq 4(n - 1)(a_1^2 + a_2^2 + \dots + a_n^2 - n).$$

**3.204.** If  $a_1, a_2, \dots, a_n$  are positive real numbers such that

$$a_1 + a_2 + \dots + a_n = n, \quad a_2, a_3, \dots, a_n \geq 1,$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \geq a_1^2 + a_2^2 + \dots + a_n^2.$$

**3.205.** Let  $a_1, a_2, \dots, a_n$  be nonnegative real numbers such that

$$a_1 + a_2 + \dots + a_n = n.$$

Prove that

$$(n+1)(a_1^2 + a_2^2 + \dots + a_n^2) \geq n^2 + a_1^3 + a_2^3 + \dots + a_n^3.$$

**3.206.** Let  $a_1, a_2, \dots, a_n$  be nonnegative real numbers such that

$$a_1 + a_2 + \dots + a_n = n.$$

Prove that

$$(n-1)(a_1^3 + a_2^3 + \dots + a_n^3) + n^2 \geq (2n-1)(a_1^2 + a_2^2 + \dots + a_n^2).$$

**3.207.** Let  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) be nonnegative real numbers such that

$$a_1 + a_2 + \dots + a_n = n.$$

Prove that

$$a_1^2 + a_2^2 + \dots + a_n^2 - n \geq \frac{n}{n-1}(1 - a_1 a_2 \dots a_n).$$

**3.208.** If  $a_1, a_2, \dots, a_n$  are positive numbers such that  $a_1 + a_2 + \dots + a_n = n$  and

$$a_1 a_2 \dots a_n \leq \frac{1}{(n-1)^{n-2}},$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \geq a_1^2 + a_2^2 + \dots + a_n^2.$$

**3.209.** If  $a_1 \geq a_2 \geq \cdots \geq a_n > 0$  such that  $a_1 + a_2 + \cdots + a_n = n$ , then

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n \geq \frac{4(n-1)^2}{n^3}(a_1 - a_2)^2.$$

**3.210.** If  $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ , then

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n} \leq (n-1)(\sqrt{a_1} - \sqrt{a_n})^2.$$

**3.211.** If  $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ , then

$$a_1 + a_2 + \cdots + a_n + \frac{k(a_1 - a_2)^2}{a_1 + a_2} \leq \sqrt{n(a_1^2 + a_2^2 + \cdots + a_n^2)},$$

where  $k = \frac{1}{1 + \sqrt{\frac{n}{n-1}}}$ .

**3.212.** For given  $n \geq 2$ , find the largest integer  $k$  such that

$$\left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right)^2 \geq \frac{a_1^2 + a_2^2 + \cdots + a_k^2}{k}$$

for all nonnegative numbers  $a_i$  satisfying  $a_1 \leq a_2 \leq \cdots \leq a_n$ .

**3.213.** For given  $n \geq 2$ , find the largest integer  $k$  such that

$$\left( \frac{a_1 + a_2 + \cdots + a_k}{k} \right)^2 \geq \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}$$

for all nonnegative numbers  $a_i$  satisfying  $a_1 \geq a_2 \geq \cdots \geq a_n$ .

**3.214.** If  $a_1 \geq a_2 \geq \cdots \geq a_9 \geq 0$ , then

$$\left( \frac{5a_1 + a_2 + \cdots + a_8}{12} \right)^2 \geq \frac{a_1^2 + a_2^2 + \cdots + a_9^2}{9}.$$

**3.215.** If  $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ , then

$$\begin{aligned} \text{(a)} \quad & a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{1}{3} (\sqrt{a_1} + \sqrt{a_2} - 2\sqrt{a_n})^2; \\ \text{(b)} \quad & a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{1}{4} (2\sqrt{a_1} - \sqrt{a_{n-1}} - \sqrt{a_n})^2. \end{aligned}$$

**3.216.** If  $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ ,  $n \geq 3$ , then

$$a_1 + a_2 + \cdots + a_n - n \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n-1}{2n} (\sqrt{a_{n-2}} + \sqrt{a_{n-1}} - 2\sqrt{a_n})^2.$$

**3.217.** Let  $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ . If

$$1 \leq k < j \leq n, \quad k + j \geq n + 1,$$

then

$$a_1 + a_2 + \cdots + a_n - n \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{2k(n-j+1)}{n+k-j+1} (\sqrt{a_k} - \sqrt{a_j})^2.$$

**3.218.** If  $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ ,  $n \geq 4$ , then

$$(a) \quad a_1 + a_2 + \cdots + a_n - n \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{1}{2} \left(1 - \frac{1}{n}\right) (\sqrt{a_{n-2}} - 3\sqrt{a_{n-1}} + 2\sqrt{a_n})^2;$$

$$(b) \quad a_1 + a_2 + \cdots + a_n - n \sqrt[n]{a_1 a_2 \cdots a_n} \geq \left(1 - \frac{2}{n}\right) (2\sqrt{a_{n-2}} - 3\sqrt{a_{n-1}} + \sqrt{a_n})^2.$$

**3.219.** Let  $a_1, a_2, \dots, a_n$  ( $n \geq 4$ ) be nonnegative real numbers such that

$$a_1 \geq a_2 \geq \cdots \geq a_n$$

and

$$(a_1 + a_2 + \cdots + a_n)^2 = 4(a_1^2 + a_2^2 + \cdots + a_n^2).$$

Prove that

$$1 \leq \frac{a_1 + a_2}{a_3 + a_4 + \cdots + a_n} \leq 1 + \sqrt{\frac{2n-8}{n-2}}.$$

**3.220.** Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be real numbers. Prove that

$$a_1 b_1 + \cdots + a_n b_n + \sqrt{(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)} \geq \frac{2}{n} (a_1 + \cdots + a_n)(b_1 + \cdots + b_n).$$

**3.221.** Let  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) be positive real numbers such that

$$a_1 \leq a_2 \leq \cdots \leq a_n,$$

$$(a_1 + a_2 + \cdots + a_n) \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) = k.$$

(a) If  $n^2 < k \leq n^2 + \frac{i(n-i)}{2}$ ,  $i \in \{2, 3, \dots, n-1\}$ , then  $a_{i-1}$ ,  $a_i$  and  $a_{i+1}$  are the lengths of the sides of a non-degenerate or degenerate triangle.

(b) If  $n^2 < k \leq \alpha_n$ , where  $\alpha_n = \frac{9n^2}{8}$  for even  $n$ , and  $\alpha_n = \frac{9n^2-1}{8}$  for odd  $n$ , then there exist three numbers  $a_i$  which are the lengths of the sides of a non-degenerate or degenerate triangle.

**3.222.** Let  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) be positive real numbers such that

$$a_1 \leq a_2 \leq \dots \leq a_n,$$

$$(a_1 + a_2 + \dots + a_n)^2 = k(a_1^2 + a_2^2 + \dots + a_n^2).$$

(a) If  $\frac{(2n-i)^2}{4n-3i} \leq k < n$ ,  $i \in \{2, 3, \dots, n-1\}$ , then  $a_{i-1}$ ,  $a_i$  and  $a_{i+1}$  are the lengths of the sides of a non-degenerate or degenerate triangle.

(b) If  $\frac{8n+1}{9} \leq k < n$ , then there exist three numbers  $a_i$  which are the lengths of the sides of a non-degenerate or degenerate triangle.

**3.223.** Let  $q$  be a fixed real number such that there exist three nonnegative real numbers  $a, b, c$  satisfying

$$a \geq 1 \geq b \geq c \geq 0, \quad a + b + c = 3, \quad ab + bc + ca = q.$$

Prove that the product  $r = abc$  has the maximum value for  $b = c$ , and the minimum value for  $b = 1$  or  $c = 0$ .

**3.224.** Let  $p$  and  $q$  be fixed real numbers such that there exist three nonnegative real numbers  $a, b, c$  satisfying

$$a \geq 1 \geq b \geq c \geq 0, \quad a + b + c = p, \quad ab + bc + ca = q.$$

Prove that the product  $r = abc$  has the maximum value for  $b = c$ , and the minimum value for  $a = 1$  or  $b = 1$  or  $c = 0$ .

**3.225.** (a) If  $p$  and  $q$  are fixed real numbers such that there exist three positive real numbers  $a, b, c$  satisfying

$$a \geq b \geq c \geq 1, \quad a + b + c = p, \quad ab + bc + ca = q,$$

then the product  $r = abc$  has the maximum value for  $b = c$ , and the minimum value for  $a = b$  or  $c = 1$ .

(b) If  $p$  and  $q$  are fixed real numbers such that there exist three nonnegative real numbers  $a, b, c$  satisfying

$$1 \geq a \geq b \geq c \geq 0, \quad a + b + c = p, \quad ab + bc + ca = q,$$

then the product  $r = abc$  has the maximum value for  $a = 1$  or  $b = c$ , and the minimum value for  $a = b$  or  $c = 0$ .

**3.226.** Let  $q$  be a fixed real number such that there exist three nonnegative real numbers  $a, b, c$  satisfying

$$a \geq b \geq 1 \geq c \geq 0, \quad a + b + c = 3, \quad ab + bc + ca = q.$$

Prove that the product  $r = abc$  has the maximum value for  $b = 1$ , and has the minimum value for  $a = b$  or  $c = 0$ .

**3.227.** Let  $p$  and  $q$  be fixed real numbers such that there exist three nonnegative real numbers  $a, b, c$  satisfying

$$a \geq b \geq 1 \geq c \geq 0, \quad a + b + c = p, \quad ab + bc + ca = q.$$

Prove that the product  $r = abc$  has the maximum value for  $b = 1$  or  $c = 1$ , and the minimum value for  $a = b$  or  $c = 0$ .

**3.228.** (a) If  $p$  is a real number such that there exist three positive real numbers  $a, b, c$  satisfying

$$a \geq 1 \geq b \geq c, \quad a + b + c = p, \quad abc = 1,$$

then the sum  $q = ab + bc + ca$  has the maximum value for  $b = 1$ , and the minimum value for  $b = c$ .

(b) If  $p$  is a real number such that there exist three positive real numbers  $a, b, c$  satisfying

$$a \geq b \geq 1 \geq c, \quad a + b + c = p, \quad abc = 1,$$

then the sum  $q = ab + bc + ca$  has the maximum value for  $a = b$ , and the minimum value for  $b = 1$ .

**3.229.** If  $a \geq 1 \geq b \geq c \geq 0$  such that  $a + b + c = 3$ , then

$$abc + \frac{9}{ab + bc + ca} \geq 4.$$

**3.230.** If  $a \geq 1 \geq b \geq c \geq 0$  such that  $a + b + c = 3$ , then

$$abc + \frac{2}{ab + bc + ca} \geq \frac{5}{a^2 + b^2 + c^2}.$$

**3.231.** If  $a \geq b \geq 1 \geq c > 0$  such that  $a + b + c = 3$ , then

$$\frac{1}{abc} + 2 \geq \frac{9}{ab + bc + ca}.$$

**3.232.** If  $a \geq b \geq 1 \geq c > 0$  such that  $a + b + c = 3$ , then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 11 \geq 4(a^2 + b^2 + c^2).$$

**3.233.** If  $a \geq b \geq 1 \geq c > 0$  such that  $a + b + c = 3$ , then

$$\frac{1}{abc} + \frac{2}{a^2 + b^2 + c^2} \geq \frac{5}{ab + bc + ca}.$$

**3.234.** If  $a \geq b \geq 1 \geq c \geq 0$  such that  $a + b + c = 3$ , then

$$\frac{9}{a^3 + b^3 + c^3} + 2 \leq \frac{15}{a^2 + b^2 + c^2}.$$

**3.235.** If  $a \geq b \geq 1 \geq c \geq 0$  such that  $a + b + c = 3$ , then

$$\frac{36}{a^3 + b^3 + c^3} + 9 \leq \frac{65}{a^2 + b^2 + c^2}.$$

**3.236.** If  $a, b, c$  are real numbers such that

$$a \geq b \geq 1 \geq c, \quad a^2 + b^2 + c^2 = 3,$$

then

$$(a) \quad 1 - abc \leq 2(b - c)^2;$$

$$(b) \quad 1 - abc \geq 2(a - b)^2;$$

$$(c) \quad 1 - abc \geq \frac{1}{2}(a - c)^2;$$

$$(d) \quad 1 - abc \leq \frac{3}{4}(a - c)^2.$$

**3.237.** If  $a, b, c$  are real numbers such that

$$a \geq 1 \geq b \geq c, \quad a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \leq \frac{2}{3}(a - c)^2.$$



**3.238.** If  $a \geq 1 \geq b \geq c \geq 0$  and  $a^2 + b^2 + c^2 = 3$ , then

$$1 - abc \leq \frac{1}{\sqrt{2}}(a - c).$$

**3.239.** If  $a \geq 1 \geq b \geq c \geq 0$  and  $a^2 + b^2 + c^2 = 3$ , then

$$1 - abc \leq (1 + \sqrt{2})(a - b).$$

**3.240.** If  $a \geq 1 \geq b \geq c \geq 0$  and  $a^2 + b^2 + c^2 = 3$ , then

$$1 - abc \leq (3 + 2\sqrt{2})(a - b)^2.$$

**3.241.** If  $a, b, c, d$  are nonnegative real numbers such that

$$a \geq b \geq 1 \geq c \geq d, \quad a + b + c + d = 3,$$

then

$$a^2 + b^2 + c^2 + d^2 + 10abcd \leq 5.$$

**3.242.** If  $a, b, c, d$  are nonnegative real numbers such that

$$a \geq b \geq 1 \geq c \geq d, \quad a + b + c + d = 6,$$

then

$$a^2 + b^2 + c^2 + d^2 + 4abcd \leq 26.$$

**3.243.** If  $a, b, c, d$  are nonnegative real numbers such that

$$a \geq b \geq 1 \geq c \geq d, \quad a + b + c + d = 4,$$

then

$$a^2 + b^2 + c^2 + d^2 + 6abcd \leq 10.$$

**3.244.** If  $a, b, c, d$  are nonnegative real numbers such that

$$a \geq b \geq 1 \geq c \geq d, \quad a + b + c + d = 4,$$

then

$$a^2 + b^2 + c^2 + d^2 + 6\sqrt{abcd} \leq 10.$$

**3.245.** If  $a, b, c, d$  are positive real numbers such that

$$a + b + c + d = 4, \quad a \leq b \leq 1 \leq c \leq d,$$

then

$$9 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \geq 4 + 8(a^2 + b^2 + c^2 + d^2).$$

**3.246.** Let  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) be real numbers such that  $a_1 + a_2 + \dots + a_n = n$ .

(a) If  $a_1 \geq 1 \geq a_2 \geq \dots \geq a_n$ , then

$$a_1^3 + a_2^3 + \dots + a_n^3 + 2n \geq 3(a_1^2 + a_2^2 + \dots + a_n^2).$$

(b) If  $a_1 \leq 1 \leq a_2 \leq \dots \leq a_n$ , then

$$a_1^3 + a_2^3 + \dots + a_n^3 + 2n \leq 3(a_1^2 + a_2^2 + \dots + a_n^2).$$

**3.247.** Let  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) be nonnegative real numbers such that  $a_1 + a_2 + \dots + a_n = n$ .

(a) If  $a_1 \geq 1 \geq a_2 \geq \dots \geq a_n$ , then

$$a_1^4 + a_2^4 + \dots + a_n^4 + 5n \geq 6(a_1^2 + a_2^2 + \dots + a_n^2).$$

(b) If  $a_1 \leq 1 \leq a_2 \leq \dots \leq a_n$ , then

$$a_1^4 + a_2^4 + \dots + a_n^4 + 6n \leq 7(a_1^2 + a_2^2 + \dots + a_n^2).$$

**3.248.** If  $a_1, a_2, \dots, a_n$  are positive real numbers such that

$$a_1 \geq 1 \geq a_2 \geq \dots \geq a_n, \quad \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} = n,$$

then

$$a_1^2 + a_2^2 + \dots + a_n^2 + 2n \geq 3(a_1 + a_2 + \dots + a_n).$$

**3.249.** If  $a_1, a_2, \dots, a_n$  are real numbers such that

$$a_1 \leq 1 \leq a_2 \leq \dots \leq a_n, \quad a_1 + a_2 + \dots + a_n = n,$$

then

$$(1 + a_1^2)(1 + a_2^2) \cdots (1 + a_n^2) \geq 2^n.$$

**3.250.** If  $a_1, a_2, \dots, a_n$  are positive real numbers such that

$$a_1 \geq 1 \geq a_2 \geq \dots \geq a_n, \quad a_1 a_2 \cdots a_n = 1,$$

then

$$a_1^n + a_2^n + \dots + a_n^n - n \geq n^2 \left( \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} - n \right).$$

**3.251.** If  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) are real numbers such that

$$a_1 + a_2 + \dots + a_n = n, \quad a_1 \geq a_2 \geq 1 \geq a_3 \geq \dots \geq a_n,$$

then

$$a_1^4 + a_2^4 + \dots + a_n^4 - n \geq \frac{14}{3}(a_1^2 + a_2^2 + \dots + a_n^2 - n).$$

**3.252.** If  $a_1, a_2, \dots, a_n$  are nonnegative real numbers such that at most one of them is larger than 1 and  $a_1 a_2 \cdots a_n = 1$ , then

$$a_1 + a_2 + \dots + a_n \geq \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}.$$



## 3.2 Solutions

**P 3.1.** If  $a, b, c$  are positive real numbers, then

$$a^2 + b^2 + c^2 + 2abc + 1 \geq 2(ab + bc + ca).$$

(Darij Grinberg, 2004)

**First Solution.** Setting

$$a = x^3, \quad b = y^3, \quad c = z^3, \quad x, y, z > 0,$$

we need to prove that

$$x^6 + y^6 + z^6 + 2x^3y^3z^3 + 1 \geq 2(x^3y^3 + y^3z^3 + z^3x^3).$$

Using Schur's inequality and then the AM-GM inequality, we have

$$x^6 + y^6 + z^6 + 3x^2y^2z^2 \geq \sum x^2y^2(x^2 + y^2) \geq 2 \sum x^3y^3.$$

Thus, it suffices to show that

$$2x^3y^3z^3 - 3x^2y^2z^2 + 1 \geq 0,$$

which is equivalent to

$$(xyz - 1)^2(2xyz + 1) \geq 0.$$

The equality holds for  $a = b = c = 1$ .

**Second Solution.** Among the numbers  $1 - a$ ,  $1 - b$  and  $1 - c$  there are always two with the same sign; let

$$(1 - b)(1 - c) \geq 0.$$

We have

$$\begin{aligned} a^2 + b^2 + c^2 + 2abc + 1 - 2(ab + bc + ca) &= \\ &= (a - 1)^2 + (b - c)^2 + 2a + 2abc - 2a(b + c) \\ &= (a - 1)^2 + (b - c)^2 + 2a(1 - b)(1 - c) \geq 0. \end{aligned}$$

**Remark.** The following generalization holds:

- Let  $a, b, c$  be positive real numbers. If  $0 \leq k \leq 1$ , then

$$a^2 + b^2 + c^2 + 2kabc + k \geq (k + 1)(ab + bc + ca).$$

Since the both sides of the inequality are linear of  $k$ , it suffices to prove it for only  $k = 0$  and  $k = 1$ . For  $k = 0$ , the inequality reduces to

$$a^2 + b^2 + c^2 \geq ab + bc + ca,$$

which is well-known. □

**P 3.2.** Let  $a, b, c$  be nonnegative real numbers. If  $0 \leq k \leq \sqrt{2}$ , then

$$a^2 + b^2 + c^2 + kabc + 2k + 3 \geq (k + 2)(a + b + c).$$

**Solution.** Since the both sides of the inequality are linear of  $k$ , it suffices to prove the inequality for  $k = 0$  and  $k = \sqrt{2}$ . For  $k = 0$ , the inequality reduces to

$$(a - 1)^2 + (b - 1)^2 + (c - 1)^2 \geq 0.$$

Consider further that  $k = \sqrt{2}$ , and write the inequality as

$$(a - 1)^2 + (b - 1)^2 + (c - 1)^2 \geq \sqrt{2} (a + b + c - 2 - abc).$$

Using the substitution

$$x = a - 1, \quad y = b - 1, \quad z = c - 1,$$

we need to show that

$$x^2 + y^2 + z^2 + \sqrt{2} (xyz + xy + yz + zx) \geq 0$$

for  $x, y, z \geq -1$ . Among the numbers  $x, y$  and  $z$  there are always two of them with the same sign; let us say

$$yz \geq 0.$$

Since

$$y^2 + z^2 \geq \frac{1}{2}(y + z)^2$$

and

$$xyz + xy + yz + zx = (x + 1)yz + x(y + z) \geq x(y + z),$$

it suffices to prove that

$$x^2 + \frac{1}{2}(y + z)^2 + \sqrt{2} x(y + z) \geq 0,$$

which is equivalent to

$$\left[ x + \frac{1}{\sqrt{2}}(y + z) \right]^2 \geq 0.$$

Thus, the proof is completed. The equality holds for  $a = b = c = 1$ . In addition, if  $k = \sqrt{2}$ , then the equality holds also for  $a = 0$  and  $b = c = 1 + 1/\sqrt{2}$  (or any cyclic permutation).  $\square$

**P 3.3.** If  $a, b, c$  are positive real numbers, then

$$abc(a + b + c) + 2(a^2 + b^2 + c^2) + 3 \geq 4(ab + bc + ca).$$

**First Solution.** Applying the AM-GM inequality two times, we get

$$abc(a+b+c) + 3 \geq 2\sqrt{3abc(a+b+c)} \geq \frac{18abc}{a+b+c}.$$

Therefore, it suffices to prove that

$$a^2 + b^2 + c^2 + \frac{18abc}{a+b+c} \geq 2(ab+bc+ca),$$

which is just Schur's inequality of third degree. The equality holds for  $a = b = c = 1$ .

**Second Solution.** Applying the AM-GM, we get

$$\begin{aligned} abc(a+b+c) + 3 &= (a^2bc + 1) + (ab^2c + 1) + (abc^2 + 1) \\ &\geq 2a\sqrt{bc} + 2b\sqrt{ca} + 2c\sqrt{ab}. \end{aligned}$$

Thus, it suffices to prove that

$$a^2 + b^2 + c^2 + a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \geq 2(ab+bc+ca).$$

Substituting

$$x = \sqrt{a}, \quad y = \sqrt{b}, \quad z = \sqrt{c},$$

we need to show that

$$x^4 + y^4 + z^4 + xyz(x+y+z) \geq 2(x^2y^2 + y^2z^2 + z^2x^2).$$

This inequality can be obtained by summing Schur's inequality of degree four

$$x^4 + y^4 + z^4 + xyz(x+y+z) \geq xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2)$$

to

$$xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) \geq 2(x^2y^2 + y^2z^2 + z^2x^2).$$

The last inequality is equivalent to

$$xy(x-y)^2 + yz(y-z)^2 + zx(z-x)^2 \geq 0.$$

□

**P 3.4.** If  $a, b, c$  are positive real numbers, then

$$a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2) + 3 \geq 3(ab + bc + ca).$$

**Solution.** Write the inequality as follows

$$(a + b + c)(ab + bc + ca) + 3 \geq 3(abc + ab + bc + ca).$$

$$(a + b + c - 3)(ab + bc + ca) + 3 \geq 3abc.$$

Using the known inequality

$$(a + b + c)(ab + bc + ca) \geq 9abc,$$

it suffices to show that

$$3(a + b + c - 3)(ab + bc + ca) + 9 \geq (a + b + c)(ab + bc + ca),$$

which is equivalent to

$$[2(a + b + c) - 9](ab + bc + ca) + 9 \geq 0.$$

For the nontrivial case  $2(a + b + c) - 9 < 0$ , using the known inequality

$$(a + b + c)^2 \geq 3(ab + bc + ca),$$

it is enough to show that

$$[2(a + b + c) - 9](a + b + c)^2 + 27 \geq 0.$$

This inequality is equivalent to the obvious inequality

$$(a + b + c - 3)^2[2(a + b + c) + 3] \geq 0.$$

The equality holds for  $a = b = c = 1$ .

□

**P 3.5.** If  $a, b, c$  are positive real numbers, then

$$\left(\frac{a^2 + b^2 + c^2}{3}\right)^3 \geq a^2b^2c^2 + (a - b)^2(b - c)^2(c - a)^2.$$

(Vasile Cîrtoaje, 2011)

**Solution** (by Vo Quoc Ba Can). Assume that

$$a = \min\{a, b, c\}.$$

By virtue of the AM-GM inequality, we have

$$\begin{aligned} \left(\frac{a^2 + b^2 + c^2}{3}\right)^3 &= \left[\frac{(a^2 + b^2 + c^2 - 2bc) + bc + bc}{3}\right]^3 \\ &\geq (a^2 + b^2 + c^2 - 2bc)b^2c^2 \\ &= a^2b^2c^2 + (b - c)^2b^2c^2. \end{aligned}$$



Thus, it suffices to prove that

$$(b-c)^2 b^2 c^2 \geq (b-c)^2 (b-a)^2 (c-a)^2.$$

This is obvious, because

$$b^2 > (b-a)^2, \quad c^2 > (c-a)^2.$$

The equality occurs for  $a = b = c$ .

□

**P 3.6.** If  $a, b, c$  are nonnegative real numbers, then

$$[ab(a+b) + bc(b+c) + ca(c+a)]^2 \geq 4(ab+bc+ca)(a^2b^2 + b^2c^2 + c^2a^2).$$

(Vasile Cîrtoaje and Vo Quoc Ba Can, 2011)

**First Solution.** Assume that  $a \geq b \geq c$ . For the nontrivial case  $b > 0$ , by the AM-GM inequality, we have

$$4(ab+bc+ca)(a^2b^2 + b^2c^2 + c^2a^2) \leq \left[ b(ab+bc+ca) + \frac{a^2b^2 + b^2c^2 + c^2a^2}{b} \right]^2.$$

Thus, it suffices to prove that

$$ab(a+b) + bc(b+c) + ca(c+a) \geq b(ab+bc+ca) + \frac{a^2b^2 + b^2c^2 + c^2a^2}{b}.$$

This inequality reduces to the obvious form

$$ac(a-b)(b-c) \geq 0.$$

The equality holds for  $a = b = c$ , for  $b = c = 0$  (or any cyclic permutation), and for  $a = 0$  and  $b = c$  (or any cyclic permutation).

**Second Solution.** We will prove the stronger inequality

$$[ab(a+b) + bc(b+c) + ca(c+a)]^2 \geq 4(ab+bc+ca)(a^2b^2 + b^2c^2 + c^2a^2) + A,$$

where

$$A = (a-b)^2(b-c)^2(c-a)^2.$$

Let

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

Since

$$(a-b)^2(b-c)^2(c-a)^2 = -27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3,$$

we can write this inequality as

$$(pq - 3r)^2 \geq 4q(q^2 - 2pr) - 27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3,$$

which reduces to

$$r(p^3 + 9r - 4pq) \geq 0.$$

This is true since

$$p^3 + 9r - 4pq \geq 0$$

is just the third degree Schur's inequality. □

**P 3.7.** If  $a, b, c$  are positive real numbers, then

$$(a) \quad a^3 + b^3 + c^3 + ab + bc + ca + 9 \geq 5(a + b + c);$$

$$(b) \quad a^3 + b^3 + c^3 + 4(ab + bc + ca) + 18 \geq 11(a + b + c).$$

(Vasile Cîrtoaje, 2010)

**Solution.** Let

$$p = a + b + c, \quad q = ab + bc + ca.$$

From

$$a(a-1)^2 + b(b-1)^2 + c(c-1)^2 \geq 0,$$

we get

$$a^3 + b^3 + c^3 \geq 2(a^2 + b^2 + c^2) - a - b - c = 2p^2 - p - 4q.$$

(a) Using the result above and the known inequality  $p^2 \geq 3q$ , we have

$$\begin{aligned} a^3 + b^3 + c^3 + ab + bc + ca + 9 - 5(a + b + c) &\geq \\ &\geq (2p^2 - p - 4q) + q + 9 - 5p \\ &= 2p^2 - 6p + 9 - 3q \\ &\geq 2p^2 - 6p + 9 - p^2 \\ &= (p - 3)^2 \geq 0. \end{aligned}$$

The equality holds for  $a = b = c = 1$ .

(b) Using the result above, we have

$$\begin{aligned} a^3 + b^3 + c^3 + 4(ab + bc + ca) + 18 - 11(a + b + c) &\geq \\ &\geq (2p^2 - p - 4q) + 4q + 18 - 11p \\ &= 2(p - 3)^2 \geq 0. \end{aligned}$$

The equality holds for  $a = b = c = 1$ . □

**P 3.8.** If  $a, b, c$  are positive real numbers, then

$$(a) \quad a^3 + b^3 + c^3 + abc + 8 \geq 4(a + b + c);$$

$$(b) \quad 4(a^3 + b^3 + c^3) + 15abc + 54 \geq 27(a + b + c).$$

**Solution.** Let

$$p = a + b + c, \quad q = ab + bc + ca.$$

By Schur's inequality of third degree, we have

$$p^3 + 9abc \geq 4pq,$$

$$abc \geq \frac{p(4q - p^2)}{9}.$$

a) We get

$$\begin{aligned} a^3 + b^3 + c^3 + abc &= 4abc + p(p^2 - 3q) \\ &\geq \frac{4p(4q - p^2)}{9} + p(p^2 - 3q) \\ &= \frac{p(5p^2 - 11q)}{9}. \end{aligned}$$

Then, it suffices to prove that

$$\frac{p(5p^2 - 11q)}{9} + 8 \geq 4p,$$

which is equivalent to

$$5p^3 - 36p + 72 \geq 11pq.$$

Since  $p^2 \geq 3q$ , we have

$$\begin{aligned} 3(5p^3 - 36p + 72 - 11pq) &\geq 3(5p^3 - 36p + 72) - 11p^2 \\ &= 4(p^3 - 27p + 54) \\ &= 4(p - 3)^2(p + 6) \geq 0. \end{aligned}$$

The equality holds for  $a = b = c = 1$ .

(b) We get

$$\begin{aligned} 4(a^3 + b^3 + c^3) + 15abc &= 27abc + 4p(p^2 - 3q) \\ &\geq 3p(4q - p^2) + 4p(p^2 - 3q) = p^3. \end{aligned}$$

Then, it suffices to prove that

$$p^3 + 54 \geq 27p,$$

which is equivalent to the obvious inequality

$$(p-3)^2(p+6) \geq 0.$$

The equality holds for  $a = b = c = 1$ , and also for  $a = 0$  and  $b = c = 3/2$  (or any cyclic permutation).

**Remark.** Similarly, we can prove the following generalization (Vasile Cîrtoaje, 2010):

- Let  $a, b, c$  be nonnegative real numbers. If  $0 \leq k \leq 27/4$ , then

$$a^3 + b^3 + c^3 + (k-3)abc + 2k \geq k(a+b+c).$$

□

**P 3.9.** Let  $a, b, c$  be nonnegative real numbers such that

$$a + b + c = a^2 + b^2 + c^2.$$

Prove that

$$ab + bc + ca \geq a^2b^2 + b^2c^2 + c^2a^2.$$

(Vasile Cîrtoaje, 2006)

**Solution** (by Michael Rozenberg). From the hypothesis condition, by squaring, we get

$$a^4 + b^4 + c^4 - a^2 - b^2 - c^2 = 2(ab + bc + ca - a^2b^2 - b^2c^2 - c^2a^2).$$

Therefore, we can write the required inequality as

$$a^4 + b^4 + c^4 \geq a^2 + b^2 + c^2.$$

This inequality has the homogeneous form

$$(a+b+c)^2(a^4+b^4+c^4) \geq (a^2+b^2+c^2)^3,$$

which follows immediately from Hölder's inequality. The equality holds for  $a = b = c = 1$ , for  $a = b = c = 0$ , for  $(a, b, c) = (0, 1, 1)$  (or any cyclic permutation), and for  $(a, b, c) = (1, 0, 0)$  (or any cyclic permutation).

□

**P 3.10.** If  $a, b, c$  are nonnegative real numbers, then

$$(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) \geq (ab + bc + ca)^3.$$

(Vasile Cîrtoaje, 2006)

**Solution.** We have

$$(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) = 9a^2b^2c^2 + 2 \sum a^3b^3 + 4abc \sum a^3$$

and

$$(ab + bc + ca)^3 = 6a^2b^2c^2 + \sum a^3b^3 + 3abc \sum ab(a + b).$$

So, we can rewrite the inequality as

$$3a^2b^2c^2 + \sum a^3b^3 + 4abc \sum a^3 \geq 3abc \sum ab(a + b).$$

Since  $\sum a^3b^3 \geq 3a^2b^2c^2$  (by the AM-GM inequality), it suffices to prove that

$$6abc + 4 \sum a^3 \geq 3 \sum ab(a + b).$$

We can get this inequality by summing the inequalities

$$\frac{1}{3} \sum a^3 \geq abc$$

and

$$3abc + \sum a^3 \geq \sum ab(a + b).$$

The first inequality follows from the AM-GM inequality, while the second is just the third degree Schur's inequality. The equality holds when  $a = b = c$ , and also when two of  $a, b, c$  are zero.

**Remark.** Similarly, we can also prove the following inequality

$$(2a^2 + 7bc)(2b^2 + 7ca)(2c^2 + 7ab) \geq 27(ab + bc + ca)^3.$$

□

**P 3.11.** If  $a, b, c$  are nonnegative real numbers, then

$$(2a^2 + bc)(2b^2 + ca)(2c^2 + ab) \geq (ab + bc + ca)^3.$$

(Vasile Cîrtoaje, 2006)

**First Solution.** Since

$$(2a^2 + bc)(2b^2 + ca)(2c^2 + ab) = 9a^2b^2c^2 + 4 \sum a^3b^3 + 2abc \sum a^3$$

and

$$(ab + bc + ca)^3 = 6a^2b^2c^2 + \sum a^3b^3 + 3abc \sum ab(a + b),$$

the inequality is equivalent to

$$3a^2b^2c^2 + 3 \sum a^3b^3 + 2abc \sum a^3 \geq 3abc \sum ab(a + b).$$

We can get this inequality by summing

$$\frac{2}{3}abc \sum a^3 \geq 2a^2b^2c^2$$

and

$$\sum a^3b^3 + 3a^2b^2c^2 \geq abc \sum ab(a+b).$$

The first inequality follows from the AM-GM inequality, while the second is just the third degree Schur's inequality applied to the numbers  $ab$ ,  $bc$  and  $ca$ . The equality holds when  $a = b = c$ , and also when two of  $a, b, c$  are zero.

**Second Solution.** By Hölder's inequality, we have

$$(a^2 + bc + a^2)(b^2 + b^2 + ca)(ab + c^2 + c^2) \geq (ab + bc + ca)^3,$$

from which the desired inequality follows.

**Remark.** Using the first method, we can also prove the following inequality

$$(5a^2 + bc)(5b^2 + ca)(5c^2 + ab) \geq 8(ab + bc + ca)^3.$$

□

**P 3.12.** Let  $a, b, c$  be nonnegative real numbers such that

$$a + b + c = 2.$$

Prove that

$$(a) \quad (a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \leq (a + b)(b + c)(c + a);$$

$$(b) \quad (a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \leq 2.$$

**Solution.** Assume that

$$a = \min\{a, b, c\}.$$

It is easy to check that the equality holds in both inequalities for  $a = 0$  and  $b = c = 1$ .

(a) Since

$$a^2 + b^2 \leq b(a + b)$$

and

$$c^2 + a^2 \leq c(c + a),$$

it suffices to show that

$$bc(b^2 + c^2) \leq b + c.$$

By the AM-GM inequality, we have

$$2bc(b^2 + c^2) \leq \left[ \frac{2bc + (b^2 + c^2)}{2} \right]^2 = \frac{(b + c)^4}{4} \leq 2(b + c).$$

The equality holds for  $a = 0$  and  $b = c = 1$  (or any cyclic permutation).

(b) **First Solution.** Since

$$a^2 + b^2 \leq b(a + b)$$

and

$$c^2 + a^2 \leq c(a + c),$$

it suffices to show that

$$bc(a + b)(a + c)(b^2 + c^2) \leq 2.$$

By the AM-GM inequality, we have

$$4bc(a + b)(a + c)(b^2 + c^2) \leq \left[ \frac{2b(a + c) + 2c(a + b) + (b^2 + c^2)}{3} \right]^3.$$

Therefore, we only need to show that

$$b^2 + c^2 + 4bc + 2ab + 2ac \leq 6.$$

This is true since

$$\begin{aligned} 12 - 2(b^2 + c^2 + 4bc + 2ab + 2ac) &= \\ &= 3(a + b + c)^2 - 2(b^2 + c^2 + 4bc + 2ab + 2ac) \\ &= 2a^2 + b^2 + c^2 - 2bc + 2ab + 2ac \\ &= 2a(a + b + c) + (b - c)^2 \geq 0. \end{aligned}$$

The equality holds for  $a = 0$  and  $b = c = 1$  (or any cyclic permutation).

**Second Solution.** Let us denote

$$F(a, b, c) = (a^2 + b^2)(b^2 + c^2)(c^2 + a^2).$$

We will show that

$$F(a, b, c) \leq F(0, b + a/2, c + a/2) \leq 2.$$

The left inequality,

$$(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \leq (b + a/2)^2[(b + a/2)^2 + (c + a/2)^2](c + a/2)^2,$$

is true since

$$\begin{aligned} a^2 + b^2 &\leq (b + a/2)^2, \\ b^2 + c^2 &\leq (b + a/2)^2 + (c + a/2)^2, \end{aligned}$$

$$c^2 + a^2 \leq (c + a/2)^2.$$

The right inequality holds if the original inequality holds for  $a = 0$ ; that is,

$$b^2 c^2 (b^2 + c^2) \leq 2$$

for  $b + c = 2$ . Indeed, by virtue of the AM-GM inequality, we have

$$bc \leq \left( \frac{b+c}{2} \right)^2 = 1,$$

hence

$$\begin{aligned} 2b^2 c^2 (b^2 + c^2) &\leq 2bc(b^2 + c^2) \\ &\leq \left[ \frac{2bc + (b^2 + c^2)}{2} \right]^2 \\ &= \frac{(b+c)^4}{4} = 4. \end{aligned}$$

□

**P 3.13.** Let  $a, b, c$  be nonnegative real numbers such that

$$a + b + c = 2.$$

Prove that

$$(a^3 + b^3)(b^3 + c^3)(c^3 + a^3) \leq 2.$$

**Solution.** Due to symmetry, we may assume that

$$a = \min\{a, b, c\}.$$

It is easy to check that the equality holds for  $a = 0$  and  $b = c = 1$ . Write the inequality as

$$\left[ \prod (a+b) \right] \left[ \prod (a^2 - ab + b^2) \right] \leq 2.$$

Since

$$\prod (a+b) \leq (a+b+c)(ab+bc+ca) = 2(ab+bc+ca),$$

it suffices to show that

$$(ab+bc+ca) \prod (a^2 - ab + b^2) \leq 1.$$

Since

$$a^2 - ab + b^2 \leq b^2$$

and

$$c^2 - ca + a^2 \leq c^2,$$



it suffices to show that

$$b^2c^2(ab+bc+ca)(b^2-bc+c^2) \leq 1.$$

In virtue of the AM-GM inequality, we have

$$b^2c^2(ab+bc+ca)(b^2-bc+c^2) \leq \left[ \frac{bc+bc+(ab+bc+ca)+(b^2-bc+c^2)}{4} \right]^4.$$

Therefore, it remains to show that

$$b^2+c^2+2bc+ab+ca \leq 4.$$

This is true since

$$\begin{aligned} 4 - (b^2 + c^2 + 2bc + ab + ca) &= (a + b + c)^2 - (b^2 + c^2 + 2bc + ab + ca) \\ &= a(a + b + c) \geq 0. \end{aligned}$$

The equality holds for  $a = 0$  and  $b = c = 1$  (or any cyclic permutation).

□

**P 3.14.** Let  $a, b, c$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 = 2.$$

Prove that

$$(a^3 + b^3)(b^3 + c^3)(c^3 + a^3) \leq 2.$$

(Vasile Cîrtoaje, 2011)

**Solution.** Let

$$x = a^2, \quad y = b^2, \quad z = c^2, \quad x + y + z = 2.$$

Since

$$(a^3 + b^3)^2 \leq (a^2 + b^2)(a^4 + b^4) = (x + y)(x^2 + y^2),$$

it suffices to prove that

$$(x + y)(y + z)(z + x)(x^2 + y^2)(y^2 + z^2)(z^2 + x^2) \leq 4.$$

Due to symmetry, we may assume that

$$x = \min\{x, y, z\}.$$

It is easy to check that the equality holds for  $x = 0$  and  $y = z = 1$ . Since

$$(x + y)(y + z)(z + x) \leq (x + y + z)(xy + yz + zx) = 2(xy + yz + zx)$$

and

$$x^2 + y^2 \leq y(x + y), \quad z^2 + x^2 \leq z(x + z),$$

it suffices to show that

$$yz(xy + yz + zx)(x + y)(x + z)(y^2 + z^2) \leq 2.$$

Write this inequality as

$$(2yz)[2(xy + yz + zx)][2(x + y)(x + z)](y^2 + z^2) \leq 16.$$

By the AM-GM inequality, it suffices to show that

$$\left[ \frac{2yz + 2(xy + yz + zx) + 2(x + y)(x + z) + (y^2 + z^2)}{4} \right]^4 \leq 16.$$

This inequality is equivalent to

$$\begin{aligned} 2x^2 + y^2 + z^2 + 6yz + 4xy + 4zx &\leq 8, \\ 2x^2 + y^2 + z^2 + 6yz + 4xy + 4zx &\leq 2(x + y + z)^2, \\ (y - z)^2 &\geq 0. \end{aligned}$$

The equality holds for  $a = 0$  and  $b = c = 1$  (or any cyclic permutation).

□

**P 3.15.** If  $a, b, c$  are nonnegative real numbers such that

$$a + b + c = 2,$$

then

$$(3a^2 - 2ab + 3b^2)(3b^2 - 2bc + 3c^2)(3c^2 - 2ca + 3a^2) \leq 36.$$

(Vasile Cîrtoaje, 2011)

**Solution.** Due to symmetry, assume that

$$a = \min\{a, b, c\}.$$

On the other hand, we can check that the equality holds for  $(a, b, c) = (0, 1, 1)$ . Since

$$0 \leq 3a^2 - 2ab + 3b^2 \leq b(a + 3b)$$

and

$$0 < 3c^2 - 2ca + 3a^2 \leq c(a + 3c),$$

it suffices to show that

$$bc(a + 3b)(a + 3c)(3b^2 - 2bc + 3c^2) \leq 36.$$

Write this inequality as

$$[4b(a+3c)][4c(a+3b)][3(3b^2-2bc+3c^2)] \leq 12^3.$$

By virtue of the AM-GM inequality, it suffices to show that

$$\left[ \frac{4b(a+3c) + 4c(a+3b) + 3(3b^2-2bc+3c^2)}{3} \right]^3 \leq 12^3.$$

This is equivalent to

$$9(b+c)^2 + 4a(b+c) \leq 36.$$

We have

$$\begin{aligned} 36 - 9(b+c)^2 - 4a(b+c) &= 9(a+b+c)^2 - 9(b+c)^2 - 4a(b+c) \\ &= a(9a+14b+14c) \geq 0. \end{aligned}$$

The equality holds for  $a=0$  and  $b=c=1$  (or any cyclic permutation).

**Remark.** Similarly, we can prove the following more general statement.

- Let  $a, b, c$  be nonnegative real numbers. If  $\frac{2}{3} \leq k \leq 2$ , then

$$(a^2 - kab + b^2)(b^2 - kbc + c^2)(c^2 - kca + a^2) \leq \frac{4}{27(2+k)^2}(a+b+c)^6,$$

with equality for  $a=0$  and  $\frac{b}{c} + \frac{c}{b} = 1 + \frac{3k}{2}$  (or any cyclic permutation).

□

**P 3.16.** Let  $a, b, c$  be nonnegative real numbers such that

$$a+b+c=3.$$

Prove that

$$(a^2 - 4ab + b^2)(b^2 - 4bc + c^2)(c^2 - 4ca + a^2) \leq 3.$$

(Vasile Cîrtoaje, 2011)

**Solution.** Assume that

$$a \leq b \leq c.$$

If  $c^2 - 4ca + a^2 \leq 0$ , then

$$b \leq c \leq (2 + \sqrt{3})a \leq (2 + \sqrt{3})b.$$

From

$$b \leq (2 + \sqrt{3})a, \quad c \leq (2 + \sqrt{3})b,$$

it follows that

$$a^2 - 4ab + b^2 \leq 0, \quad b^2 - 4bc + c^2 \leq 0.$$

Since

$$(a^2 - 4ab + b^2)(b^2 - 4bc + c^2)(c^2 - 4ca + a^2) \leq 0,$$

the desired inequality is trivial. Consider further that  $c^2 - 4ca + a^2 \geq 0$ . There are only two cases when the left hand side of the desired inequality is nonnegative:

$$a^2 - 4ab + b^2 \geq 0, \quad b^2 - 4bc + c^2 \geq 0, \quad c^2 - 4ca + a^2 \geq 0$$

and

$$a^2 - 4ab + b^2 \leq 0, \quad b^2 - 4bc + c^2 \leq 0, \quad c^2 - 4ca + a^2 \geq 0.$$

*Case 1:*  $a^2 - 4ab + b^2 \geq 0$ ,  $b^2 - 4bc + c^2 \geq 0$ ,  $c^2 - 4ca + a^2 \geq 0$ . Since

$$a^2 - 4ab + b^2 \leq b^2, \quad c^2 - 4ca + a^2 \leq c^2,$$

it suffices to prove that

$$b^2 c^2 (b^2 - 4bc + c^2) \leq 3.$$

It is easy to show that the homogeneous inequality

$$(a^2 - 4ab + b^2)(b^2 - 4bc + c^2)(c^2 - 4ca + a^2) \leq 3 \left( \frac{a+b+c}{3} \right)^6$$

becomes an equality for

$$a = 0, \quad b^2 + c^2 = 7bc.$$

Thus, we apply the AM-GM inequality as follows:

$$\begin{aligned} b^2 c^2 (b^2 - 4bc + c^2) &= \frac{1}{9} (3bc)(3bc)(b^2 - 4bc + c^2) \\ &\leq \frac{1}{9} \left[ \frac{3bc + 3bc + (b^2 - 4bc + c^2)}{3} \right]^3 \\ &= 3 \left( \frac{b+c}{3} \right)^6 \leq 3 \left( \frac{a+b+c}{3} \right)^6 = 3. \end{aligned}$$

*Case 2:*  $a^2 - 4ab + b^2 \leq 0$ ,  $b^2 - 4bc + c^2 \leq 0$ ,  $c^2 - 4ca + a^2 \geq 0$ . By the AM-GM inequality, we have

$$\begin{aligned} (4ab - a^2 - b^2)(4bc - b^2 - c^2)(c^2 - 4ca + a^2) &\leq \\ &\leq \left[ \frac{(4ab - a^2 - b^2) + (4bc - b^2 - c^2) + (c^2 - 4ca + a^2)}{3} \right]^3 \\ &= \frac{8}{27} (2ab + 2bc - 2ca - b^2)^3; \end{aligned}$$

therefore, it suffices to prove that

$$(2ab + 2bc - 2ca - b^2)^3 \leq \frac{81}{8},$$

which is equivalent to the homogeneous inequality

$$2\sqrt[3]{9}(2ab + 2bc - 2ca - b^2) \leq (a + b + c)^2.$$

Since  $2\sqrt[3]{9} < \frac{21}{5}$ , we only need to show that

$$21(2ab + 2bc - 2ca - b^2) \leq 5(a + b + c)^2.$$

Write this inequality as  $f(a) \geq 0$ , where

$$f(a) = 5a^2 + 4(13c - 8b)a + 26b^2 + 5c^2 - 32bc.$$

From  $a^2 - 4ab + b^2 \leq 0$ , it follows that

$$a \geq (2 - \sqrt{3})b,$$

which involves  $4a \geq b$  and  $5a^2 \geq \frac{b^2}{20}$ ; therefore,

$$f(a) \geq \frac{b^2}{20} + (13c - 8b)b + 26b^2 + 5c^2 - 32bc = \frac{1}{20}(19b - 10c)^2 \geq 0.$$

This completes the proof. The equality holds for  $a = 0$ ,  $b = \frac{3 - \sqrt{5}}{2}$  and  $c = \frac{3 + \sqrt{5}}{2}$  (or any permutation). □

**P 3.17.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = 3,$$

then

$$abc + \frac{12}{ab + bc + ca} \geq 5.$$

**Solution.** By the third degree Schur's inequality

$$(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca),$$

we get  $3abc \geq 4(ab + bc + ca) - 9$ . Thus, it suffices to prove that

$$4(ab + bc + ca) - 9 + \frac{36}{ab + bc + ca} \geq 15.$$

This inequality is equivalent to

$$(ab + bc + ca - 3)^2 \geq 0.$$

The equality holds for  $a = b = c = 1$ . □

**P 3.18.** If  $a, b, c$  are positive real numbers such that

$$a^2 + b^2 + c^2 = 3,$$

then

$$5(a + b + c) + \frac{3}{abc} \geq 18.$$

**Solution.** Let

$$x = \frac{a + b + c}{3}.$$

From

$$2(ab + bc + ca) = (a + b + c)^2 - (a^2 + b^2 + c^2) = 3(3x^2 - 1),$$

we get  $x > 1/\sqrt{3}$ . By the known inequality

$$(ab + bc + ca)^2 \geq 3abc(a + b + c),$$

we get

$$\frac{1}{abc} \geq \frac{4x}{(3x^2 - 1)^2}.$$

Then, it suffices to prove that

$$5x + \frac{4x}{(3x^2 - 1)^2} \geq 6,$$

which is equivalent to

$$15x^5 - 18x^4 - 10x^3 + 12x^2 + 3x - 2 \geq 0,$$

$$(x - 1)^2(15x^3 + 12x^2 - x - 2) \geq 0.$$

We still have to show that

$$15x^3 + 12x^2 - x - 2 \geq 0.$$

Since  $x > 1/\sqrt{3}$ , we get

$$15x^3 + 12x^2 - x - 2 > x^2\left(12 - \frac{1}{x} - \frac{2}{x^2}\right) > x^2(12 - \sqrt{3} - 6) > 0.$$

The equality holds for  $a = b = c = 1$ .

□

**P 3.19.** If  $a, b, c$  are positive real numbers such that

$$a^2 + b^2 + c^2 = 3,$$

then

$$12 + 9abc \geq 7(ab + bc + ca).$$

(Vasile Cîrtoaje, 2005)

**Solution.** Denote  $x = (a + b + c)/3$ . Since

$$2(ab + bc + ca) = (a + b + c)^2 - (a^2 + b^2 + c^2) = 3(3x^2 - 1),$$

we can write the inequality as

$$5 + 2abc \geq 7x^2.$$

By Schur's inequality of degree three, we get

$$(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca),$$

$$3x^3 + abc \geq 2x(3x^2 - 1),$$

$$abc \geq 3x^3 - 2x.$$

Then,

$$5 + 2abc - 7x^2 \geq 5 + 2(3x^3 - 2x) - 7x^2 = (x - 1)^2(6x + 5) \geq 0.$$

The equality holds for  $a = b = c = 1$ .

□

**P 3.20.** If  $a, b, c$  are positive real numbers such that

$$a^2 + b^2 + c^2 = 3,$$

then

$$21 + 18abc \geq 13(ab + bc + ca).$$

(Vasile Cîrtoaje, 2005)

**Solution.** Let

$$p = a + b + c, \quad q = ab + bc + ca.$$

From

$$2q = (a + b + c)^2 - (a^2 + b^2 + c^2) = p^2 - 3,$$

we get  $p > \sqrt{3}$ . In addition, from Schur's inequality of degree four, we have

$$abc \geq \frac{(p^2 - q)(4q - p^2)}{6p} = \frac{(p^2 + 3)(p^2 - 6)}{12p}.$$

Therefore,

$$\begin{aligned} 21 + 18abc - 13(ab + bc + ca) &\geq 21 + \frac{3(p^2 + 3)(p^2 - 6)}{2p} - \frac{13(p^2 - 3)}{2} \\ &= \frac{(p - 3)^2(3p^2 + 5p - 6)}{2p} \geq 0. \end{aligned}$$

The equality holds for  $a = b = c = 1$ .

□

**P 3.21.** If  $a, b, c$  are positive real numbers such that

$$a^2 + b^2 + c^2 = 3,$$

then

$$(2 - ab)(2 - bc)(2 - ca) \geq 1.$$

(Vasile Cîrtoaje, 2005)

**First Solution.** Let  $p = a + b + c$ . From

$$3(a^2 + b^2 + c^2) \geq (a + b + c)^2,$$

we get  $p \leq 3$ . Since

$$\begin{aligned} (2 - ab)(2 - bc)(2 - ca) &= 8 - 4(ab + bc + ca) + 2abc(a + b + c) - a^2b^2c^2 \\ &= 8 - 2(p^2 - 3) + 2abcp - a^2b^2c^2 \\ &= 14 - p^2 - (p - abc)^2, \end{aligned}$$

we can write the inequality as

$$13 - p^2 - (p - abc)^2 \geq 0.$$

Clearly,

$$3(p - abc) = (a^2 + b^2 + c^2)(a + b + c) - 3abc > 0.$$

By Schur's inequality

$$(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca),$$

we get

$$\begin{aligned} p^3 + 9abc &\geq 2p(p^2 - 3), \\ abc &\geq \frac{p(p^2 - 6)}{9}. \end{aligned}$$

Since

$$0 < p - abc \leq p - \frac{p(p^2 - 6)}{9} = \frac{p(15 - p^2)}{9},$$

it suffices to prove that

$$13 - p^2 - \frac{p^2(15 - p^2)^2}{81} \geq 0.$$

Setting

$$p = 3\sqrt{x}, \quad 0 < x \leq 1,$$

this inequality becomes

$$13 - 34x + 30x^2 - 9x^3 \geq 0.$$



It is true because

$$\begin{aligned} 13 - 34x + 30x^2 - 9x^3 &= (1 - x)(13 - 21x + 9x^2) \\ &= (1 - x)[1 + 3(1 - x)(4 - 3x)] \geq 0. \end{aligned}$$

The equality holds for  $a = b = c = 1$ .

**Second Solution.** We use the mixing variables technique. Assume that  $a \leq 1$  and show that

$$(2 - ab)(2 - bc)(2 - ca) \geq (2 - x^2)(2 - ax)^2 \geq 1,$$

where

$$x = \sqrt{\frac{b^2 + c^2}{2}} = \sqrt{\frac{3 - a^2}{2}}, \quad x < \sqrt{\frac{3}{2}}.$$

Since

$$2 - bc \geq 2 - \frac{1}{2}(b^2 + c^2) \geq 2 - \frac{3}{2} > 0$$

and, similarly,  $2 - ca > 0$ ,  $2 - ab > 0$ , we can prove the left inequality by multiplying the inequalities

$$2 - bc \geq 2 - x^2$$

and

$$(2 - ca)(2 - ab) \geq (2 - ax)^2.$$

The last inequality is true because

$$\begin{aligned} (2 - ca)(2 - ab) - (2 - ax)^2 &= 2a(2x - b - c) - a^2(x^2 - bc) \\ &= \frac{2a(b - c)^2}{2x + b + c} - \frac{a^2(b - c)^2}{2} \\ &= \frac{a(b - c)^2[4 - a(2x + b + c)]}{2(2x + b + c)} \end{aligned}$$

and

$$\begin{aligned} 4 - a(2x + b + c) &\geq 4(1 - ax) = 2(2 - a\sqrt{6 - 2a^2}) \\ &= \frac{4(1 - a^2)(2 - a^2)}{2 + a\sqrt{6 - 2a^2}} \geq 0. \end{aligned}$$

The right inequality,  $(2 - x^2)(2 - ax)^2 \geq 1$ , is equivalent to

$$(1 + a^2)(2 - ax)^2 \geq 2.$$

Since  $2(1 + a^2) \geq (1 + a)^2$  and  $2 - ax \geq 2 - x > 0$ , it suffices to show that

$$(1 + a)(2 - ax) \geq 2.$$

Indeed,

$$\begin{aligned}(1+a)(2-ax) - 2 &= a(2-x-ax) = \frac{a(a^4 + 2a^3 - 2a^2 - 6a + 5)}{2(2+x+ax)} \\ &= \frac{a(a-1)^2(a^2 + 4a + 5)}{2(2+x+ax)} \geq 0.\end{aligned}$$

□

**P 3.22.** Let  $a, b, c$  be positive real numbers such that

$$abc = 1.$$

Prove that

$$\left(\frac{a+b+c}{3}\right)^5 \geq \frac{a^2+b^2+c^2}{3}.$$

**First Solution.** Write the inequality in the homogeneous form

$$(a+b+c)^5 \geq 81abc(a^2+b^2+c^2).$$

Using to the known inequality

$$(ab+bc+ca)^2 \geq 3abc(a+b+c),$$

it suffices to show that

$$(a+b+c)^6 \geq 27(ab+bc+ca)^2(a^2+b^2+c^2).$$

Setting  $p = a+b+c$  and  $q = ab+bc+ca$ , we have

$$\begin{aligned}(a+b+c)^6 - 27(ab+bc+ca)^2(a^2+b^2+c^2) &= p^6 - 27q^2(p^2 - 2q) \\ &= (p^2 - 3q)^2(p^2 + 6q) \geq 0.\end{aligned}$$

The equality occurs for  $a = b = c = 1$ .

**Second Solution.** Use the mixing variables method. We show that

$$E(a, b, c) \geq E(a, x, x) \geq 0,$$

where

$$E(a, b, c) = (a+b+c)^5 - 81abc(a^2+b^2+c^2), \quad x = \frac{b+c}{2}.$$

Indeed, we have

$$\begin{aligned}\frac{E(a, b, c) - E(a, x, x)}{81} &= a^3(x^2 - bc) + a[2x^4 - bc(b^2 + c^2)] \\ &= \frac{1}{4}a^3(b-c)^2 + \frac{1}{8}a(b-c)^4 \geq 0\end{aligned}$$

and

$$\begin{aligned} E(a, x, x) &= (a + b + c)^5 - \frac{81}{8}a(b + c)^2[2a^2 + (b + c)^2] \\ &= \frac{1}{8}(2a - b - c)^2[2a^3 + 12a^2(b + c) - 9a(b + c)^2 + 8(b + c)^3] \geq 0, \end{aligned}$$

since

$$\begin{aligned} &2a^3 + 12a^2(b + c) - 9a(b + c)^2 + 8(b + c)^3 > \\ &> 6a^2(b + c) - 12a(b + c)^2 + 6(b + c)^3 \\ &= 6(b + c)(a - b - c)^2 \geq 0. \end{aligned}$$

□

**P 3.23.** If  $a, b, c$  are positive real numbers such that

$$abc = 1,$$

then

$$a^3 + b^3 + c^3 + a^{-3} + b^{-3} + c^{-3} + 21 \geq 3(a + b + c)(a^{-1} + b^{-1} + c^{-1}).$$

**Solution.** Since

$$a^3 + b^3 + c^3 + a^{-3} + b^{-3} + c^{-3} + 3 = \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right)$$

and

$$(a + b + c)(a^{-1} + b^{-1} + c^{-1}) = \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) + \left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) + 3,$$

we can write the desired inequality in the homogeneous form

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) + 9 \geq 3 \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) + 3 \left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right),$$

or

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} - 3\right) \left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c} - 3\right) \geq 0.$$

This is true because, by the AM-GM inequality, we have

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3, \quad \frac{b}{a} + \frac{c}{b} + \frac{a}{c} \geq 3.$$

The equality holds for  $a = b = c = 1$ .

□

**P 3.24.** If  $a, b, c$  are positive real numbers such that

$$abc = 1,$$

then

$$a^2 + b^2 + c^2 - ab - bc - ca \geq \frac{9}{4}(a + b + c - 3).$$

**Solution.** Write the inequality in the form

$$3(4x^2 - 3x + 3) \geq 4(ab + bc + ca),$$

where

$$x = \frac{a + b + c}{3}.$$

The third degree Schur's inequality states that

$$(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca),$$

which is equivalent to

$$4(ab + bc + ca) \leq \frac{3(3x^3 + 1)}{x}.$$

Therefore, it suffices to show that

$$3(4x^2 - 3x + 3) \geq \frac{3(3x^3 + 1)}{x}.$$

This inequality reduces to

$$(x - 1)^3 \geq 0,$$

which is true because

$$x \geq \sqrt[3]{abc} = 1.$$

The equality holds for  $a = b = c = 1$ .

□

**P 3.25.** If  $a, b, c$  are positive real numbers such that

$$abc = 1,$$

then

$$a^2 + b^2 + c^2 + a + b + c \geq 2(ab + bc + ca).$$

**Solution.** Let

$$p = a + b + c, \quad q = ab + bc + ca.$$

By virtue of the AM-GM inequality, we have

$$p \geq 3\sqrt[3]{abc} = 3,$$

and by Schur's inequality

$$p^3 + 9abc \geq 4pq,$$

we get

$$4q \leq \frac{p^3 + 9}{p}.$$

Therefore,

$$\begin{aligned} a^2 + b^2 + c^2 + a + b + c - 2(ab + bc + ca) &= p^2 + p - 4q \\ &\geq p^2 + p - \frac{p^3 + 9}{p} \\ &= \frac{(p - 3)(p + 3)}{p} \geq 0. \end{aligned}$$

The equality holds for  $a = b = c = 1$ .

□

**P 3.26.** If  $a, b, c$  are positive real numbers such that

$$abc = 1,$$

then

$$a^2 + b^2 + c^2 + 15(ab + bc + ca) \geq 16(a + b + c).$$

**Solution.** Write the inequality as  $F(a, b, c) \geq 0$ , where

$$F(a, b, c) = a^2 + b^2 + c^2 + 15\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 16(a + b + c).$$

Assume that  $a \geq b \geq c$  and denote

$$t = \sqrt{bc}, \quad 0 < t \leq 1, \quad at^2 = 1.$$

We will show that

$$F(a, b, c) \geq F(a, t, t) \geq 0.$$

Since

$$F(a, b, c) - F(a, t, t) = b^2 + c^2 - 2t^2 + 15\left(\frac{1}{b} + \frac{1}{c} - \frac{2}{t}\right) - 16(b + c - 2t)$$

$$\begin{aligned}
&= (b - c)^2 + 15 \left( \frac{1}{\sqrt{b}} - \frac{1}{\sqrt{c}} \right)^2 - 16(\sqrt{b} - \sqrt{c})^2 \\
&= (\sqrt{b} - \sqrt{c})^2 \left[ (\sqrt{b} + \sqrt{c})^2 + \frac{15}{bc} - 16 \right] \geq (\sqrt{b} - \sqrt{c})^2 \left( 4\sqrt{bc} + \frac{15}{bc} - 16 \right),
\end{aligned}$$

it suffices to show that

$$4t + \frac{15}{t^2} - 16 \geq 0.$$

Indeed,

$$4t + \frac{15}{t^2} - 16 > t + \frac{15}{t} - 16 = \frac{(1-t)(15-t)}{t} \geq 0.$$

The inequality  $F(a, t, t) \geq 0$  is equivalent to

$$(t-1)^2(17t^4 + 2t^3 - 13t^2 + 2t + 1) \geq 0.$$

We have

$$\begin{aligned}
17t^4 + 2t^3 - 13t^2 + 2t + 1 &= (2t-1)^4 + t(t^3 + 34t^2 - 37t + 10) \\
&= (2t-1)^4 + \frac{t}{4}[t(2t-1)^2 + 140t^2 - 149t + 40] > 0
\end{aligned}$$

since  $D = 149^2 - 4 \cdot 140 \cdot 40 = -199$ . The equality holds for  $a = b = c = 1$ .

□

**P 3.27.** If  $a, b, c$  are positive real numbers such that

$$abc = 1,$$

then

$$\frac{2}{a+b+c} + \frac{1}{3} \geq \frac{3}{ab+bc+ca}.$$

**Solution.** Let

$$x = \frac{ab+bc+ca}{3}.$$

By virtue of the AM-GM inequality, we have

$$x \geq \sqrt[3]{ab \cdot bc \cdot ca} = 1.$$

The third degree Schur's inequality applied to  $ab, bc, ca$ , states that

$$(ab+bc+ca)^3 + 9a^2b^2c^2 \geq 4abc(a+b+c)(ab+bc+ca),$$

which is equivalent to

$$\frac{3}{a+b+c} \geq \frac{4x}{3x^3+1}.$$

Therefore,

$$\begin{aligned} 3 \left( \frac{2}{a+b+c} + \frac{1}{3} - \frac{3}{ab+bc+ca} \right) &\geq \frac{8x}{3x^3+1} + 1 - \frac{3}{x} \\ &= \frac{3x^4 - 9x^3 + 8x^2 + x - 3}{x(3x^3+1)} = \frac{(x-1)(3x^3 - 6x^2 + 2x + 3)}{x(3x^3+1)}. \end{aligned}$$

Since  $x \geq 1$ , we need to show that

$$3x^3 - 6x^2 + 2x + 3 \geq 0.$$

For  $x \geq 2$ , we have

$$3x^3 - 6x^2 + 2x + 3 > 3x^3 - 6x^2 = 3x^2(x-2) \geq 0,$$

and for  $1 \leq x < 2$ , we have

$$3x^3 - 6x^2 + 2x + 3 = 3x(x-1)^2 + 3 - x > 0.$$

The equality holds for  $a = b = c = 1$ .

□

**P 3.28.** Let  $a, b, c$  be the lengths of the sides of a triangle. If

$$a^2 + b^2 + c^2 = 3,$$

then

$$a + b + c \geq 2 + abc.$$

(Vasile Cîrtoaje, 2005)

**Solution.** Let

$$p = a + b + c, \quad q = ab + bc + ca.$$

We need to show that

$$p^2 - 2q = 3$$

involves

$$p \geq 2 + abc.$$

Since

$$pq \geq 9abc,$$

it suffices to show that  $p^2 - 2q = 3$  implies

$$9p \geq 18 + pq;$$

that is,

$$18p \geq 36 + p(p^2 - 3),$$

$$\begin{aligned} p^3 - 21p + 36 &\leq 0, \\ (p - 3)(p^2 + 3p - 12) &\leq 0. \end{aligned}$$

Since

$$p^2 \leq 3(a^2 + b^2 + c^2) = 9,$$

we need to show that

$$p^2 + 3p - 12 \geq 0.$$

From

$$\begin{aligned} 2(ab + bc + ca) - a^2 - b^2 - c^2 &= \\ = (\sqrt{a} + \sqrt{b} + \sqrt{c})(-\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{a} - \sqrt{b} + \sqrt{c})(\sqrt{a} + \sqrt{b} - \sqrt{c}) &\geq 0, \end{aligned}$$

we get

$$p^2 \geq 2(a^2 + b^2 + c^2) = 6,$$

hence

$$p^2 + 3p - 12 \geq 6 + 3\sqrt{6} - 12 > 0.$$

The equality occurs for  $a = b = c$ .

□

**P 3.29.** If  $a, b, c$  are positive real numbers such that

$$abc = 1,$$

then

$$\sqrt[3]{(1+a)(1+b)(1+c)} \geq \sqrt[4]{4(1+a+b+c)}.$$

(Pham Huu Duc, 2008)

**Solution.** Since

$$\begin{aligned} (1+a)(1+b)(1+c) &= (1+a+b+c) + (1+ab+bc+ca) \\ &\geq 2\sqrt{(1+a+b+c)(1+ab+bc+ca)}, \end{aligned}$$

it suffices to prove that

$$(1+ab+bc+ca)^2 \geq 4(1+a+b+c),$$

which is equivalent to

$$(1+q)^2 \geq 4(1+p),$$

where

$$p = a + b + c, \quad q = ab + bc + ca.$$

Setting  $x = bc$ ,  $y = ca$ ,  $z = ab$  in Schur's inequality

$$(x+y+z)^3 + 9xyz \geq 4(x+y+z)(xy+yz+zx),$$



we get

$$q^3 + 9 \geq 4pq.$$

Since

$$\begin{aligned} (1+q)^2 - 4(1+p) &\geq (1+q)^2 - 4 - \frac{q^3+9}{q} \\ &= \frac{(q-3)(2q+3)}{q}, \end{aligned}$$

it suffices to show that  $q \geq 3$ . Indeed, by the AM-GM inequality, we have

$$q = ab + bc + ca \geq 3\sqrt[3]{a^2b^2c^2} = 3.$$

The equality holds for  $a = b = c = 1$

□

**P 3.30.** If  $a, b, c$  are positive real numbers, then

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq 18(a^2 - bc)(b^2 - ca)(c^2 - ab).$$

(Vasile Cîrtoaje, 2007)

**Solution.** Due to homogeneity, we may assume that  $abc = 1$ , when the inequality can be written as

$$a^6 + b^6 + c^6 - 3 \geq 18(a^3 + b^3 + c^3 - a^3b^3 - b^3c^3 - c^3a^3).$$

Substituting  $a^3, b^3, c^3$  by  $a, b, c$ , respectively, we need to show that  $abc = 1$  implies  $F(a, b, c) \geq 0$ , where

$$F(a, b, c) = a^2 + b^2 + c^2 - 3 - 18(a + b + c - ab - bc - ca).$$

To do this, we use the mixing variables method. Without loss of generality, assume that  $a \geq 1$ . We claim that

$$F(a, b, c) \geq F(a, \sqrt{bc}, \sqrt{bc}) \geq 0.$$

We have

$$\begin{aligned} F(a, b, c) - F(a, \sqrt{bc}, \sqrt{bc}) &= (b-c)^2 - 18(\sqrt{b} - \sqrt{c})^2 + 18a(\sqrt{b} - \sqrt{c})^2 \\ &= (b-c)^2 + 18(a-1)(\sqrt{b} - \sqrt{c})^2 \geq 0. \end{aligned}$$

Also, putting  $\sqrt{bc} = t$ , we have

$$\begin{aligned} F(a, \sqrt{bc}, \sqrt{bc}) &= F\left(\frac{1}{t^2}, t, t\right) = \frac{1}{t^4} + 20t^2 - 3 - \frac{18}{t^2} - 36t + \frac{36}{t} \\ &= \frac{(t-1)^2(2t-1)^2(t+1)(5t+1)}{t^4} \geq 0. \end{aligned}$$

The equality holds for  $a = b = c$ , and for  $a/2 = b = c$  (or any cyclic permutation).

□

**P 3.31.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = 3,$$

then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq a^2 + b^2 + c^2.$$

(Vasile Cîrtoaje, 2006)

**First Solution.** Since

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca},$$

it suffices to prove that

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq a^2 + b^2 + c^2,$$

which is equivalent to

$$abc(a^2 + b^2 + c^2) \leq 3.$$

Let  $x = (ab + bc + ca)/3$ . From the known inequality

$$(ab + bc + ca)^2 \geq 3abc(a + b + c),$$

we get

$$abc \leq x^2.$$

On the other hand, we have

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = 9 - 6x.$$

Then,

$$abc(a^2 + b^2 + c^2) - 3 \leq x^2(9 - 6x) - 3 = -3(x - 1)^2(2x + 1) \leq 0.$$

The equality holds for  $a = b = c = 1$ .

**Second Solution.** Since  $a + b + c = 3$ , we can write the inequality as

$$\sum \left( \frac{1}{a^2} - a^2 + 4a - 4 \right) \geq 0,$$

which is equivalent to

$$\sum \frac{(1 - a)^2(1 + 2a - a^2)}{a^2} \geq 0.$$

Without loss of generality, assume that  $a = \max\{a, b, c\}$ . We have two cases to consider.

*Case 1:*  $a \leq 1 + \sqrt{2}$ . Since  $a, b, c \leq 1 + \sqrt{2}$ , we have

$$1 + 2a - a^2 \geq 0, \quad 1 + 2b - b^2 \geq 0, \quad 1 + 2c - c^2 \geq 0.$$

Thus, the conclusion follows.

Case 2:  $a > 1 + \sqrt{2}$ . Since  $b + c = 3 - a < 2 - \sqrt{2} < \frac{2}{3}$ , we have

$$bc \leq \frac{1}{4}(b + c)^2 < \frac{1}{9},$$

and hence

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} > \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{2}{bc} > 18 > (a + b + c)^2 > a^2 + b^2 + c^2.$$

□

**P 3.32.** If  $a, b, c$  are positive real numbers such that

$$ab + bc + ca = 3,$$

then

$$a^3 + b^3 + c^3 + 7abc \geq 10.$$

(Vasile Cîrtoaje, 2005)

**Solution.** Let

$$x = \frac{a + b + c}{3}.$$

By the well-known inequality

$$(a + b + c)^2 \geq 3(ab + bc + ca),$$

we get  $x \geq 1$ . Since

$$\begin{aligned} a^3 + b^3 + c^3 &= 3abc + (a + b + c)^3 - 3(a + b + c)(ab + bc + ca) \\ &= 3abc + 27x^3 - 27x, \end{aligned}$$

we can write the inequality as

$$10abc + 27x^3 - 27x - 10 \geq 0.$$

For  $x \geq \frac{4}{3}$ , this inequality is true since

$$27x^3 - 27x - 10 = 27x(x^2 - 1) - 10 \geq 36\left(\frac{16}{9} - 1\right) - 10 = 18.$$

For  $1 \leq x \leq \frac{4}{3}$ , we use Schur's inequality

$$(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca),$$

which is equivalent to

$$abc + 3x^3 - 4x \geq 0.$$

Therefore,

$$\begin{aligned} 10abc + 27x^3 - 27x - 10 &\geq 10(-3x^3 + 4x) + 27x^3 - 27x - 10 \\ &= (x - 1)[4 - 3x + 3(2 - x^2)] \geq 0. \end{aligned}$$

The equality holds for  $a = b = c = 1$ .

□

**P 3.33.** If  $a, b, c$  are nonnegative real numbers such that

$$a^3 + b^3 + c^3 = 3,$$

then

$$a^4b^4 + b^4c^4 + c^4a^4 \leq 3.$$

(Vasile Cîrtoaje, 2003)

**Solution.** By virtue of the AM-GM inequality, we have

$$ab \leq \frac{a^3 + b^3 + 1}{3} = \frac{4 - c^3}{3},$$

hence

$$a^4b^4 = a^3b^3 \cdot ab \leq \frac{4a^3b^3 - a^3b^3c^3}{3}.$$

Similarly,

$$b^4c^4 \leq \frac{4b^3c^3 - a^3b^3c^3}{3}, \quad c^4a^4 \leq \frac{4c^3a^3 - a^3b^3c^3}{3}.$$

Summing these inequalities, we obtain

$$a^4b^4 + b^4c^4 + c^4a^4 \leq \frac{4(a^3b^3 + b^3c^3 + c^3a^3)}{3} - a^3b^3c^3.$$

Using the substitutions  $x = a^3$ ,  $y = b^3$ ,  $z = c^3$ , it suffices to prove that

$$4(xy + yz + zx) \leq 3xyz + 9,$$

where  $x, y, z$  are nonnegative real numbers satisfying  $x + y + z = 3$ . This follows immediately from Schur's inequality

$$4(x + y + z)(xy + yz + zx) \leq 9xyz + (x + y + z)^3.$$

The equality holds for  $a = b = c = 1$ .

**Remark 1.** We may write the inequality in the homogeneous form

$$\left(\frac{a^4b^4 + b^4c^4 + c^4a^4}{3}\right)^3 \leq \left(\frac{a^3 + b^3 + c^3}{3}\right)^8.$$

From this, we get the reverse statement.

- If  $a, b, c$  are nonnegative real numbers such that  $a^4b^4 + b^4c^4 + c^4a^4 = 3$ , then

$$a^3 + b^3 + c^3 \geq 3.$$

**Remark 2.** The inequality in P 3.33 is a particular case of the following statement:

- Let  $a, b, c$  be nonnegative real numbers such that  $a + b + c = 3$ . If  $0 < k \leq k_0$ , where

$$k_0 = \frac{\ln 3}{\ln 9 - \ln 4} \approx 1.355,$$

then

$$a^kb^k + b^kc^k + c^ka^k \leq 3.$$

□

**P 3.34.** If  $a, b, c$  are nonnegative real numbers, then

$$(a+1)^2(b+1)^2(c+1)^2 \geq 4(a+b+c)(ab+bc+ca) + 28abc.$$

(Vasile Cîrtoaje, 2011)

**Solution.** By the AM-GM inequality, we have

$$\begin{aligned} (a+1)(b+1)(c+1) &= (abc+1) + (a+b+c) + (ab+bc+ca) \\ &\geq 2\sqrt{abc} + 2\sqrt{(a+b+c)(ab+bc+ca)}. \end{aligned}$$

Thus, it suffices to prove that

$$\left[\sqrt{abc} + \sqrt{(a+b+c)(ab+bc+ca)}\right]^2 \geq (a+b+c)(ab+bc+ca) + 7abc,$$

which can be written as

$$\sqrt{abc(a+b+c)(ab+bc+ca)} \geq 3abc.$$

This is true if

$$\sqrt{(a+b+c)(ab+bc+ca)} \geq 3\sqrt{abc}.$$

Indeed, we have

$$(a+b+c)(ab+bc+ca) - 9abc = a(b-c)^2 + b(c-a)^2 + c(a-b)^2 \geq 0.$$

The equality holds for  $a = b = c = 1$ .

□

**P 3.35.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = 3,$$

then

$$1 + 8abc \geq 9 \min\{a, b, c\}.$$

(Vasile Cîrtoaje, 2007)

**Solution.** Without loss of generality, assume that

$$a = \min\{a, b, c\}, \quad a \leq 1,$$

when the inequality becomes

$$1 + 8abc \geq 9a.$$

From  $(a - b)(a - c) \geq 0$ , we get

$$bc \geq a(b + c) - a^2 = a(3 - a) - a^2 = a(3 - 2a^2).$$

Therefore,

$$1 + 8abc - 9a \geq 1 + 8a^2(3 - 2a^2) - 9a = (1 - a)(1 - 4a)^2 \geq 0.$$

The equality holds for  $a = b = c = 1$ , and also for  $(a, b, c) = \left(\frac{1}{4}, \frac{1}{4}, \frac{5}{2}\right)$  (or any cyclic permutation). □

**P 3.36.** If  $a, b, c$  are positive real numbers such that

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 + 4abc \geq 5 \min\{a, b, c\}.$$

(Vasile Cîrtoaje, 2007)

**Solution.** Without loss of generality, assume that

$$a = \min\{a, b, c\}, \quad a \leq 1.$$

The inequality can be written as

$$1 + 4abc \geq 5a.$$

From  $(a^2 - b^2)(a^2 - c^2) \geq 0$ , we get

$$bc \geq a\sqrt{b^2 + c^2 - a^2} = a\sqrt{3 - 2a^2}.$$

Therefore, it suffices to prove that

$$4a^2\sqrt{3-2a^2} \geq 5a-1.$$

We consider two cases.

*Case 1:*  $0 < a \leq 1/3$ . Since

$$\sqrt{3-2a^2} \geq \frac{5}{3} > \frac{25}{16},$$

it is enough to show that

$$\frac{25}{4}a^2 \geq 5a-1.$$

This inequality is equivalent to  $(5a-2)^2 \geq 0$ .

*Case 2:*  $1/3 < a \leq 1$ . Since

$$\sqrt{3-2a^2} \geq 2-a,$$

which is equivalent to the obvious inequality

$$(1-a)(3a-1) \geq 0,$$

it suffices to show that

$$4a^2(2-a) \geq 5a-1.$$

Indeed, we have

$$4a^2(2-a) - 5a + 1 = (1-a)(2a-1)^2 \geq 0.$$

The proof is completed. The equality holds for  $a = b = c = 1$ .

□

**P 3.37.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = abc,$$

then

$$(1-a)(1-b)(1-c) + (\sqrt{3}-1)^3 \geq 0.$$

**Solution.** Without loss of generality, assume that

$$a \geq b \geq c.$$

The product  $(1-a)(1-b)(1-c)$  is negative for either  $a > 1 > b \geq c$  or  $a \geq b \geq c > 1$ . Since  $a > 1 > b \geq c$  involves the contradiction

$$0 = a + b + c - abc > a(1-bc) > 0,$$

it suffices to consider only the case  $a \geq b \geq c > 1$ . Setting

$$x = a - 1, \quad y = b - 1, \quad z = c - 1,$$

we need to show that

$$xyz \leq (\sqrt{3} - 1)^3$$

for all  $x, y, z > 0$  such that

$$xy + yz + zx + xyz = 2.$$

Let

$$t = \sqrt[3]{xyz}, \quad t > 0.$$

By the AM-GM inequality, we have

$$2 = xy + yz + zx + xyz \geq 3\sqrt[3]{x^2y^2z^2} + xyz = 3t^2 + t^3,$$

hence

$$\begin{aligned} t^3 + 3t^2 - 2 &\leq 0, \\ (t + 1)(t^2 + 2t - 2) &\leq 0, \\ t^2 + 2t - 2 &\leq 0, \\ t &\leq \sqrt{3} - 1, \\ xyz &\leq (\sqrt{3} - 1)^3. \end{aligned}$$

The equality holds for  $a = b = c = \sqrt{3}$ .

□

**P 3.38.** If  $a, b, c$  are nonnegative real numbers such that

$$a + b + c = 2,$$

then

$$(a^2 + bc)(b^2 + ca)(c^2 + ab) \leq 1.$$

(Vasile Cîrtoaje, 2005)

**Solution.** Without loss of generality, assume that

$$a \geq b \geq c.$$

Since

$$a^2 + bc \leq (a + \frac{c}{2})^2$$

and

$$(b^2 + ca)(c^2 + ab) \leq \frac{1}{4}(b^2 + ca + c^2 + ab)^2,$$

it suffices to show that

$$(2a + c)(b^2 + c^2 + ab + ac) \leq 4.$$

Let

$$E(a, b, c) = (2a + c)(b^2 + c^2 + ab + ac).$$



We will show that

$$E(a, b, c) \leq E(a, b + c, 0) \leq 4.$$

Indeed,

$$E(a, b, c) - E(a, b + c, 0) = c(b^2 + c^2 + ac - 3ab) \leq 0$$

and

$$\begin{aligned} E(a, b + c, 0) - 4 &= 2a(a + b + c)(b + c) - 4 \\ &= 4a(2 - a) - 4 = -4(a - 1)^2 \leq 0. \end{aligned}$$

The equality occurs for  $a = b = 1$  and  $c = 0$  (or any cyclic permutation).

□

**P 3.39.** If  $a, b, c$  are nonnegative real numbers, then

$$(8a^2 + bc)(8b^2 + ca)(8c^2 + ab) \leq (a + b + c)^6.$$

**Solution.** We use the mixing variables technique. Without loss of generality, assume that  $a \leq b \leq c$ . Let

$$x = \frac{b + c}{2}, \quad x \geq a,$$

and

$$E(a, b, c) = (8a^2 + bc)(8b^2 + ca)(8c^2 + ab) - (a + b + c)^6.$$

We will prove that

$$E(a, b, c) \leq E(a, x, x) \leq 0.$$

The left inequality is equivalent to

$$(8a^2 + x^2)(8x^2 + ax)^2 \geq (8a^2 + bc)(8b^2 + ca)(8c^2 + ab),$$

which follows by multiplying the inequalities

$$8a^2 + x^2 \geq 8a^2 + bc$$

and

$$(8x^2 + ax)^2 \geq (8b^2 + ca)(8c^2 + ab).$$

The first inequality is obvious, and the last inequality is equivalent to

$$64(x^4 - b^2c^2) + a^2(x^2 - bc) - 8a(b^3 + c^3 - 2x^3) \geq 0.$$

Since

$$b^3 + c^3 - 2x^3 = \frac{3(b + c)(b - c)^2}{4} = 6x(x^2 - bc) \geq 0,$$

we need to show that

$$64(x^2 + bc) + a^2 - 48ax \geq 0.$$

This is true, since

$$64(x^2 + bc) + a^2 - 48ax \geq 48x(x - a) \geq 0.$$

The right inequality,  $E(a, x, x) \leq 0$ , is equivalent to

$$\begin{aligned} (8a^2 + x^2)(8x^2 + ax)^2 - (a + 2x)^6 &\leq 0, \\ 176x^5 - 273ax^4 + 32a^2x^3 + 52a^3x^2 + 12a^4x + a^5 &\geq 0, \\ (x - a)^2(176x^3 + 79ax^2 + 14a^2x + a^3) &\geq 0. \end{aligned}$$

The equality holds for  $a = b = c$ , and for  $a = 0$  and  $b = c$  (or any cyclic permutation).

□

**P 3.40.** If  $a, b, c$  are positive real numbers such that

$$a^2b^2 + b^2c^2 + c^2a^2 = 3,$$

then

$$a + b + c \geq abc + 2.$$

(Vasile Cîrtoaje, 2006)

**Solution.** Without loss of generality, assume that

$$a \geq b \geq c.$$

From  $a^2b^2 + b^2c^2 + c^2a^2 = 3$ , it follows that

$$1 \leq ab < \sqrt{3}.$$

We have

$$\begin{aligned} a + b + c - abc - 2 &= a + b - 2 - (ab - 1)c \\ &\geq 2\sqrt{ab} - 2 - (ab - 1)c \\ &= (\sqrt{ab} - 1) \left[ 2 - (\sqrt{ab} + 1)c \right]. \end{aligned}$$

So, we need to prove that

$$2 \geq (\sqrt{ab} + 1)c.$$

Since

$$\sqrt{ab} + 1 \leq 2\sqrt{ab},$$

it suffices to show that  $2\sqrt{ab} \leq 2\sqrt{ab}$ ; that is,

$$1 \geq abc^2.$$

Indeed, we have

$$c^2 = \frac{3 - a^2b^2}{a^2 + b^2} \leq \frac{3 - a^2b^2}{2ab},$$

hence

$$1 - abc^2 \geq 1 - \frac{3 - a^2b^2}{2} = \frac{a^2b^2 - 1}{2} \geq 0.$$

The equality holds for  $a = b = c = 1$ .

□

**P 3.41.** Let  $a, b, c$  be nonnegative real numbers such that

$$a + b + c = 5.$$

Prove that

$$(a^2 + 3)(b^2 + 3)(c^2 + 3) \geq 192.$$

**First Solution.** Without loss of generality, assume that

$$a = \min\{a, b, c\}, \quad a \leq \frac{5}{3}.$$

By virtue of the Cauchy-Schwarz inequality, we have

$$(b^2 + 3)(c^2 + 3) = (b^2 + 3)(3 + c^2) \geq 3(b + c)^2 = 3(5 - a)^2.$$

Therefore, it suffices to show that

$$(a^2 + 3)(5 - a)^2 \geq 64$$

for  $0 \leq a \leq \frac{5}{3}$ . Indeed,

$$(a^2 + 3)(5 - a)^2 - 64 = (a - 1)^2(a^2 - 8a + 11) \geq 0,$$

since

$$a^2 - 8a + 11 = \left(\frac{5}{3} - a\right) \left(\frac{19}{3} - a\right) + \frac{4}{9} > 0.$$

The equality holds for  $a = 3$  and  $b = c = 1$  (or any cyclic permutation).

**Second Solution.** Without loss of generality, assume that

$$a = \max\{a, b, c\}.$$

First, we show that

$$(b^2 + 3)(c^2 + 3) \geq (x^2 + 3)^2,$$

where

$$x = \frac{b+c}{2}, \quad 0 \leq x \leq \frac{5}{3}.$$

This inequality is equivalent to

$$(b-c)^2(6-bc-x^2) \geq 0,$$

which is true because

$$6-bc-x^2 \geq 2(3-x^2) > 0.$$

Thus, it suffices to prove that

$$(a^2+3)(x^2+3)^2 \geq 192,$$

which is equivalent to

$$[(5-2x)^2+3](x^2+3)^2 \geq 192,$$

$$(x^2-5x+7)(x^2+3)^2 \geq 48,$$

$$(x-1)^2(x^4-3x^3+6x^2-15x+15) \geq 0.$$

This inequality is true since

$$x^4-3x^3+6x^2-15x+15 = x^2 \left(x - \frac{3}{2}\right)^2 + 15 \left(\frac{x}{2} - 1\right)^2 > 0.$$

□

**P 3.42.** *If  $a, b, c$  are nonnegative real numbers, then*

$$a^2 + b^2 + c^2 + abc + 2 \geq a + b + c + ab + bc + ca.$$

(Michael Rozenberg, 2012)

**Solution.** Among the numbers

$$1-a, \quad 1-b, \quad 1-c,$$

there are always two with the same sign; let us say  $(1-b)(1-c) \geq 0$ , that is

$$bc \geq b+c-1.$$

Thus, it suffices to show that

$$a^2 + b^2 + c^2 + a(b+c-1) + 2 \geq a + b + c + ab + bc + ca,$$

which is equivalent to

$$a^2 - 2a + b^2 + c^2 - bc - (b+c) + 2 \geq 0.$$

Since

$$b^2 + c^2 - bc \geq \frac{1}{4}(b + c)^2,$$

it suffices to show that

$$a^2 - 2a + \frac{1}{4}(b + c)^2 - (b + c) + 2 \geq 0,$$

which can be written in the obvious form

$$(a - 1)^2 + \left( \frac{b + c}{2} - 1 \right)^2 \geq 0.$$

The equality holds for  $a = b = c = 1$ .

□

**P 3.43.** If  $a, b, c$  are nonnegative real numbers, then

$$\sum a^3(b + c)(a - b)(a - c) \geq 3(a - b)^2(b - c)^2(c - a)^2.$$

**Solution.** Without loss of generality, assume that

$$a = \min\{a, b, c\}.$$

Since

$$a^3(b + c)(a - b)(a - c) \geq 0$$

and

$$\begin{aligned} & b^3(c + a)(b - c)(b - a) + c^3(a + b)(c - a)(c - b) = \\ &= (b - c)[bc(b^3 - c^3) + (b - c)(b^3 + c^3)a - (b^3 - c^3)a^2] \\ &= (b - c)^2[(b^2 + bc + c^2)(bc - a^2) + (b^3 + c^3)a] \\ &\geq (b - c)^2(b^2 + bc + c^2)(bc - a^2), \end{aligned}$$

it suffices to show that

$$(b^2 + bc + c^2)(bc - a^2) \geq 3(a - b)^2(c - a)^2.$$

Since

$$bc - a^2 = (a - b)(a - c) + a(b + c - 2a) \geq (a - b)(a - c),$$

it suffices to show that

$$b^2 + bc + c^2 \geq 3(a - b)(a - c),$$

which is equivalent to the obvious inequality

$$(b - c)^2 + 3a(b + c - a) \geq 0.$$

The equality holds for  $a = b = c$ , for  $a = 0$  and  $b = c$  (or any cyclic permutation), and for  $b = c = 0$  (or any cyclic permutation).

□

**P 3.44.** Prove that 2 is the greatest real number  $k$  such that

$$a + b + c + 4abc \geq k(ab + bc + ca)$$

for all  $a, b, c \in [0, 1]$ .

**Solution.** Setting  $a = 0$  and  $b = c = 1$ , we get  $k \leq 2$ . So, we claim that  $k = 2$  is the greatest real number  $k$ . To prove this, we only need to show that

$$a + b + c + 4abc \geq 2(ab + bc + ca)$$

for all  $a, b, c \in [0, 1]$ . Write the inequality as

$$a(1 + 4bc - 2b - 2c) + b + c - 2bc \geq 0.$$

Since

$$b + c - 2bc = b(1 - c) + c(1 - b) \geq 0,$$

the inequality is clearly true for  $1 + 4bc - 2b - 2c \geq 0$ . Consider further that  $1 + 4bc - 2b - 2c < 0$ , when it suffices to show that

$$(1 + 4bc - 2b - 2c) + b + c - 2bc \geq 0.$$

This is equivalent to the obvious inequality

$$bc + (1 - b)(1 - c) \geq 0.$$

Thus, the proof is completed. If  $k = 2$ , then the equality holds for  $a = b = c = 0$ , and also for  $a = 0$  and  $b = c = 1$  (or any cyclic permutation).

**Remark.** From the proof above it follows that the following stronger inequality holds for all  $a, b, c \in [0, 1]$ :

$$a + b + c + 3abc \geq 2(ab + bc + ca),$$

with equality for  $a = b = c = 0$ , for  $a = b = c = 1$ , and also for  $a = 0$  and  $b = c = 1$  (or any cyclic permutation). □

**P 3.45.** If  $a, b, c \geq \frac{2}{3}$  such that

$$a + b + c = 3,$$

then

$$a^2b^2 + b^2c^2 + c^2a^2 \geq ab + bc + ca.$$

**Solution.** We use the mixing variables method. Assume that  $a = \max\{a, b, c\}$  and denote

$$x = \frac{b+c}{2}, \quad 2/3 \leq x \leq 1.$$

We will show that

$$E(a, b, c) \geq E(a, x, x) \geq 0,$$

where

$$E(a, b, c) = a^2b^2 + b^2c^2 + c^2a^2 - ab - bc - ca.$$

We have

$$\begin{aligned} E(a, b, c) - E(a, x, x) &= a^2(b^2 + c^2 - 2x^2) - (x^4 - b^2c^2) + (x^2 - bc) \\ &= (x^2 - bc)(2a^2 - x^2 - bc + 1) \\ &= \frac{1}{4}(b-c)^2[a^2 + (a^2 - bc) + (1 - x^2)] \geq 0 \end{aligned}$$

and

$$E(a, x, x) = 2a^2x^2 + x^4 - 2ax - x^2.$$

Since  $a + 2x = 3$ , we get

$$\begin{aligned} 9E(a, x, x) &= 18a^2x^2 + 9x^4 - (2ax + x^2)(a + 2x)^2 \\ &= x(5x^3 - 12ax^2 + 9a^2x - 2a^3) \\ &= x(x-a)^2(5x-2a) \\ &= 3x(x-a)^2(3x-2) \geq 0. \end{aligned}$$

The equality holds for  $a = b = c = 1$ , and also for  $a = 5/3$  and  $b = c = 2/3$  (or any cyclic permutation).

□

**P 3.46.** If  $a, b, c$  are positive real numbers such that

$$a \leq 1 \leq b \leq c, \quad a + b + c = 3,$$

then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq a^2 + b^2 + c^2.$$

**Solution.** Let

$$F(a, b, c) = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - a^2 - b^2 - c^2.$$

We will show that

$$F(a, b, c) \geq F(a, 1, b+c-1) \geq 0.$$

The left inequality is true since

$$\begin{aligned}
 F(a, b, c) - F(a, 1, b + c - 1) &= \\
 &= \left( \frac{1}{b} + \frac{1}{c} - 1 - \frac{1}{b + c - 1} \right) + 1 + (b + c - 1)^2 - b^2 - c^2 \\
 &= (b + c) \left( \frac{1}{bc} - \frac{1}{b + c - 1} \right) + 2(b - 1)(c - 1) \\
 &= (b - 1)(c - 1) \left[ 2 - \frac{b + c}{bc(b + c - 1)} \right]
 \end{aligned}$$

and

$$\begin{aligned}
 2bc(b + c - 1) - b - c &= (2bc - 1)(b + c) - 2bc \\
 &\geq 2(2bc - 1)\sqrt{bc} - 2bc \\
 &= 2\sqrt{bc} (\sqrt{bc} - 1)(2\sqrt{bc} + 1) \geq 0.
 \end{aligned}$$

The right inequality  $F(a, 1, b + c - 1) \geq 0$  is equivalent to  $F(a, 1, x) \geq 0$ , where  $x > 0$  and  $x + a = 2$ . We have

$$F(a, 1, x) = \frac{1}{a} + \frac{1}{x} - a^2 - x^2 = \frac{(x + a)^4}{8ax} - a^2 - x^2 = \frac{(x - a)^4}{8ax} \geq 0.$$

The equality holds for  $a = b = c = 1$ .

□

**P 3.47.** If  $a, b, c$  are positive real numbers such that

$$a \leq 1 \leq b \leq c, \quad a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c},$$

then

$$a^2 + b^2 + c^2 \leq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

(Vasile Cîrtoaje, 2008)

**Solution.** Write the inequality as

$$b^2 - \frac{1}{b^2} \leq (a^2 + c^2) \left( \frac{1}{a^2 c^2} - 1 \right).$$

From  $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ , we have

$$b - \frac{1}{b} = (a + c) \left( \frac{1}{ac} - 1 \right) \geq 0.$$



Thus, the desired inequality holds if

$$(a + c) \left( b + \frac{1}{b} \right) \leq (a^2 + c^2) \left( \frac{1}{ac} + 1 \right).$$

On the other hand, we have

$$b + \frac{1}{b} - c - \frac{1}{c} = (b - c) - \frac{b - c}{bc} = (b - c) \left( 1 - \frac{1}{bc} \right) \leq 0.$$

Then, it suffices to prove that

$$(a + c) \left( c + \frac{1}{c} \right) \leq (a^2 + c^2) \left( \frac{1}{ac} + 1 \right),$$

which is equivalent to the obvious inequality

$$c(1 - a^2)(a - c) \leq 0.$$

The proof is completed. The equality holds for  $b = 1$  and  $ac = 1$ .

□

**P 3.48.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c},$$

then

$$(abc - 1) \left( a^n + b^n + c^n - \frac{1}{a^n} - \frac{1}{b^n} - \frac{1}{c^n} \right) \leq 0$$

for any integer  $n \geq 2$ .

(Vasile Cîrtoaje, 2007)

**Solution.** Since the statement remains unchanged by substituting  $a, b, c$  with  $1/a, 1/b, 1/c$ , respectively, it suffices to prove that

$$a^n + b^n + c^n - \frac{1}{a^n} - \frac{1}{b^n} - \frac{1}{c^n} \leq 0$$

for

$$abc \geq 1, \quad a + b + c = 1/a + 1/b + 1/c.$$

It is easy to check that  $a + b + c = 1/a + 1/b + 1/c$  is equivalent to

$$(ab - 1)(bc - 1)(ca - 1) = a^2b^2c^2 - 1,$$

and the desired inequality is equivalent to

$$(a^n b^n - 1)(b^n c^n - 1)(c^n a^n - 1) \geq a^{2n} b^{2n} c^{2n} - 1.$$

Setting

$$x = bc, \quad y = ca, \quad z = ab,$$

we need to show that

$$(x-1)(y-1)(z-1) = xyz - 1 \geq 0$$

involves

$$(x^n - 1)(y^n - 1)(z^n - 1) \geq x^n y^n z^n - 1.$$

This inequality holds if

$$\begin{aligned} (x^{n-1} + x^{n-2} + \cdots + 1)(y^{n-1} + y^{n-2} + \cdots + 1)(z^{n-1} + z^{n-2} + \cdots + 1) &\geq \\ &\geq x^{n-1}y^{n-1}z^{n-1} + x^{n-2}y^{n-2}z^{n-2} + \cdots + 1. \end{aligned}$$

Since the last inequality is clearly true, the proof is completed. The equality occurs for  $a = bc = 1$  (or any cyclic permutation). □

**P 3.49.** Let  $a, b, c$  be positive real numbers, and let

$$E(a, b, c) = a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b).$$

Prove that

$$(a) \quad (a+b+c)E(a, b, c) \geq ab(a-b)^2 + bc(b-c)^2 + ca(c-a)^2;$$

$$(b) \quad 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)E(a, b, c) \geq (a-b)^2 + (b-c)^2 + (c-a)^2.$$

**Solution.** (a) Using Schur's inequality of degree four

$$\sum a^2(a-b)(a-c) \geq 0,$$

we have

$$\begin{aligned} (a+b+c)E(a, b, c) &= \sum a^2(a-b)(a-c) + \sum a(b+c)(a-b)(a-c) \\ &\geq \sum a(b+c)(a-b)(a-c) \\ &= \sum ab(a-b)(a-c) + \sum ac(a-b)(a-c) \\ &= \sum ab(a-b)(a-c) + \sum ba(b-c)(b-a) \\ &= \sum ab(a-b)^2 \geq 0. \end{aligned}$$

The equality holds for  $a = b = c$ . If  $a, b, c$  are nonnegative real numbers, then the equality also holds for  $a = 0$  and  $b = c$  (or any cyclic permutation).

(b) Since

$$\begin{aligned}
 (ab + bc + ca)E(a, b, c) &= \\
 &= abc \sum (a - b)(a - c) + \sum (a^2b + a^2c)(a - b)(a - c) \\
 &= \frac{1}{2}abc \sum (a - b)^2 + \sum [a^2b(a - b)(a - c) + b^2a(b - c)(b - a)] \\
 &= \frac{1}{2}abc \sum (a - b)^2 + \sum ab(a - b)^2(a + b - c),
 \end{aligned}$$

the required inequality is equivalent to

$$\sum ab(a - b)^2(a + b - c) \geq 0.$$

Without loss of generality, assume that  $a \geq b \geq c$ . Then,

$$\begin{aligned}
 \sum ab(a - b)^2(a + b - c) &\geq bc(b - c)^2(b + c - a) + ac(a - c)^2(a + c - b) \\
 &\geq bc(b - c)^2(b + c - a) + ac(b - c)^2(a + c - b) \\
 &= c(b - c)^2[(a - b)^2 + c(a + b)] \geq 0.
 \end{aligned}$$

The equality holds for  $a = b = c$ .

□

**P 3.50.** Let  $a \geq b \geq c$  be nonnegative real numbers. Schur's inequalities of third and fourth degree state that

$$(a) \quad a(a - b)(a - c) + b(b - c)(b - a) + c(c - a)(c - b) \geq 0;$$

$$(b) \quad a^2(a - b)(a - c) + b^2(b - c)(b - a) + c^2(c - a)(c - b) \geq 0.$$

Prove that (a) is sharper than (b) if

$$\sqrt{b} + \sqrt{c} \leq \sqrt{a},$$

and (b) is sharper than (a) if

$$\sqrt{b} + \sqrt{c} \geq \sqrt{a}.$$

(Vasile Cîrtoaje, 2005)

**Solution.** Let

$$p = a + b + c, \quad q = ab + bc + ca.$$

If we rewrite Schur's inequalities as

$$abc \geq f(p, q)$$

and

$$abc \geq g(p, q),$$

respectively, then (a) is sharper than (b) if  $f(p, q) \geq g(p, q)$ , while (b) is sharper than (a) if  $g(p, q) \geq f(p, q)$ . Therefore, we need to show that

$$(\sqrt{b} + \sqrt{c} - \sqrt{a})[g(p, q) - f(p, q)] \geq 0.$$

From the known relation

$$4q - p^2 = (\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{b} + \sqrt{c} - \sqrt{a})(\sqrt{c} + \sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b} - \sqrt{c}),$$

it follows that  $4q - p^2$  and  $\sqrt{b} + \sqrt{c} - \sqrt{a}$  has the same sign. Therefore, it suffices to prove that

$$(4q - p^2)[g(p, q) - f(p, q)] \geq 0.$$

In order to find  $f(p, q)$ , write the inequality in (a) as follows

$$\begin{aligned} a^3 + b^3 + c^3 + 3abc &\geq ab(a + b) + bc(b + c) + ca(c + a) \\ (a + b + c)^3 + 9abc &\geq 4(a + b + c)(ab + bc + ca), \end{aligned}$$

from which

$$f(p, q) = \frac{p(4q - p^2)}{9}.$$

Analogously, write the inequality in (b) as follows

$$\begin{aligned} a^4 + b^4 + c^4 + abc(a + b + c) &\geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2), \\ a^4 + b^4 + c^4 + 2abc(a + b + c) &\geq (ab + bc + ca)(a^2 + b^2 + c^2). \end{aligned}$$

Since

$$a^2 + b^2 + c^2 = p^2 - 2q$$

and

$$\begin{aligned} a^4 + b^4 + c^4 &= (a^2 + b^2 + c^2)^2 - 2(a^2b^2 + b^2c^2 + c^2a^2) \\ &= (p^2 - 2q)^2 - 2q^2 + 4abcp, \end{aligned}$$

we get

$$g(p, q) = \frac{(p^2 - q)(4q - p^2)}{6p}.$$

Therefore, we have

$$g(p, q) - f(p, q) = \frac{(p^2 - 3q)(4q - p^2)}{18p},$$

hence

$$(4q - p^2)[g(p, q) - f(p, q)] = \frac{(p^2 - 3q)(4q - p^2)^2}{18p} \geq 0.$$

**Remark.** If  $a, b, c$  are the lengths of the sides of a triangle, then Schur's inequality of degree four is always stronger than Schur's inequality of degree three.

□

**P 3.51.** If  $a, b, c$  are nonnegative real numbers such that

$$(a+b)(b+c)(c+a) = 8,$$

then

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca.$$

(Vasile Cîrtoaje, 2010)

**First Solution.** Assume that  $a \geq b \geq c$ , and write the inequality in the equivalent homogeneous forms

$$(\sqrt{a} + \sqrt{b} + \sqrt{c})\sqrt{(a+b)(b+c)(c+a)} \geq 2\sqrt{2}(ab + bc + ca),$$

$$\sum \sqrt{a(b+c)}[\sqrt{(a+b)(a+c)} - \sqrt{2a(b+c)}] \geq 0,$$

$$\sum \frac{(a-b)(a-c)\sqrt{a(b+c)}}{\sqrt{(a+b)(a+c)} + \sqrt{2a(b+c)}} \geq 0.$$

Since  $(c-a)(c-b) \geq 0$ , it suffices to prove that

$$\frac{(a-b)(a-c)\sqrt{a(b+c)}}{\sqrt{(a+b)(a+c)} + \sqrt{2a(b+c)}} + \frac{(b-c)(b-a)\sqrt{b(c+a)}}{\sqrt{(b+c)(b+a)} + \sqrt{2b(c+a)}} \geq 0,$$

which is true if

$$\frac{(a-c)\sqrt{a(b+c)}}{\sqrt{(a+b)(a+c)} + \sqrt{2a(b+c)}} \geq \frac{(b-c)\sqrt{b(c+a)}}{\sqrt{(b+c)(b+a)} + \sqrt{2b(c+a)}}.$$

Since  $\sqrt{a} \geq \sqrt{b}$ ,

$$\sqrt{(a+b)(a+c)} \geq \sqrt{2a(b+c)}$$

and

$$\sqrt{(b+c)(b+a)} \leq \sqrt{2b(a+c)},$$

it suffices to show that

$$\frac{(a-c)\sqrt{b+c}}{\sqrt{(a+b)(a+c)}} \geq \frac{(b-c)\sqrt{c+a}}{\sqrt{(b+c)(b+a)}}.$$

This is equivalent to the obvious inequality

$$c(a-b) \geq 0.$$

The equality holds for  $a = b = c = 1$ , and for  $a = 0$  and  $b = c = \sqrt[3]{4}$  (or any cyclic permutation).

**Second Solution.** Let

$$p = a + b + c, \quad q = ab + bc + ca.$$

By squaring, the inequality becomes

$$p + 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}) \geq q^2.$$

Since

$$\begin{aligned}\sqrt{ab} &\geq \frac{2ab}{a+b} = \frac{ab(b+c)(c+a)}{4} = \frac{ab(q+c^2)}{4}, \\ \sqrt{bc} &\geq \frac{bc(q+c^2)}{4}, \quad \sqrt{ca} \geq \frac{ca(q+c^2)}{4},\end{aligned}$$

we have

$$2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}) \geq \frac{q(ab+bc+ca) + abc(a+b+c)}{2} = \frac{q^2 + abcp}{2}.$$

Using this result, it suffices to show that

$$p + \frac{q^2 + abcp}{2} \geq q^2,$$

which is equivalent to

$$p(2 + abc) \geq q^2.$$

Having in view the hypothesis  $pq - abc = 8$ , we can write this inequality in the homogeneous forms

$$p \left( \frac{pq - abc}{4} + abc \right) \geq q^2,$$

$$q(p^2 - 4q) + 3abcp \geq 0.$$

For the non-trivial case where  $p^2 - 4q \leq 0$ , using Schur's inequality

$$p^3 + 3abc \geq 4pq$$

gives

$$3(p^2q - 4q^2 + 3abcp) \geq 3p^2q - 12q^2 + (4pq - p^3)p = (p^2 - 3q)(4q - p^2) \geq 0.$$

□

**P 3.52.** If  $a, b, c \in [1, 4 + 3\sqrt{2}]$ , then

$$9(ab + bc + ca)(a^2 + b^2 + c^2) \geq (a + b + c)^4.$$

(Vasile Cîrtoaje, 2005)

**Solution.** Let

$$A = a^2 + b^2 + c^2, \quad B = ab + bc + ca.$$

Since

$$\begin{aligned} 9(ab + bc + ca)(a^2 + b^2 + c^2) - (a + b + c)^4 &= 9AB - (A + 2B)^2 \\ &= (A - B)(4B - A) \end{aligned}$$

and

$$2(A - B) = (a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0,$$

we need to show that  $4B - A \geq 0$ ; that is, to show that  $E(a, b, c) \leq 0$ , where

$$E(a, b, c) = a^2 + b^2 + c^2 - 4(ab + bc + ca).$$

We claim that  $E(a, b, c)$  has the maximum value for  $a, b, c \in \{1, w\}$ , where  $w = 4 + 3\sqrt{2}$ . For the sake of contradiction, assume that there exists a triple  $(a, b, c)$  with  $a \in (1, w)$  such that

$$E(a, b, c) \geq \max\{E(1, b, c), E(w, b, c)\}.$$

From

$$E(a, b, c) - E(1, b, c) = (a - 1)(a + 1 - 4b - 4c) \geq 0,$$

we get

$$a - 4(b + c) \geq -1,$$

and from

$$E(a, b, c) - E(w, b, c) = (a - w)(a + w - 4b - 4c) \geq 0,$$

we get

$$a - 4(b + c) \leq -w.$$

These results involve  $w \leq 1$ , which is false. Therefore, since  $E(a, b, c)$  is symmetric, we have

$$\begin{aligned} E(a, b, c) &\leq \max\{E(1, 1, 1), E(1, 1, w), E(1, w, w), E(w, w, w)\} \\ &= \max\{-9, w^2 - 8w - 2, 1 - 2w^2 - 8w, -9w^2\} \\ &= \max\{-9, 0, -99 - 72\sqrt{2}, -306 - 216\sqrt{2}\} = 0. \end{aligned}$$

This completes the proof. The equality holds for  $a = b = c$ , and also for  $a = b = 1$  and  $c = 4 + 3\sqrt{2}$  (or any cyclic permutation). □

**P 3.53.** If  $a, b, c$  are nonnegative real numbers such that

$$a + b + c + abc = 4,$$

then

$$(a) \quad a^2 + b^2 + c^2 + 12 \geq 5(ab + bc + ca);$$

$$(b) \quad 3(a^2 + b^2 + c^2) + 13(ab + bc + ca) \geq 48.$$

**Solution.** Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

(a) We need to show that

$$p^2 + 12 \geq 7q$$

for  $p + r = 4$ . By Schur's inequality of degree three, we have  $p^3 + 9r \geq 4pq$ . Therefore, we get

$$\begin{aligned} 4p(p^2 + 12 - 7q) &\geq 4p^3 + 48p - 7(p^3 + 9r) \\ &= -3(p^3 - 37p + 84) \\ &= 3(p - 3)(4 - p)(7 + p). \end{aligned}$$

Since  $4 - p = r \geq 0$ , we only need to show that  $p \geq 3$ . By virtue of the AM-GM inequality, we get

$$\begin{aligned} p^3 &\geq 27r, \\ p^3 &\geq 27(4 - p), \\ (p - 3)(p^2 + 3p + 36) &\geq 0, \\ p &\geq 3. \end{aligned}$$

The equality holds for  $a = b = c = 1$ , and for  $a = 0$  and  $b = c = 2$  (or any cyclic permutation).

(b) We need to show that

$$3p^2 + 7q \geq 48$$

for  $p + r = 4$ . Using the known inequality  $pq \geq 9r$ , we get

$$\begin{aligned} p(3p^2 + 7q - 48) &\geq 3(p^3 + 21r - 16p) \\ &= 3(p^3 - 37p + 84) \\ &= 3(p - 3)(4 - p)(7 + p) \geq 0. \end{aligned}$$

The equality holds for  $a = b = c = 1$ .

□

**P 3.54.** Let  $a, b, c$  be the lengths of the sides of a triangle. If

$$a + b + c = 3,$$

then

$$a^2b^2 + b^2c^2 + c^2a^2 \geq ab + bc + ca.$$



**Solution.** Write the inequality as follows:

$$\begin{aligned}
9(a^2b^2 + b^2c^2 + c^2a^2) &\geq (ab + bc + ca)(a + b + c)^2; \\
3[3(a^2b^2 + b^2c^2 + c^2a^2) - (ab + bc + ca)^2] &\geq \\
&\geq (ab + bc + ca)[(a + b + c)^2 - 3(ab + bc + ca)]; \\
6[a^2(b - c)^2 + b^2(c - a)^2 + c^2(a - b)^2] &\geq \\
&\geq (ab + bc + ca)[(b - c)^2 + (c - a)^2 + (a - b)^2] \geq 0; \\
\sum S_a(b - c)^2 &\geq 0,
\end{aligned}$$

where

$$S_a = 6a^2 - ab - bc - ca.$$

Without loss of generality, assume that  $a \geq b \geq c$ . It suffices to show that

$$S_b(a - c)^2 + S_c(a - b)^2 \geq 0.$$

Since

$$(a - c)^2 \geq (a - b)^2,$$

$$S_b = 6b^2 - bc - a(b + c) \geq 6b^2 - bc - (b + c)^2 > 0$$

and

$$\begin{aligned}
S_b + S_c &= 6(b^2 + c^2) - 2bc - 2a(b + c) \geq 6(b^2 + c^2) - 2bc - 2(b + c)^2 \\
&= 4(b - c)^2 + 2bc > 0,
\end{aligned}$$

we get

$$S_b(a - c)^2 + S_c(a - b)^2 \geq (S_b + S_c)(a - b)^2 \geq 0.$$

The equality holds for an equilateral triangle.

□

**P 3.55.** Let  $a, b, c$  be the lengths of the sides of a triangle. If

$$a^2 + b^2 + c^2 = 3,$$

then

$$ab + bc + ca \geq 1 + 2abc.$$

(Vasile Cîrtoaje, 2005)

**Solution.** Write the inequality as

$$3(ab + bc + ca) - a^2 - b^2 - c^2 \geq 6abc,$$

where the both sides are homogeneous. From

$$3(a^2 + b^2 + c^2) \geq (a + b + c)^2,$$

we get

$$a + b + c \leq 3.$$

Therefore, it suffices to prove the homogeneous inequality

$$(a + b + c)[3(ab + bc + ca) - a^2 - b^2 - c^2] \geq 18abc.$$

This is equivalent to

$$2ab(a + b) + 2bc(b + c) + 2ca(c + a) \geq a^3 + b^3 + c^3 + 9abc.$$

Using the known substitution

$$a = y + z, \quad b = z + x, \quad c = x + y, \quad x, y, z \geq 0,$$

the inequality can be written as

$$x^3 + y^3 + z^3 + 3xyz \geq xy(x + y) + yz(y + z) + zx(z + x),$$

which is just the third degree Schur's inequality. The equality holds for an equilateral triangle.

□

**P 3.56.** Let  $a, b, c$  be the lengths of the sides of a triangle. If

$$a + b + c = 3,$$

then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{41}{6} \geq 3(a^2 + b^2 + c^2).$$

(Vasile Cîrtoaje, 2010)

**Solution** (by Vo Quoc Ba Can). Using the substitution

$$a = \frac{y + z}{2}, \quad b = \frac{z + x}{2}, \quad c = \frac{x + y}{2},$$

where  $x, y, z \geq 0$  such that  $x + y + z = 3$ , the inequality becomes as follow

$$\frac{1}{y + z} + \frac{1}{z + x} + \frac{1}{x + y} + \frac{41}{12} \geq \frac{3}{8}[(y + z)^2 + (z + x)^2 + (x + y)^2],$$

$$\begin{aligned}\sum \frac{x+y+z}{y+z} + \frac{41}{4} &\geq \frac{9}{4} \left( \sum x^2 + \sum xy \right), \\ \sum \frac{x}{y+z} + 3 + \frac{41}{4} &\geq \frac{9}{4} \left( 9 - \sum xy \right), \\ \sum \frac{x}{y+z} &\geq 7 - \frac{9}{4} \sum xy.\end{aligned}$$

Let us denote  $t = xy + yz + zx$ . Since

$$\begin{aligned}\sum \frac{x}{y+z} &= \frac{1}{t} \sum \frac{x(xy+yz+zx)}{y+z} \geq \frac{1}{t} \sum \frac{x(xy+zx)}{y+z} \\ &= \frac{1}{t} \sum x^2 = \frac{9-2t}{t},\end{aligned}$$

it suffices to show that

$$\frac{9-2t}{t} \geq 7 - \frac{9}{4}t,$$

which is equivalent to

$$(t-2)^2 \geq 0.$$

The equality holds for a degenerate triangle having  $a = 3/2$ ,  $b = 1$ ,  $c = 1/2$  (or any permutation thereof).

□

**P 3.57.** Let  $p$  and  $q$  be fixed real numbers such that there exist three nonnegative real numbers  $a, b, c$  satisfying

$$a \geq b \geq c \geq 0, \quad a + b + c = p, \quad ab + bc + ca = q.$$

Then,

(a) the product  $r = abc$  has the minimum value only when  $a = b$  or  $c = 0$ , and the maximum value only when  $b = c$ ;

(b) if  $a, b, c$  are the lengths of the sides of a triangle (non-degenerate or degenerate), then the product  $r = abc$  has the minimum value only when  $a = b \geq c$ , and the maximum value only when  $b = c \geq \frac{a}{2}$  or  $b + c = a$ .

(Vasile Cîrtoaje, 2005)

**Solution.** (a) If  $a = c$ , then  $a = b = c$ . Assume further  $a > c$  and consider  $a, c$  and  $r$  as functions of  $b$ . Differentiating the equality constraints yields

$$a' + c' + 1 = 0, \quad a'(b+c) + c'(a+b) + a + c = 0,$$

hence

$$a'(b) = \frac{-(b-c)}{a-c} \leq 0, \quad c'(b) = \frac{-(a-b)}{a-c} \leq 0.$$

Since  $a'(b) \leq 0$  and  $c'(b) \leq 0$ , the functions  $a(b)$  and  $c(b)$  are decreasing. From

$$\begin{aligned} r'(b) &= ac + bca' + abc' = ac - \frac{bc(b-c)}{a-c} - \frac{ab(a-b)}{a-c} \\ &= ac - b(a+c-b) = -(a-b)(b-c) \leq 0, \end{aligned}$$

it follows that  $r(b)$  is decreasing. Thus,  $r(b)$  has the minimum value when  $b$  is maximum ( $a$  and  $c$  are minimum), i.e. when  $b = a$  or  $c = 0$ , and has the maximum value when  $b$  is minimum ( $c$  is maximum), i.e. when  $b = c$ .

(b) Using the known substitution

$$a = y + z, \quad b = z + x, \quad c = x + y,$$

where  $0 \leq x \leq y \leq z$ , from

$$\begin{aligned} a + b + c &= 2(x + y + z), \\ ab + bc + ca &= (x + y + z)^2 + xy + yz + zx, \\ abc &= (x + y + z)(xy + yz + zx) - xyz, \end{aligned}$$

it follows that

$$x + y + z = \frac{p}{2}, \quad xy + yz + zx = q - \frac{p^2}{4}, \quad abc = \frac{p}{2} \left( q - \frac{p^2}{4} \right) - xyz.$$

Therefore, the product  $abc$  has the minimum value when  $xyz$  has the maximum value; that is, according to (a), only when  $x = y$ , which is equivalent to  $a = b \geq c$ . Also, the product  $abc$  has the maximum value when  $xyz$  has the minimum value; that is, according to (a), only when  $y = z$  or  $x = 0$ , which is equivalent to  $b = c \geq a/2$  or  $b + c = a$ .

**Remark 1.** Using the result in (a), we can prove by the contradiction method (as in Remark 1 from P 2.53) the following generalization:

• If  $p$  and  $S$  are fixed real numbers such that there exist  $n$  nonnegative real numbers  $a_1, a_2, \dots, a_n$  satisfying

$$a_1 \geq a_2 \geq \dots \geq a_n \geq 0, \quad a_1 + a_2 + \dots + a_n = p, \quad a_1^2 + a_2^2 + \dots + a_n^2 = S,$$

then the product

$$r = a_1 a_2 \dots a_n$$

has the minimum value for  $a_1 = \dots = a_{n-1} \geq a_n$  or  $a_n = 0$ , and the maximum value for  $a_1 \geq a_2 = \dots = a_n$ .

**Remark 2.** The statements for real variables in Remark 2 from P 2.53 are also valid for nonnegative variables  $a, b, c \in [m, M]$  and  $a_1, a_2, \dots, a_n \in [m, M]$ , where  $0 \leq m < M$ .

• If  $p$  and  $q$  are fixed real numbers such that there exist three nonnegative real numbers  $a, b, c$  satisfying

$$M \geq a \geq b \geq c \geq m \geq 0, \quad a + b + c = p, \quad ab + bc + ca = q,$$

then the product

$$r = abc$$

has the minimum value when  $a = b \geq c$  or  $c = m$ , and the maximum value when  $a \geq b = c$  or  $a = M$ .

• If  $p$  and  $S$  are fixed real numbers such that there exist  $n$  nonnegative real numbers  $a_1, a_2, \dots, a_n$  satisfying

$$M \geq a_1 \geq a_2 \geq \dots \geq a_n \geq m \geq 0, \quad a_1 + a_2 + \dots + a_n = p, \quad a_1^2 + a_2^2 + \dots + a_n^2 = S,$$

then the product

$$r = a_1 a_2 \dots a_n$$

has the minimum value for  $a_1 = \dots = a_{n-1} \geq a_n$  or  $a_n = m$ , and the maximum value for  $a_1 \geq a_2 = \dots = a_n$  or  $a_1 = M$ . □

**P 3.58.** Let  $p$  and  $r$  be fixed real numbers such that there exist three nonnegative real numbers  $a, b, c$  satisfying

$$a \geq b \geq c \geq 0, \quad a + b + c = p, \quad abc = r.$$

Then, the sum

$$q = ab + bc + ca$$

has the minimum value only when  $b = c$ , and the maximum value only when  $a = b$ .

(Vasile Cîrtoaje, 2005)

**Solution.** If  $a = c$ , then  $a = b = c$ . Assume further  $a > c$ . If  $r = 0$ , then  $c=0$ ,  $p = a + b$  and  $q = ab$ . Clearly,  $q$  has the minimum value only when  $b = c = 0$ . Since  $p^2 - 4q = (a - b)^2 \geq 0$ ,  $q$  has the maximum value only when  $a = b$ . For  $r > 0$ , consider  $a, c$  and  $q$  as functions of  $b$ . By differentiating the equality constraints, we get

$$a' + c' + 1 = 0, \quad a'bc + c'ab + ac = 0,$$

hence

$$a'(b) = \frac{-a(b-c)}{b(a-c)} \leq 0, \quad c'(b) = \frac{-c(a-b)}{b(a-c)} \leq 0.$$

Since  $a'(b) \leq 0$  and  $c'(b) \leq 0$ , the functions  $a(b)$  and  $c(b)$  are decreasing. From

$$\begin{aligned} q'(b) &= a + c + (b+c)a' + (a+b)c' = a + c - \frac{a(b^2 - c^2)}{b(a-c)} - \frac{c(a^2 - b^2)}{b(a-c)} \\ &= a + c - \frac{ac + b^2}{b} = \frac{(a-b)(b-c)}{b} \geq 0, \end{aligned}$$

it follows that  $q(b)$  is increasing. Thus,  $q(b)$  has the minimum value when  $b$  is minimum ( $c$  is maximum), i.e. when  $b = c$ , and has the maximum value when  $b$  is maximum ( $a$  is minimum), i.e. when  $b = a$ .

**Remark 1.** Since

$$q = r \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)$$

and

$$q = \frac{1}{2}(p^2 - a^2 - b^2 - c^2),$$

we can extend the statement of P 3.58 as follows:

• Let  $a \geq b \geq c$  be nonnegative (positive) real numbers. If  $a + b + c$  and  $abc$  are fixed, then the sums

$$q = ab + bc + ca, \quad p_1 = -(a^2 + b^2 + c^2)$$

has the minimum values only when  $b = c$ , and the maximum value only when  $a = b$ .

**Remark 2.** Replacing  $a, b, c$  in P 3.58 with  $1/a, 1/b, 1/c$ , respectively, we get the following statement:

• Let  $a \leq b \leq c$  be nonnegative real numbers. If  $ab + bc + ca$  and  $abc$  are fixed, then the sum

$$p = a + b + ca$$

has the minimum value only when  $b = c$ , and the maximum value only when  $a = b$ .

**Remark 3.** By the contradiction method, we can prove (as in Remark 1 from P 2.53) the following generalization:

• Let  $a_1 \geq a_2 \geq \dots \geq a_n$  be nonnegative (positive) real numbers. If  $a_1 + a_2 + \dots + a_n$  and  $a_1 a_2 \dots a_n$  are fixed, then the sum

$$p_1 = -(a_1^2 + a_2^2 + \dots + a_n^2)$$

have the minimum and maximum values when  $n-1$  of  $a_1, a_2, \dots, a_n$  are equal, more precisely, they have the minimum value when  $a_1 \geq a_2 = \dots = a_n$ , and the maximum values when  $a_1 = \dots = a_{n-1} \geq a_n$ .

**Remark 4.** If  $M \geq a \geq b \geq c \geq m \geq 0$ , then  $q(b)$  has the minimum value when  $b$  is minimum ( $a$  and  $c$  are maximum), i.e. when  $b = c$  and  $a = M$ , and has the maximum value when  $b$  is maximum ( $a$  and  $c$  are minimum), i.e. when  $b = a$  and  $c = m$ . Thus, the following statement holds:

• Let  $0 \leq m < M$  and  $a, b, c \in [m, M]$  such that  $a \geq b \geq c$ . If  $a + b + c$  and  $abc$  are fixed, then the sums

$$q = ab + bc + ca, \quad S = -(a^2 + b^2 + c^2)$$

have the minimum value when  $b = c$  or  $a = M$ , and the maximum value when  $a = b$  or  $c = m$ .

We can generalize this result as follows:

• Let  $0 < m < M$  and  $a_1, a_2, \dots, a_n \in [m, M]$  such that  $a_1 \geq a_2 \geq \dots \geq a_n$ . If  $a_1 + a_2 + \dots + a_n$  and  $a_1 a_2 \dots a_n$  are fixed, then the sum

$$S = -(a_1^2 + a_2^2 + \dots + a_n^2)$$

have the minimum value when  $a_1 \geq a_2 = \dots = a_n$  or  $a_1 = M$ , and the maximum value when  $a_1 = \dots = a_{n-1} \geq a_n$  or  $a_n = m$ .

□

**P 3.59.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = 3,$$

then

$$\frac{9}{abc} + 16 \geq \frac{75}{ab + bc + ca}.$$

(Vasile Cîrtoaje, 2005)

**Solution.** Let

$$q = ab + bc + ca.$$

For fixed  $q$ , the product  $abc$  has the maximum value when two of  $a, b, c$  are equal - see P 3.57-(a). Therefore, it suffices to prove the inequality for

$$a = b, \quad c = 3 - 2a, \quad a < 3/2.$$

We have

$$\begin{aligned} \frac{9}{abc} + 16 - \frac{75}{ab + bc + ca} &= \frac{9}{a^2c} + 16 - \frac{75}{a(a + 2c)} \\ &= \frac{9}{a^2(3 - 2a)} + 16 - \frac{25}{a(2 - a)} \\ &= \frac{2(16a^4 - 56a^3 + 73a^2 - 42a + 9)}{a^2(3 - 2a)(2 - a)} \\ &= \frac{2(a - 1)^2(4a - 3)^2}{a^2(3 - 2a)(2 - a)} \geq 0, \end{aligned}$$

as desired. The equality holds for  $(a, b, c) = (1, 1, 1)$ , and also for  $(a, b, c) = \left(\frac{3}{4}, \frac{3}{4}, \frac{3}{2}\right)$  or any cyclic permutation.

□

**P 3.60.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = 3,$$

then

$$8 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) + 9 \geq 10(a^2 + b^2 + c^2).$$

(Vasile Cîrtoaje, 2006)

**First Solution.** Putting

$$q = ab + bc + ca,$$

we can write the inequality as

$$\frac{8q}{abc} + 20q \geq 81.$$

By P 3.57-(a), the product  $abc$  has the maximum value for fixed  $q$  when two of  $a, b, c$  are equal. Therefore, it suffices to prove the inequality for

$$a = b, \quad c = 3 - 2a, \quad a < 3/2.$$

We have

$$\begin{aligned} \frac{8q}{abc} + 20q - 81 &= \frac{24a(2-a)}{a^2(3-2a)} + 60a(2-a) - 81 \\ &= \frac{3a(40a^4 - 140a^3 + 174a^2 - 89a + 16)}{a^2(3-2a)} \\ &= \frac{3a(2a-1)^2(10a^2 - 25a + 16)}{a^2(3-2a)}. \end{aligned}$$

Since

$$10a^2 - 25a + 16 = 10 \left( a - \frac{5}{4} \right)^2 + \frac{3}{8} > 0,$$

the proof is completed. The equality holds for  $(a, b, c) = \left( \frac{1}{2}, \frac{1}{2}, 2 \right)$  or any cyclic permutation.

**Second Solution** (by Vo Quoc Ba Can). It is easy to check that the equality holds when two of  $a, b, c$  are  $1/2$ . Then, let us define

$$f(x) = \frac{8}{x} - 10x^2 - \alpha x - \beta,$$

such that  $(2x-1)^2$  divides  $f(x)$ . From  $f(1/2) = 0$ , we get  $\alpha + 2\beta = 27$ . Therefore,

$$f(x) = \frac{8}{x} - 10x^2 - (27 - 2\beta)x - \beta = \frac{(1-2x)h(x)}{x},$$

where

$$h(x) = 5x^2 - (\beta - 16)x + 8.$$



From  $h(1/2) = 0$ , we get  $\beta = 69/2$ , then  $\alpha = 27 - 2\beta = -42$ ; therefore,

$$f(x) = \frac{8}{x} - 10x^2 + 42x - \frac{69}{2} = \frac{(1-2x)^2(16-5x)}{2x}.$$

From

$$f(a) + f(b) + f(c) = 8 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - 10(a^2 + b^2 + c^2) + \frac{45}{2},$$

it follows that the desired inequality is equivalent to

$$f(a) + f(b) + f(c) \geq \frac{27}{2},$$

$$f(b) + f(c) \geq \frac{27}{2} - f(a).$$

Assume that

$$a = \max\{a, b, c\}, \quad a \geq 1.$$

Since

$$\frac{27}{2} - f(a) = \frac{2(a-2)^2(5a-1)}{a},$$

we can rewrite the inequality as

$$\frac{(1-2b)^2(16-5b)}{b} + \frac{(1-2c)^2(16-5c)}{c} \geq \frac{4(a-2)^2(5a-1)}{a}.$$

Since

$$16-5b > 0, \quad 16-5c > 0,$$

the Cauchy-Schwarz inequality yields

$$\begin{aligned} \frac{(1-2b)^2(16-5b)}{b} + \frac{(1-2c)^2(16-5c)}{c} &\geq \frac{(1-2b+1-2c)^2}{\frac{b}{16-5b} + \frac{c}{16-5c}} \\ &= \frac{4(a-2)^2}{\frac{b}{16-5b} + \frac{c}{16-5c}}. \end{aligned}$$

Therefore, it suffices to prove that

$$\frac{1}{\frac{b}{16-5b} + \frac{c}{16-5c}} \geq \frac{5a-1}{a},$$

which is equivalent to

$$\frac{a}{5a-1} \geq \frac{b}{16-5b} + \frac{c}{16-5c}.$$

Indeed,

$$\begin{aligned} \frac{a}{5a-1} - \frac{b}{16-5b} - \frac{c}{16-5c} &\geq \frac{a}{5a-1} - \frac{b+c}{16-5a} \\ &= \frac{3}{(5a-1)(16-5a)} > 0. \end{aligned}$$

**Remark.** The following more general inequality is true:

- If  $x_1, x_2, \dots, x_n$  are positive real numbers such that

$$x_1 + x_2 + \dots + x_n = n,$$

then

$$(n+1)^2 \left( \frac{1}{x_1} + \dots + \frac{1}{x_n} \right) \geq n(n^2 - 3n - 6) + 4(n+2)(x_1^2 + x_2^2 + \dots + x_n^2),$$

with equality for  $x_1 = (n+1)/2$  and  $x_2 = \dots = x_n = 1/2$  (or any cyclic permutation). □

**P 3.61.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = 3,$$

then

$$7(a^2 + b^2 + c^2) + 8(a^2b^2 + b^2c^2 + c^2a^2) + 4a^2b^2c^2 \geq 49.$$

**Solution.** Let

$$q = ab + bc + ca.$$

Since

$$a^2 + b^2 + c^2 = 9 - 2q,$$

$$a^2b^2 + b^2c^2 + c^2a^2 = q^2 - 6abc,$$

we can rewrite the inequality as

$$7(9 - 2q) + 8(q^2 - 6abc) + 4a^2b^2c^2 \geq 49,$$

$$2(6 - abc)^2 + 4q^2 - 7q - 65 \geq 0.$$

Since

$$abc \leq \left( \frac{a+b+c}{3} \right)^3 = 1,$$

we have  $6 - abc > 0$ . By P 3.57-(a), the product  $abc$  has the maximum value for fixed  $q$  when two of  $a, b, c$  are equal. Therefore, it suffices to prove the inequality for  $a = b$ , when  $c = 3 - 2a$ . Since

$$q = a^2 + 2ac = a^2 + 2a(3 - 2a) = 3a(2 - a), \quad abc = a^2c = a^2(3 - 2a),$$

we have

$$\begin{aligned} 2(6 - abc)^2 + 4q^2 - 7q - 65 &= 8a^6 - 24a^5 + 54a^4 - 96a^3 + 83a^2 - 42a + 7 \\ &= (a - 1)^2(2a - 1)^2(2a^2 + 7) \geq 0. \end{aligned}$$

The equality holds for  $a = b = c = 1$ , and for  $(a, b, c) = \left(\frac{1}{2}, \frac{1}{2}, 2\right)$  or any cyclic permutation.  $\square$

**P 3.62.** If  $a, b, c$  are nonnegative real numbers, then

$$(a^3 + b^3 + c^3 + abc)^2 \geq 2(a^2 + b^2)(b^2 + c^2)(c^2 + a^2).$$

(Aleksandar Bulj, 2011)

**First Solution.** Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

Using the identities

$$a^3 + b^3 + c^3 = 3r + p^3 - 3pq$$

and

$$\begin{aligned} (a^2 + b^2)(b^2 + c^2)(c^2 + a^2) &= (a^2 + b^2 + c^2)(a^2b^2 + b^2c^2 + c^2a^2) - a^2b^2c^2 \\ &= (p^2 - 2q)(q^2 - 2pr) - r^2, \end{aligned}$$

we can write the required inequality as  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = 18r^2 + 4p(3p^2 - 8q)r + p^6 - 6p^4q + 7p^2q^2 + 4q^3.$$

Since

$$3p^2 - 8q = 3(p^2 - 3q) + q \geq 0,$$

for fixed  $p$  and  $q$ ,  $f_6$  is an increasing function of  $r$ . Therefore, it suffices to prove the inequality  $f_6(a, b, c) \geq 0$  for the case when  $r$  has the minimum value; that is, when one of  $a, b, c$  is zero or two of  $a, b, c$  are equal (see P 3.57). For  $a = 0$  and for  $b = c$ , the original inequality becomes

$$(b - c)^2(b^4 + 2b^3c + b^2c^2 + 2bc^3 + c^4) \geq 0$$

and

$$a(a - b)^2(a^3 + 2a^2b + ab^2 + 4b^3) \geq 0,$$

respectively. The equality holds for  $a = b = c$ , and also for  $a = 0$  and  $b = c$  (or any cyclic permutation).

**Second Solution.** Without loss of generality, assume that  $a = \min\{a, b, c\}$ . From

$$a(a - b)(a - c) \geq 0,$$

we get

$$a^3 + abc \geq a^2(b + c),$$

and hence

$$a^3 + b^3 + c^3 + abc \geq a^2(b + c) + b^3 + c^3 = (b + c)(a^2 + b^2 + c^2 - bc).$$

On the other hand,

$$2(a^2 + b^2)(c^2 + a^2) \leq \frac{1}{2}(2a^2 + b^2 + c^2)^2.$$

Therefore, it suffices to prove that

$$(b + c)^2(a^2 + b^2 + c^2 - bc)^2 \geq \frac{1}{2}(b^2 + c^2)(2a^2 + b^2 + c^2)^2.$$

We can obtain this inequality by multiplying the obvious inequality

$$a^2 + b^2 + c^2 - bc \geq \frac{1}{2}(2a^2 + b^2 + c^2)$$

and

$$(b + c)^2(a^2 + b^2 + c^2 - bc) \geq (b^2 + c^2)(2a^2 + b^2 + c^2).$$

The last inequality is equivalent to

$$(b - c)^2(bc - a^2) \geq 0,$$

which is also true.

**Remark.** Using the first method, we can prove the following stronger inequality (Vasile Cîrtoaje, 2011).

- If  $a, b, c$  are nonnegative real numbers then

$$(a^3 + b^3 + c^3 + abc)^2 \geq 2(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) + 7(a - b)^2(b - c)^2(c - a)^2.$$

Since

$$(a - b)^2(b - c)^2(c - a)^2 = -27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3,$$

we can write this inequality as  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = 207r^2 + 2(20p^3 - 79pq)r + p^6 - 6p^4q + 32q^3.$$

We will show that for fixed  $p$  and  $q$ ,  $f_6$  is an increasing function of  $r$ . From

$$f_6 = 27r^2 + 2pqr + 20g(r) + p^6 - 6p^4q + 32q^3,$$

where

$$g(r) = 9r^2 + 2(p^3 - 4pq)r,$$

it suffices to show that  $g(r)$  is increasing. Indeed, by the third degree Schur's inequality, we have

$$g'(r) = 2(9r + p^3 - 4pq) \geq 0.$$

Therefore, it suffices to prove the inequality  $f_6(a, b, c) \geq 0$  for the case when  $r$  has the minimum value; that is, when  $a = 0$  or  $b = c$  (see P 3.57). In these cases, the original inequality becomes

$$(b - c)^4(b^2 + 4bc + c^2) \geq 0$$

and

$$a(a - b)^2(a^3 + 2a^2b + ab^2 + 4b^3) \geq 0,$$

respectively. □

**P 3.63.** If  $a, b, c$  are positive real numbers, then

$$(a + b + c - 3)(ab + bc + ca - 3) \geq 3(abc - 1)(a + b + c - ab - bc - ca).$$

(Vasile Cîrtoaje, 2011)

**Solution.** Setting

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc,$$

the inequality becomes

$$(p - 3)(q - 3) \geq 3(r - 1)(p - q),$$

or

$$3r(q - p) + pq - 6q + 9 \geq 0.$$

**First Solution.** For fixed  $p$  and  $q$ , the linear function  $f(r) = 3r(q - p) + pq - 6q + 9$  has the minimum value when  $r$  is either minimum or maximum. Thus, according to P 3.57-(a), we need only to prove that  $f(r) \geq 0$  for  $a = 0$  and for  $b = c$ .

For  $a = 0$ , the inequality becomes

$$(b + c)bc - 6bc + 9 \geq 0.$$

Putting  $x = \sqrt{bc}$ , we have

$$(b + c)bc - 6bc + 9 \geq 2x^3 - 6x^2 + 9 = 2(x + 1)(x - 2)^2 + 1 > 0.$$

For  $b = c$ , since  $p = a + 2b$ ,  $q = 2ab + b^2$  and  $r = ab^2$ , we need to show that

$$3ab^2(2ab + b^2 - a - 2b) + (a + 2b - 6)(2ab + b^2) + 9 \geq 0;$$

that is,

$$Aa^2 + Ba + C \geq 0,$$

where

$$A = b(6b^2 - 3b + 2), \quad B = b(3b^3 - 6b^2 + 5b - 12), \quad C = 2b^3 - 6b^2 + 9.$$

We have  $A > 0$  and

$$C = 2(b^3 - 3b^2 + 4) + 1 = 2(b + 1)(b - 2)^2 + 1 > 0.$$

Consider two cases.

*Case 1:*  $b \geq 12/5$ . Since

$$B = 3b^2(b - 2) + b(5b - 12) > 0,$$

we have  $Aa^2 + Ba + C > 0$ .

*Case 2:*  $0 < b < 12/5$ . Since

$$Aa^2 + Ba + C = (Aa^2 + C) + Ba \geq a(2\sqrt{AC} + B),$$

it suffices to show that  $4AC \geq B^2$ , which is equivalent to

$$4b(6b^2 - 3b + 2)(2b^3 - 6b^2 + 9) \geq b^2(3b^3 - 6b^2 + 5b - 12)^2,$$

$$b(b - 1)^4(8 + 4b - b^3) \geq 0.$$

This inequality is true since

$$8 + 4b - b^3 = 8 + 4b - 3b^2 + b^2(3 - b) > 8 + 4b - 3b^2 > 0.$$

The equality holds for  $a = b = c = 1$ .

**Second Solution.** Consider the following two cases.

*Case 1:*  $p \geq q$ . We have

$$3r(q - p) + pq - 6q + 9 = (q - 3r)(p - q) + (q - 3)^2 \geq 0$$

since

$$q - 3r \geq q - \frac{q^2}{p} \geq 0.$$

*Case 2:*  $p \leq q$ . For  $p \geq 6$ , we have

$$3r(q - p) + pq - 6q + 9 = 3r(q - p) + q(p - 6) + 9 > 0.$$

Consider further that  $p < 6$ . From  $p^2 \geq 3q \geq 3p$ , we get  $p \geq 3$ . From Schur's inequality

$$p^3 + 9r \geq 4pq,$$

we get

$$p^3 + p^2r \geq 4pq,$$

hence

$$pr \geq 4q - p^2.$$

Using this result, we have

$$\begin{aligned} p[3r(q-p) + pq - 6q + 9] &\geq 3(4q - p^2)(q-p) + p(pq - 6q + 9) \\ &= 12q^2 - 2p(p+9)q + 3p(p^2 + 3) \\ &= 12 \left( q - \frac{p^2 + 9p}{12} \right)^2 + \frac{p(12-p)(p-3)^2}{12} \geq 0. \end{aligned}$$

□

**P 3.64.** Let  $a, b, c$  be nonnegative real numbers such that

$$ab + bc + ca = 3.$$

Prove that

$$4(a^3 + b^3 + c^3) + 7abc + 125 \geq 48(a + b + c).$$

(Vasile Cîrtoaje, 2011)

**Solution.** Since

$$a^3 + b^3 + c^3 = 3abc + (a + b + c)^3 - 9(a + b + c),$$

we can write the inequality as

$$19abc + 4(a + b + c)^3 - 84(a + b + c) + 125 \geq 0.$$

As it is known, for fixed  $a + b + c$ , the product  $abc$  has the minimum value when  $a = 0$  or  $b = c$  (see P 3.57). Therefore it suffices to consider these cases.

*Case 1:*  $a = 0$ . We need to show that  $bc = 3$  yields

$$4(b^3 + c^3) + 125 \geq 48(b + c).$$

Since

$$b^3 + c^3 = (b + c)^3 - 3bc(b + c) = (b + c)^3 - 9(b + c)$$

and

$$b + c \geq 2\sqrt{bc} = 2\sqrt{3},$$

we have

$$4(b^3 + c^3) + 125 - 48(b + c) = 4(b + c)^3 - 84(b + c) + 125 > 0.$$

*Case 2:*  $b = c$ . We need to show that  $2ab + b^2 = 3$  yields

$$4(a^3 + 2b^3) + 7ab^2 + 125 \geq 48(a + 2b).$$

This inequality is equivalent to

$$8b^6 - 114b^4 + 250b^3 - 171b^2 + 27 \geq 0,$$

$$(b - 1)^2(2b - 3)^2(2b^2 + 10b + 3) \geq 0.$$

The equality holds for  $a = b = c = 1$ , and also for  $a = 1/4$  and  $b = c = 3/2$  (or any cyclic permutation). □

**P 3.65.** *If  $a, b, c$  are nonnegative real numbers such that*

$$a + b + c = 3,$$

*then*

$$4(a^4 + b^4 + c^4) + 45 \geq 19(a^2 + b^2 + c^2).$$

(Vasile Cîrtoaje, 2009)

**Solution.** We use the mixing variables method. Write the inequality as

$$F(a, b, c) \geq 0,$$

where

$$F(a, b, c) = 4(a^4 + b^4 + c^4) + 45 - 19(a^2 + b^2 + c^2).$$

Due to symmetry, we may assume that  $a \leq b \leq c$ . Let us denote

$$x = \frac{b + c}{2}, \quad 1 \leq x \leq 3/2.$$

We will show that

$$F(a, b, c) \geq F(a, x, x) \geq 0.$$

We have

$$\begin{aligned} F(a, b, c) - F(a, x, x) &= 4(b^4 + c^4 - 2x^4) - 19(b^2 + c^2 - 2x^2) \\ &= 4[(b^2 + c^2)^2 - 4x^4] + 8(x^4 - b^2c^2) - 19(b^2 + c^2 - 2x^2) \\ &= (b^2 + c^2 - 2x^2)[4(b^2 + c^2 + 2x^2) - 19] + 8(x^2 - bc)(x^2 + bc). \end{aligned}$$

Since

$$b^2 + c^2 - 2x^2 = 2(x^2 - bc) = \frac{1}{2}(b - c)^2,$$



we get

$$\begin{aligned} F(a, b, c) - F(a, x, x) &= \frac{1}{2}(b - c)^2[4(b^2 + c^2 + 2x^2) - 19 + 4(x^2 + bc)] \\ &= \frac{1}{2}(b - c)^2[4(x^2 - bc) + 24x^2 - 19] \geq 0. \end{aligned}$$

Also,

$$F(a, x, x) = F(3 - 2x, x, x) = 6(x - 1)^2(3 - 2x)(11 - 6x) \geq 0.$$

This completes the proof. The equality holds for  $a = b = c = 1$ , and for  $a = 0$  and  $b = c = 3/2$  (or any cyclic permutation).

**Remark.** Similarly, we can prove the following more general statement (Vasile Cîrtoaje and Le Huu Dien Khue, 2008).

- Let  $\alpha, \beta, \gamma$  be real numbers such that

$$1 + \alpha + \beta = 2\gamma.$$

The inequality

$$\sum a^4 + \alpha \sum a^2 b^2 + \beta abc \sum a \geq \gamma \sum ab(a^2 + b^2)$$

holds for all  $a, b, c \geq 0$  if and only if

$$\alpha \geq (\gamma - 1) \max\{2, \gamma + 1\}.$$

□

**P 3.66.** If

$$a, b, c \geq \frac{13 - 4\sqrt{10}}{3} \approx 0.117$$

such that  $a + b + c = 9$ , then

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq \sqrt{ab + bc + ca}.$$

(Vasile Cîrtoaje, 2018)

**Solution.** Denote

$$k = \frac{13 - 4\sqrt{10}}{3}$$

and

$$\sqrt{a} = x, \quad \sqrt{b} = y, \quad \sqrt{c} = z.$$

We need to show that

$$x + y + z \geq \sqrt{(xy + yz + zx)^2 - 2xyz(x + y + z)}$$

for  $x \geq y \geq z \geq \sqrt{k}$  and  $(x + y + z)^2 - 2(xy + yz + zx) = 9$ . According to Remark 2 from P 3.57, for fixed  $x + y + z$  and  $xy + yz + zx$ , the product  $xyz$  has the minimum value for  $x = y \geq z$  or  $z = \sqrt{k}$ . So, we only need to consider these cases. This means to consider the cases  $a = b \geq c \geq k$  and  $a \geq b \geq c = k$ .

*Case 1:*  $a = b \geq c \geq k$ . We need to show that  $2a + c = 9$  involves

$$2\sqrt{a} + \sqrt{c} \geq \sqrt{a^2 + 2ac},$$

that is

$$2\sqrt{2(9-c)} + 2\sqrt{c} \geq \sqrt{3(9-c)(3+c)}.$$

By squaring, the inequality becomes

$$8\sqrt{2c(9-c)} \geq -3c^2 + 22c + 9.$$

This is true if

$$128c(9-c) \geq (-3c^2 + 22c + 9)^2,$$

which is equivalent to

$$\begin{aligned} 3c^4 - 44c^3 + 186c^2 - 252c + 27 &\leq 0, \\ (c-3)^2(3c^2 - 26c + 3) &\leq 0. \end{aligned}$$

Since  $3c^2 - 26c + 3 \leq 0$  for  $c \in [k, 3]$ , the inequality is true.

*Case 2:*  $a \geq b \geq c = k$ . We need to show that

$$\sqrt{a} + \sqrt{b} + \sqrt{k} \geq \sqrt{ab + k(a+b)},$$

that is

$$\begin{aligned} \sqrt{a+b+2\sqrt{ab}} + \sqrt{k} &\geq \sqrt{ab+k(a+b)}, \\ \sqrt{9-k+2\sqrt{ab}} + \sqrt{k} &\geq \sqrt{ab+k(9-k)}. \end{aligned}$$

From

$$(a-c)(b-c) \geq 0,$$

we get

$$ab \geq c(a+b) - c^2 = c(9-2c) = k(9-2k) > 1.$$

Substituting  $\sqrt{ab} = x$ , we need to show that  $f(x) \geq 0$  for

$$1 < x \leq \frac{a+b}{2} = \frac{9-k}{2} \approx 4.4415,$$

where

$$f(x) = \sqrt{2x+9-k} + \sqrt{k} - \sqrt{x^2+k(9-k)},$$

with

$$f'(x) = \frac{1}{\sqrt{2x+9-k}} - \frac{x}{\sqrt{x^2+k(9-k)}}.$$

We show that  $f'(x) < 0$ , therefore  $f$  is decreasing. This is true if

$$x^2(2x + 9 - k) > x^2 + k(9 - k),$$

that is

$$2x^3 + (8 - k)x^2 - k(9 - k) > 0.$$

Indeed,

$$2x^3 + (8 - k)x^2 - k(9 - k) > x^2 + (8 - k)x^2 - k(9 - k) = (9 - k)(x^2 - k) > 0.$$

So, we have

$$f(x) \geq f\left(\frac{9-k}{2}\right) = 0.$$

The inequality is an equality for  $a = b = c = 3$ , and also for  $a = b = \frac{7+2\sqrt{10}}{3}$  and  $c = \frac{13-4\sqrt{10}}{3}$  (or any cyclic permutation). □

**P 3.67.** If  $a, b, c$  are positive real numbers such that  $abc = 1$ , then

$$ab + bc + ca + \frac{6}{a + b + c} \geq 5.$$

(Vasile Cîrtoaje, 2005)

**First Solution.** Denoting

$$x = \frac{ab + bc + ca}{3},$$

the inequality can be written as

$$(a + b + c)(3x - 5) + 6 \geq 0.$$

In virtue of the AM-GM inequality, we get  $x \geq 1$ . Since the inequality holds for  $x \geq 5/3$ , assume next  $1 \leq x < 5/3$ . Applying the third degree Schur's inequality to the numbers  $ab$ ,  $bc$  and  $ca$ , we have

$$(ab + bc + ca)^3 + 9a^2b^2c^2 \geq 4abc(a + b + c)(ab + bc + ca),$$

which is equivalent to

$$a + b + c \leq \frac{3(3x^3 + 1)}{4x}.$$

Since  $3x - 5 < 0$ , it suffices to prove that

$$\frac{3(3x^3 + 1)(3x - 5)}{4x} + 6 \geq 0.$$

This inequality is equivalent to

$$\begin{aligned} 9x^4 - 15x^3 + 11x - 5 &\geq 0, \\ (x-1)(9x^3 - 6x^2 - 6x + 5) &\geq 0. \end{aligned}$$

Since

$$9x^3 - 6x^2 - 6x + 5 > 9x^3 - 6x^2 - 6x + 3 = 3(x-1)(3x^2 + x - 1) \geq 0,$$

the conclusion follows. The equality holds for  $a = b = c = 1$ .

**Second Solution** (by Vo Quoc Ba Can). Among

$$a-1, \quad b-1, \quad c-1,$$

there are two with the same sign. Due to symmetry, assume that  $(b-1)(c-1) \geq 0$ ; that is,

$$b+c \leq 1+bc.$$

Then,

$$\frac{6}{a+b+c} \geq \frac{6}{a+1+bc} = \frac{6a}{a^2+a+1}.$$

On the other hand, using the AM-GM inequality yields

$$ab+bc+ca = a(b+c) + bc \geq 2a\sqrt{bc} + bc = 2\sqrt{a} + \frac{1}{a}.$$

Therefore, it suffices to prove that

$$2\sqrt{a} + \frac{1}{a} + \frac{6a}{a^2+a+1} \geq 5.$$

Setting  $\sqrt{a} = x$ , this inequality becomes as follows:

$$\begin{aligned} 2x + \frac{1}{x^2} + \frac{6x^2}{x^4+x^2+1} &\geq 5, \\ 2x + \frac{1}{x^2} - 3 &\geq 2 - \frac{6x^2}{x^4+x^2+1}, \\ \frac{(x-1)^2(2x+1)}{x^2} &\geq \frac{2(x^2-1)^2}{x^4+x^2+1}, \\ (x-1)^2(2x^5-x^4-2x^3-x^2+2x+1) &\geq 0, \\ (x-1)^2[x(x-1)^2(2x^2+3x+2)+1] &\geq 0. \end{aligned}$$

**Third Solution** Assume that  $a \geq b \geq c > 0$ . By P 3.58, for fixed  $a+b+c$  and  $abc = 3$ , the expression  $ab+bc+ca$  has the minimum value for  $b = c$ . So, it suffices to show that  $ab^2 = 1$  involves

$$2ab + b^2 + \frac{6}{a+2b} \geq 5.$$

The inequality is equivalent to

$$\begin{aligned}\frac{2}{b} + b^2 + \frac{6b^2}{2b^3 + 1} &\geq 5, \\ 2b^6 &= 10b^4 + 11b^3 - 5b + 2 \geq 0, \\ (b-1)^2(2b^4 + 4b^3 - 4b^2 - b + 2) &\geq 0, \\ (b-1)^2[2b^4 + b^3 + (b-1)^2(3b+2)] &\geq 0.\end{aligned}$$

**Remark.** Actually, the following inequality holds for any positive real numbers  $a, b, c$  (*Nguyen Van Huyen*):

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{6}{a+b+c} \geq \frac{15}{abc+2}.$$

By P 3.58, it suffices to prove the inequality for  $b = c$ , when it becomes

$$\frac{1}{a} + \frac{2}{b} + \frac{6}{a+2b} \geq \frac{15}{ab^2+2}.$$

Using the substitution  $ab^2 = d^3$ , we can write the inequality as  $f(b) \geq 0$ , where

$$f(b) = \frac{b^2}{d^3} + \frac{2}{b} + \frac{6b^2}{2b^3 + d^3} - \frac{15}{d^3 + 2}.$$

From

$$f'(b) = \frac{2b}{d^3} - \frac{2}{b^2} - \frac{12b(b^3 - d^3)}{(2b^3 + d^3)^2} = \frac{2(b^3 - d^3)[(2b^3 - d^3)^2 + 2b^3d^3]}{b^2d^3(2b^3 + d^3)^2},$$

it follows that  $f(b)$  is decreasing on  $(0, d]$  and increasing on  $[d, \infty)$ . Therefore,

$$f(b) \geq f(d) = \frac{5}{d} - \frac{15}{d^3 + 2} = \frac{5(d-1)^2(d+2)}{d(d^3 + 2)} \geq 0.$$

Another solution is based on the substitutions  $a = tx$ ,  $b = ty$  and  $c = tz$ , where  $x, y$  and  $z$  are positive real numbers such that  $xyz = 1$  and  $t > 0$  (*Leonard Giugiuc*). The inequality becomes

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{6}{x+y+z} \geq \frac{15t}{t^3+2},$$

and can be obtained by adding the inequalities

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{6}{x+y+z} \geq 5$$

and

$$5 \geq \frac{15t}{t^3+2}.$$

□

**P 3.68.** Let  $f_n(a, b, c)$  be a symmetric homogeneous polynomial of degree  $n \leq 5$ . Prove that

(a) the inequality  $f_n(a, b, c) \geq 0$  holds for all nonnegative real numbers  $a, b, c$  if and only if  $f_n(a, 1, 1) \geq 0$  and  $f_n(0, b, c) \geq 0$  for all  $a, b, c \geq 0$ ;

(b) the inequality  $f_n(a, b, c) \geq 0$  holds for all lengths  $a, b, c$  of the sides of a non-degenerate or degenerate triangle if and only if  $f_n(x, 1, 1) \geq 0$  for  $0 \leq x \leq 2$ , and  $f_n(y + z, y, z) \geq 0$  for all  $y, z \geq 0$ .

(Vasile Cîrtoaje, 2005)

**Solution.** Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

Any symmetric homogeneous polynomial  $f_n(a, b, c)$  of degree  $n \leq 5$  can be written as

$$f_n(a, b, c) = A_n(p, q)r + B_n(p, q),$$

where  $A_n(p, q)$  and  $B_n(p, q)$  are polynomial functions. For fixed  $p$  and  $q$ , the linear function

$$g_n(r) = A_n(p, q)r + B_n(p, q)$$

has the minimum value when  $r$  is either minimum or maximum.

(a) By P 3.57-(a), for fixed  $p$  and  $q$ , the product  $r$  has the minimum and value when two of  $a, b, c$  are equal or one of  $a, b, c$  is 0. Due to symmetry and homogeneity, the conclusion follows.

(b) By P 3.57-(b), for fixed  $p$  and  $q$ , the product  $r$  has the minimum and maximum value when two of  $a, b, c$  are equal or one of  $a, b, c$  is the sum of the others. Due to symmetry and homogeneity, the conclusion follows.

**Remark.** Similarly, we can prove the following statement, which does not involve the homogeneity property.

• Let  $f_n(a, b, c)$  be a symmetric polynomial function of degree  $n \leq 5$ . The inequality  $f_n(a, b, c) \geq 0$  holds for all nonnegative real numbers  $a, b, c$  if and only if it holds for  $a = 0$  and for  $b = c$ .

□

**P 3.69.** If  $a, b, c, d$  are nonnegative real numbers such that  $ab + ac + ad + bc + bd + cd \geq 6$ , then

$$(1 + a^2)(1 + b^2)(1 + c^2)(1 + d^2) \geq (a + b + c + d)^2.$$

(Vasile Cîrtoaje, *Recreatii Matematice*, 2, 2024)

**Solution.** Since

$$(a + b)(c + d) + (a + c)(b + d) + (a + d)(b + c) = 2(ab + ac + ad + bc + bd + cd) \geq 12,$$

at least one of the inequality below is true:

$$(a+b)(c+d) \geq 4, \quad (a+c)(b+d) \geq 4, \quad (a+d)(b+c) \geq 4.$$

Assume that

$$(a+c)(b+d) \geq 4,$$

and, in addition,

$$a+c \geq b+d.$$

There are two cases to consider:  $b+d \geq 2$  and  $b+d \leq 2$ .

*Case 1:*  $b+d \geq 2$ . Since

$$(a^2+1)(c^2+1) \geq (a+c)^2, \quad (b^2+1)(d^2+1) \geq (b+d)^2,$$

it suffices to show that

$$(a+c)(b+d) \geq a+b+c+d.$$

Indeed, we have

$$(a+c)(b+d) - (a+b+c+d) \geq 2(a+c) - (a+b+c+d) = (a+c) - (b+d) \geq 0.$$

*Case 2:*  $b+d \leq 2$ . Let

$$x = \frac{a+c}{2}, \quad y = \frac{b+d}{2}, \quad x \geq 1 \geq y.$$

We have

$$xy \geq 1, \quad bd \leq y^2.$$

Since

$$(a^2+1)(c^2+1) \geq (a+c)^2 = 4x^2$$

and

$$(b^2+1)(d^2+1) = (b+d)^2 + (1-bd)^2 \geq 4y^2 + (1-y^2)^2 = (1+y^2)^2,$$

it suffices to show that

$$4x^2(1+y^2)^2 \geq (2x+2y)^2,$$

which is equivalent to

$$x(1+y^2) \geq x+y, \quad y(xy-1) \geq 0.$$

The proof is completed. The equality occurs for  $a=b=c=d=1$ .

**Remark.** The inequality is also true for  $abcd \geq 1$  (see P 2.116). Indeed, if  $abcd \geq 1$ , then

$$ab+ac+ad+bc+bd+cd \geq 6\sqrt{abcd} \geq 6.$$

□

**P 3.70.** Let  $a, b, c$  be nonnegative real numbers. If  $k \leq 2$ , then

$$\sum a(a-b)(a-c)(a-kb)(a-kc) \geq 0.$$

(Vasile Cîrtoaje, 2008)

**Solution.** Let us denote

$$f_5(a, b, c) = \sum a(a-b)(a-c)(a-kb)(a-kc).$$

By P 3.68-(a), it suffices to show that  $f_5(a, 1, 1) \geq 0$  and  $f_5(0, b, c) \geq 0$  for all  $a, b, c \geq 0$ . Indeed, we have

$$f_5(a, 1, 1) = a(a-1)^2(a-k)^2 \geq 0$$

and

$$f_5(0, b, c) = (b+c)(b-c)^2(b^2-kbc+c^2) \geq 0.$$

The equality holds for  $a = b = c$ , for  $a = 0$  and  $b = c$  (or any cyclic permutation), and for  $a/k = b = c$ ,  $k > 0$  (or any cyclic permutation). □

**P 3.71.** Let  $a, b, c$  be nonnegative real numbers. If  $k \in \mathbf{R}$ , then

$$\sum (b+c)(a-b)(a-c)(a-kb)(a-kc) \geq 0.$$

(Vasile Cîrtoaje, 2008)

**Solution.** Let

$$f_5(a, b, c) = \sum (b+c)(a-b)(a-c)(a-kb)(a-kc).$$

By P 3.68-(a), it suffices to show that  $f_5(a, 1, 1) \geq 0$  and  $f_5(0, b, c) \geq 0$  for all  $a, b, c \geq 0$ . Indeed, we have

$$f_5(a, 1, 1) = 2(a-1)^2(a-k)^2 \geq 0$$

and

$$f_5(0, b, c) = k^2(b+c)b^2c^2 + bc(b+c)(b-c)^2 \geq 0.$$

The equality holds for  $a = b = c$ , for  $b = c = 0$  (or any cyclic permutation), and for  $a/k = b = c$ ,  $k > 0$  (or any cyclic permutation).

**Remark.** Similarly, we can prove the following generalization:

- Let  $a, b, c$  be nonnegative real numbers. If  $m \geq 0$  and  $m(k-2) \leq 1$ , then

$$\sum (ma+b+c)(a-b)(a-c)(a-kb)(a-kc) \geq 0.$$

□



**P 3.72.** If  $a, b, c$  are nonnegative real numbers, then

$$\sum a(a-2b)(a-2c)(a-5b)(a-5c) \geq 0.$$

(Vasile Cîrtoaje, 2008)

**Solution.** Let

$$f_5(a, b, c) = \sum a(a-2b)(a-2c)(a-5b)(a-5c).$$

By P 3.68-(a), it suffices to show that  $f_5(a, 1, 1) \geq 0$  and  $f_5(0, b, c) \geq 0$  for all  $a, b, c \geq 0$ . Indeed, we have

$$\begin{aligned} f_5(a, 1, 1) &= a^3(a-7)^2 + 20a^3 - 60a^2 + 44a + 8 \\ &\geq 20a^3 - 60a^2 + 44a + 8 \geq 0, \end{aligned}$$

since

$$20a^3 - 60a^2 + 44a + 8 > 20a^2(a-3) \geq 0$$

for  $a \geq 3$ , and

$$20a^3 - 60a^2 + 44a + 8 = 5(2a-3)^2 + 8 - a \geq 8 - a \geq 0$$

for  $a \leq 8$ . Also,

$$f_5(0, b, c) = (b+c)(b^2-4bc+c^2)^2 \geq 0.$$

The equality holds for  $a = 0$  and  $b^2 - 4bc + c^2 = 0$  (or any cyclic permutation).

□

**P 3.73.** If  $a, b, c$  are the lengths of the side of a triangle, then

$$a^4 + b^4 + c^4 + 9abc(a+b+c) \leq 10(a^2b^2 + b^2c^2 + c^2a^2).$$

**First Solution.** Let

$$f_4(a, b, c) = 10(a^2b^2 + b^2c^2 + c^2a^2) - a^4 - b^4 - c^4 - 9abc(a+b+c).$$

By P 3.68-(b), it suffices to show that  $f_4(x, 1, 1) \geq 0$  for  $0 \leq x \leq 2$  and  $f_4(y+z, y, z) \geq 0$  for  $y, z \geq 0$ . We have

$$f_4(x, 1, 1) = 8 - 18x + 11x^2 - x^4 = (2-x)(4+x)(1-x)^2 \geq 0$$

and

$$\begin{aligned} f_4(y+z, y, z) &= 8(y^2+z^2)^2 - 2yz(y^2+z^2) - 28y^2z^2 \\ &= 2(y-z)^2(4y^2+4z^2+7yz) \geq 0. \end{aligned}$$

The equality holds for an equilateral triangle and for a degenerate triangle with  $a/2 = b = c$  (or any cyclic permutation).

**Second Solution.** We use the sum-of-squares method (SOS method). Write the inequality as follows

$$\begin{aligned} 9(\sum b^2c^2 - abc \sum a) - (\sum a^4 - \sum b^2c^2) &\geq 0, \\ 9 \sum a^2(b-c)^2 - \sum (b^2-c^2)^2 &\geq 0, \\ \sum (b-c)^2(3a-b-c)(3a+b+c) &\geq 0. \end{aligned}$$

Without loss of generality, assume that  $a \geq b \geq c$ . Since

$$(b-c)^2(3a-b-c)(3a+b+c) \geq 0,$$

it suffices to show that

$$(c-a)^2(3b-c-a)(3b+c+a) + (a-b)^2(3c-a-b)(3c+a+b) \geq 0.$$

Since

$$3b-c-a \geq 2b-a \geq b+c-a \geq 0$$

and  $(c-a)^2 \geq (a-b)^2$ , it is enough to prove that

$$(3b-c-a)(3b+c+a) + (3c-a-b)(3c+a+b) \geq 0.$$

We have

$$(3b+c+a) - (3c+a+b) = 2(b-c) \geq 0,$$

hence

$$\begin{aligned} (3b-c-a)(3b+c+a) + (3c-a-b)(3c+a+b) &\geq \\ &\geq (3b-c-a)(3c+a+b) + (3c-a-b)(3c+a+b) \\ &= 2(b+c-a)(3c+a+b) \geq 0. \end{aligned}$$

□

**P 3.74.** If  $a, b, c$  are the lengths of the sides of a triangle, then

$$3(a^4 + b^4 + c^4) + 7abc(a + b + c) \leq 5 \sum ab(a^2 + b^2).$$

**Solution.** Let

$$f_4(a, b, c) = 5 \sum ab(a^2 + b^2) - 3(a^4 + b^4 + c^4) - 7abc(a + b + c).$$

By P 3.68-(b), it suffices to show that  $f_4(x, 1, 1) \geq 0$  for  $0 \leq x \leq 2$  and  $f_4(y + z, y, z) \geq 0$  for  $y, z \geq 0$ . Indeed,

$$f_4(x, 1, 1) = 4 - 4x - 7x^2 + 10x^3 - 3x^4 = (2 - x)(2 + 3x)(1 - x)^2 \geq 0$$

and

$$\begin{aligned} f_4(y + z, y, z) &= 4(y^2 + z^2)^2 + 4yz(y^2 + z^2) - 24y^2z^2 \\ &= 4(y - z)^2(y^2 + z^2 + 3yz) \geq 0. \end{aligned}$$

The equality holds for an equilateral triangle and for a degenerate triangle with  $a/2 = b = c$  (or any cyclic permutation). □

**P 3.75.** If  $a, b, c$  are the lengths of the sides of a triangle, then

$$\frac{b^2 + c^2 - 6bc}{a} + \frac{c^2 + a^2 - 6ca}{b} + \frac{a^2 + b^2 - 6ab}{c} + 4(a + b + c) \leq 0.$$

(Vasile Cîrtoaje, 2005)

**First Solution.** Write the inequality as  $f_4(a, b, c) \geq 0$ , where

$$f_4(a, b, c) = \sum bc(6bc - b^2 - c^2) - 4abc(a + b + c).$$

By P 3.68-(b), it suffices to show that  $f_4(x, 1, 1) \geq 0$  for  $0 \leq x \leq 2$  and  $f_4(y + z, y, z) \geq 0$  for  $y, z \geq 0$ . Since

$$f_4(x, 1, 1) = 2(2 - 5x + 4x^2 - x^3) = 2(1 - x)^2(2 - x) \geq 0$$

and

$$\begin{aligned} f_4(y + z, y, z) &= 4(y^2 + z^2)^2 - 2yz(y^2 + z^2) - 12y^2z^2 \\ &= 2(y - z)^2(2y^2 + 3yz + 2z^2) \geq 0, \end{aligned}$$

the proof is completed. The equality holds for an equilateral triangle and for a degenerate triangle with  $a/2 = b = c$  (or any cyclic permutation).

**Second Solution.** We use the SOS method. Write the inequality as follows:

$$\begin{aligned} \sum bc(b^2 + c^2 - 6bc) + 4abc \sum a &\leq 0, \\ \sum bc(b^2 + c^2 - 2bc) - 4(\sum b^2c^2 - abc \sum a) &\leq 0, \\ \sum bc(b - c)^2 - 2 \sum a^2(b - c)^2 &\leq 0, \\ \sum (b - c)^2(2a^2 - bc) &\geq 0. \end{aligned}$$

Without loss of generality, assume that  $a \geq b \geq c$ . Since  $(b-c)^2(2a^2-bc) \geq 0$ , it suffices to prove that

$$(c-a)^2(2b^2-ca) + (a-b)^2(2c^2-ab) \geq 0.$$

Since

$$2b^2-ca \geq 2b^2-c(b+c) = (b-c)(2b+c) \geq 0$$

and  $(c-a)^2 \geq (a-b)^2$ , it is enough to show that

$$(2b^2-ca) + (2c^2-ab) \geq 0.$$

Indeed,

$$(2b^2-ca) + (2c^2-ab) = (b-c)^2 + (b+c)(b+c-a) \geq 0.$$

□

**P 3.76.** Let  $f_6(a, b, c)$  be a sixth degree symmetric homogeneous polynomial written in the form

$$f_6(a, b, c) = Ar^2 + B(p, q)r + C(p, q),$$

where  $A$  is the highest coefficient of  $f_6$  and

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

For  $A \leq 0$ , prove that:

(a) the inequality  $f_6(a, b, c) \geq 0$  holds for all nonnegative real numbers  $a, b, c$  if and only if  $f_6(a, 1, 1) \geq 0$  and  $f_6(0, b, c) \geq 0$  for all  $a, b, c \geq 0$ ;

(b) the inequality  $f_6(a, b, c) \geq 0$  holds for all for all lengths  $a, b, c$  of the sides of a non-degenerate or degenerate triangle if and only if  $f_6(x, 1, 1) \geq 0$  for  $0 \leq x \leq 2$ , and  $f_6(y+z, y, z) \geq 0$  for all  $y, z \geq 0$ .

(Vasile Cîrtoaje, 2006)

**Solution.** For fixed  $p$  and  $q$ , the function  $f$  defined by

$$f(r) = Ar^2 + B(p, q)r + C(p, q)$$

is a quadratic concave function of  $r$ . Therefore,  $f(r)$  has the minimum value when  $r$  is minimum or maximum. According to P 3.57, the conclusion follows. As we have shown in the proof of P 2.75,  $A$  is called the *highest coefficient* of  $f_6(a, b, c)$ .

**Remark 1.** We can extend the part (a) of P 3.76 as follows:

(a1) For  $A \leq 0$ , the inequality  $f_6(a, b, c) \geq 0$  holds for all nonnegative real numbers  $a, b, c$  satisfying  $p^2 \leq 4q$  if and only if  $f_6(a, 1, 1) \geq 0$  for all  $0 \leq a \leq 4$ ;

(a2) For  $A \leq 0$ , the inequality  $f_6(a, b, c) \geq 0$  holds for all nonnegative real numbers  $a, b, c$  satisfying  $p^2 > 4q$  if and only if  $f_6(a, 1, 1) \geq 0$  for all  $a > 4$  and  $f_6(0, b, c) \geq 0$  for all  $b, c \geq 0$ .

Notice that the restriction  $0 \leq a \leq 4$  in (a1) follows by setting  $b = c = 1$  in  $p^2 \leq 4q$ . In addition, the condition  $f_6(0, b, c) \geq 0$  is not necessary in (a1) since  $a = 0$  and  $p^2 \leq 4q$  involve  $b = c$ ; therefore, the condition  $f_6(0, b, c) \geq 0$  is equivalent to  $f_6(0, 1, 1) \geq 0$ , which follows from  $f_6(a, 1, 1) \geq 0$  for all  $0 \leq a \leq 4$ .

Also, the restriction  $a > 4$  in (a2) follows by setting  $b = c = 1$  in  $p^2 > 4q$ .

**Remark 2.** In general, the statement in P 3.76-(a) and its extension in Remark 1 remains valid when  $f_6(a, b, c)$  is only a symmetric function of the form

$$f_6(a, b, c) = Ar^2 + B(p, q)r + C(p, q),$$

where  $B(p, q)$  and  $C(p, q)$  are rational or nonrational functions. If  $f_6(a, b, c)$  is not homogeneous, then the condition  $b = c = 1$  must be replaced with  $b = c$ . □

**P 3.77.** If  $a, b, c$  are nonnegative real numbers, then

$$\sum a(b+c)(a-b)(a-c)(a-2b)(a-2c) \geq (a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2008)

**Solution.** Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc,$$

and

$$f_6(a, b, c) = f(a, b, c) - (a-b)^2(b-c)^2(c-a)^2,$$

where

$$f(a, b, c) = \sum a(b+c)(a-b)(a-c)(a-2b)(a-2c).$$

Since

$$\begin{aligned} \sum a(b+c)(a-b)(a-c)(a-2b)(a-2c) &= \\ &= \sum a(p-a)(a^2+2bc-q)(a^2+6bc-2q), \end{aligned}$$

$f(a, b, c)$  has the same highest coefficient  $A_0$  as

$$P_1(a, b, c) = - \sum a^2(a^2+2bc)(a^2+6bc);$$

that is, according to Remark 2 from P 2.75,

$$A_0 = P_1(1, 1, 1) = -3(1+2)(1+6) = -63.$$

Then,  $f_6(a, b, c)$  has the highest coefficient

$$A = A_0 + 27 = -36.$$

Since  $A < 0$ , according to P 3.76-(a), it suffices to prove that  $f_6(a, 1, 1) \geq 0$  and  $f_6(0, b, c) \geq 0$  for all  $a, b, c \geq 0$ . Indeed, we have

$$f_6(a, 1, 1) = 2a(a-1)^2(a-2)^2 \geq 0$$

and

$$f_6(0, b, c) = bc(b-c)^4 \geq 0.$$

The equality holds for  $a = b = c$ , for  $a = 0$  and  $b = c$  (or any cyclic permutation), and for  $a/2 = b = c$  (or any cyclic permutation). □

**P 3.78.** Let  $a, b, c$  be nonnegative real numbers.

(a) If  $2 \leq k \leq 6$ , then

$$\sum a(a-b)(a-c)(a-kb)(a-kc) + \frac{4(k-2)(a-b)^2(b-c)^2(c-a)^2}{a+b+c} \geq 0.$$

(b) If  $k \geq 6$ , then

$$\sum a(a-b)(a-c)(a-kb)(a-kc) + \frac{(k+2)^2(a-b)^2(b-c)^2(c-a)^2}{4(a+b+c)} \geq 0.$$

(Vasile Cîrtoaje, 2009)

**Solution.** a) We need to prove that  $f_6(a, b, c) \geq 0$ , where

$$\begin{aligned} f_6(a, b, c) &= (a+b+c) \sum a(a-b)(a-c)(a-kb)(a-kc) \\ &\quad + 4(k-2)(a-b)^2(b-c)^2(c-a)^2. \end{aligned}$$

Since  $f_6(a, b, c)$  has the same highest coefficient as

$$4(k-2)(a-b)^2(b-c)^2(c-a)^2$$

and  $(a-b)^2(b-c)^2(c-a)^2$  has the highest coefficient  $-27$ , it follows that  $f_6(a, b, c)$  has the highest coefficient

$$A = -108(k-2).$$

Since  $A \leq 0$  for  $k \geq 2$ , according to P 3.76-(a), it suffices to prove that  $f_6(a, 1, 1) \geq 0$  and  $f_6(0, b, c) \geq 0$  for all  $a, b, c \geq 0$ . Indeed, we have

$$f_6(a, 1, 1) = a(a+2)(a-1)^2(a-k)^2 \geq 0$$

and

$$f_6(0, b, c) = (b-c)^6 + (6-k)bc(b-c)^4 \geq 0.$$

The equality holds for  $a = b = c$ , for  $a = 0$  and  $b = c$  (or any cyclic permutation, and for and for  $a/k = b = c$  (or any cyclic permutation).

(b) We need to prove that  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = 4(a + b + c) \sum a(a - b)(a - c)(a - kb)(a - kc) \\ + (k + 2)^2(a - b)^2(b - c)^2(c - a)^2.$$

Since  $f_6(a, b, c)$  has the same highest coefficient as

$$(k + 2)^2(a - b)^2(b - c)^2(c - a)^2$$

and  $(a - b)^2(b - c)^2(c - a)^2$  has the highest coefficient  $-27$ , it follows that  $f_6(a, b, c)$  has the highest coefficient

$$A = -27(k + 2)^2 < 0.$$

According to P 3.76-(a), it suffices to prove that  $f_6(a, 1, 1) \geq 0$  and  $f_6(0, b, c) \geq 0$  for all  $a, b, c \geq 0$ . Indeed, we have

$$f_6(a, 1, 1) = 4a(a + 2)(a - 1)^2(a - k)^2 \geq 0$$

and

$$f_6(0, b, c) = (b - c)^2[2(b^2 + c^2) - (k - 2)bc]^2 \geq 0.$$

The equality holds for  $a = b = c$ , for  $a = 0$  and  $b = c$  (or any cyclic permutation, and for and for  $a/k = b = c$  (or any cyclic permutation), and for  $a = 0$  and  $\frac{b}{c} + \frac{c}{b} = \frac{k - 2}{2}$  (or any cyclic permutation).

□

**P 3.79.** If  $a, b, c$  are nonnegative real numbers, then

$$(3a^2 + 2ab + 3b^2)(3b^2 + 2bc + 3c^2)(3c^2 + 2ca + 3a^2) \geq 8(a^2 + 3bc)(b^2 + 3ca)(c^2 + 3ab).$$

**Solution.** Let

$$p = a + b + c, \quad q = ab + bc + ca,$$

$$f(a, b, c) = (3a^2 + 2ab + 3b^2)(3b^2 + 2bc + 3c^2)(3c^2 + 2ca + 3a^2),$$

$$g(a, b, c) = (a^2 + 3bc)(b^2 + 3ca)(c^2 + 3ab).$$

We need to prove that  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = f(a, b, c) - 8g(a, b, c).$$

By Remark 2 from P 2.75,  $f(a, b, c)$  has the highest coefficient

$$A_1 = (2 - 3)^3 = -1,$$

and  $g(a, b, c)$  has the highest coefficient

$$A_2 = g(1, 1, 1) = 64.$$

Therefore,  $f_6(a, b, c)$  has the highest coefficient

$$A = A_1 - 8A_2 = -1 - 512 = -513.$$

Then, by P 3.76-(a), it suffices to prove that  $f_6(a, 1, 1) \geq 0$  and  $f_6(0, b, c) \geq 0$  for all nonnegative real  $a, b, c$ . Indeed,

$$\begin{aligned} f_6(a, 1, 1) &= 8(3a^2 + 2a + 3)^2 - 8(a^2 + 3)(3a + 1)^2 \\ &= 48(a + 1)(a - 1)^2 \geq 0, \end{aligned}$$

$$f_6(0, b, c) = 3b^2c^2(9b^2 - 2bc + 9c^2) \geq 0.$$

The equality holds for  $a = b = c$ .

□

**P 3.80.** Let  $a, b, c$  be nonnegative real numbers such that

$$a + b + c = 2.$$

If

$$\frac{-2}{3} \leq k \leq \frac{11}{8},$$

then

$$(a^2 + kab + b^2)(b^2 + kbc + c^2)(c^2 + kca + a^2) \leq k + 2.$$

(Vasile Cîrtoaje, 2011)

**Solution.** Let

$$f(a, b, c) = (a^2 + kab + b^2)(b^2 + kbc + c^2)(c^2 + kca + a^2).$$

We need to prove that  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = (k + 2)(a + b + c)^6 - 64f(a, b, c).$$

According to Remark 2 from P 2.75,  $f(a, b, c)$  has the highest coefficient

$$A_1 = (k - 1)^3.$$

Thus,  $f_6(a, b, c)$  has the highest coefficient

$$A = -64A_1 = 64(1 - k)^3.$$



Also, we have

$$f_6(a, 1, 1) = (k+2)a[(a-1)^2 + 11 - 8k][a^3 + 14a^2 + (8k+12)a + 16]$$

and

$$\begin{aligned} f_6(0, b, c) &= (k+2)(b+c)^6 - 64b^2c^2(b^2 + kbc + c^2) \\ &= (b-c)^2 [(k+2)(b^2 + c^2)^2 + 8(k+2)bc(b^2 + c^2) + 4(7k-2)b^2c^2]. \end{aligned}$$

*Case 1:*  $1 \leq k \leq \frac{11}{8}$ . Since  $A \leq 0$ , according to P 3.76-(a), it suffices to prove that  $f_6(a, 1, 1) \geq 0$  and  $f_6(0, b, c) \geq 0$  for all  $a, b, c \geq 0$ . Clearly, these conditions are satisfied. The equality holds for  $a = 0$  and  $b = c = 1$  (or any cyclic permutation). If  $k = 11/8$ , then the equality holds also for  $a = b = c = 2/3$ .

*Case 2:*  $\frac{-2}{3} \leq k < 1$ . Since  $A > 0$ , we will use the *highest coefficient cancellation method*. We will prove the sharper inequality  $g_6(a, b, c) \geq 0$ , where

$$g_6(a, b, c) = f_6(a, b, c) - 64(1-k)^3a^2b^2c^2.$$

Since  $g_6(a, b, c)$  has the highest coefficient equal to zero, it suffices to show that  $g_6(a, 1, 1) \geq 0$  and  $g_6(0, b, c) \geq 0$  for all  $a, b, c \geq 0$  (see P 3.76). The inequality  $g_6(a, 1, 1) \geq 0$  is true if

$$(k+2)[(a-1)^2 + 11 - 8k][a^3 + 14a^2 + (8k+12)a + 16] \geq 64(1-k)^3a.$$

It suffices to show that

$$(k+2)(11-8k)(6a^2 + (8k+12)a + 16) \geq 64(1-k)^3a.$$

Moreover, since  $11 - 8k > 8(1 - k)$ , we only need to show that

$$(k+2)[3a^2 + (4k+6)a + 8] \geq 4(1-k)^2a.$$

Since

$$\begin{aligned} 3a^2 + (4k+6)a + 8 &\geq 3(2a-1) + (4k+6)a + 8 \\ &= (4k+12)a + 5 > 4(k+3)a, \end{aligned}$$

it is enough to show that

$$(k+2)(k+3) \geq (1-k)^2.$$

Indeed,

$$(k+2)(k+3) - (1-k)^2 = 7k+5 = 7\left(k + \frac{2}{3}\right) + \frac{1}{3} > 0.$$

The inequality  $g_6(0, b, c) \geq 0$  is also true since

$$\begin{aligned} g_6(0, b, c) &\geq (b-c)^2[4(k+2)b^2c^2 + 16(k+2)b^2c^2 + 4(7k-2)b^2c^2] \\ &= 16(3k+2)b^2c^2(b-c)^2 \geq 0. \end{aligned}$$

Thus, the proof is completed. The equality holds for  $a = 0$  and  $b = c = 1$  (or any cyclic permutation). If  $k = 11/8$ , then the equality holds also for  $a = b = c = 2/3$ .

□

**P 3.81.** Let  $a, b, c$  be nonnegative real numbers such that

$$a + b + c = 2.$$

Prove that

$$(2a^2 + bc)(2b^2 + ca)(2c^2 + ab) \leq 4.$$

**Solution.** Write the inequality in the homogeneous form  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = (a + b + c)^6 - 16(2a^2 + bc)(2b^2 + ca)(2c^2 + ab).$$

Since  $f_6(a, b, c)$  has the same highest coefficient  $A$  as  $P_2(a, b, c)$ , where

$$P_2(a, b, c) = -16(2a^2 + bc)(2b^2 + ca)(2c^2 + ab),$$

according to Remark 2 from P 2.75, we have

$$A = P_2(1, 1, 1) = -432.$$

Since  $A < 0$ , according to P 3.76-(a), it suffices to prove that  $f_6(a, 1, 1) \geq 0$  and  $f_6(0, b, c) \geq 0$  for all  $a, b, c \geq 0$ . We have

$$\begin{aligned} f_6(a, 1, 1) &= a(a + 2)^2(a^3 + 8a^2 - 8a + 32) \\ &= a(a + 2)^2[a^3 + 4a^2 + 28 + 4(a - 1)^2] \geq 0, \end{aligned}$$

$$f_6(0, b, c) = (b + c)^6 - 64b^3c^3 \geq 0.$$

The equality holds for  $a = 0$  and  $b = c = 1$  (or any cyclic permutation).

□

**P 3.82.** Let  $a, b, c$  be nonnegative real numbers, no two of which are zero. Then,

$$\sum (a - b)(a - c)(a - 2b)(a - 2c) \geq \frac{5(a - b)^2(b - c)^2(c - a)^2}{ab + bc + ca}.$$

(Vasile Cîrtoaje, 2010)

**Solution.** Denote

$$p = a + b + c, \quad q = ab + bc + ca,$$

and write the inequality as  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = q \sum (a - b)(a - c)(a - 2b)(a - 2c) - 5(a - b)^2(b - c)^2(c - a)^2.$$

Clearly,  $f_6(a, b, c)$  has the highest coefficient

$$A = (-5)(-27) = 135.$$

Since  $A > 0$ , we will use the *highest coefficient cancellation method*. We have

$$f_6(a, 1, 1) = (2a + 1)(a - 1)^2(a - 2)^2, \quad f_6(0, b, c) = bc[(b + c)^2 - 6bc]^2.$$

Consider two cases:  $p^2 \leq 4q$  and  $p^2 > 4q$ .

*Case 1:*  $p^2 \leq 4q$ . Since

$$f_6(1, 1, 1) = 0, \quad f_6(2, 1, 1) = 0,$$

we define the symmetric homogeneous polynomial of degree three

$$P(a, b, c) = abc + Bp^3 + Cpq$$

such that  $P(1, 1, 1) = 0$  and  $P(2, 1, 1) = 0$ . We get  $B = 1/18$  and  $C = -5/18$ , hence

$$P(a, b, c) = abc + \frac{1}{18}p^3 - \frac{5}{18}pq.$$

Consider now the sharper inequality  $g_6(a, b, c) \geq 0$ , where

$$g_6(a, b, c) = f_6(a, b, c) - 135P^2(a, b, c).$$

Clearly,  $g_6(a, b, c)$  has the highest coefficient  $A = 0$ . Then, according to Remark 1 from the proof of P 3.76, it suffices to prove that  $g_6(a, 1, 1) \geq 0$  for  $0 \leq a \leq 4$ . We have

$$P(a, 1, 1) = \frac{1}{18}(a - 1)^2(a - 2),$$

hence

$$\begin{aligned} g_6(a, 1, 1) &= f_6(a, 1, 1) - 135P^2(a, 1, 1) \\ &= \frac{1}{12}(a - 1)^2(a - 2)^2(7 + 34a - 5a^2) \geq 0. \end{aligned}$$

*Case 2:*  $p^2 > 4q$ . Define the symmetric homogeneous function

$$R(a, b, c) = abc + Cpq - (9C + 1)\frac{q^2}{3p},$$

which satisfies

$$\begin{aligned} R(1, 1, 1) &= 0, \\ R(a, 1, 1) &= \frac{(a - 1)^2[3C(2a + 1) - 1]}{3(a + 2)}, \\ R(0, b, c) &= \frac{bc[3C(b + c)^2 - (9C + 1)bc]}{3(b + c)}. \end{aligned}$$

Consider further the sharper inequality  $g_6(a, b, c) \geq 0$ , where

$$g_6(a, b, c) = f_6(a, b, c) - 135R^2(a, b, c).$$

Since  $g_6(a, b, c)$  has the highest coefficient  $A = 0$ , according to Remark 2 from the proof of P 3.76, it suffices to prove that  $g_6(a, 1, 1) \geq 0$  for  $a > 4$ , and  $g_6(0, b, c) \geq 0$  for all  $b, c \geq 0$ . Since  $f_6(0, b, c) = bc[(b+c)^2 - 6bc]^2$ , the inequality  $g_6(0, b, c) \geq 0$  holds for all  $b, c \geq 0$  only if

$$\frac{9C+1}{3C} = 6;$$

that is,  $C = 1/9$ . For this value of  $C$ , we have

$$R(a, 1, 1) = \frac{2(a-1)^3}{9(a+2)}, \quad R(0, b, c) = \frac{bc[(b+c)^2 - 6bc]}{9(b+c)},$$

hence

$$g_6(a, 1, 1) = f_6(a, 1, 1) - 135R^2(a, 1, 1) = \frac{(a-1)^2}{3(a+2)^2}g(a),$$

where

$$g(a) = 3(2a+1)(a^2-4)^2 - 20(a-1)^4,$$

and

$$\begin{aligned} g_6(0, b, c) &= f_6(0, b, c) - 135R^2(0, b, c) \\ &= \frac{bc(3b^2 + bc + 3c^2)[(b+c)^2 - 6bc]^2}{3(b+c)^2} \geq 0. \end{aligned}$$

To complete the proof, we need to show that  $g(a) \geq 0$  for  $a > 4$ . This is true since  $a^2 - 4 > (a-1)^2$  and  $3(2a+1) > 20$ .

The equality holds for  $a = b = c$ , for  $a/2 = b = c$  (or any cyclic permutation), and for  $a = 0$  and  $b/c + c/b = 4$  (or any cyclic permutation).

□

**P 3.83.** If  $a, b, c$  are positive real numbers such that

$$abc = 1,$$

then

$$ab + bc + ca + \frac{50}{a+b+c+5} \geq \frac{37}{4}.$$

(Michael Rozenberg, 2013)

**Solution.** For  $abc = 1$  and fixed  $a + b + c$ , the sum  $ab + bc + ca$  has the minimum value when two of  $a, b, c$  are equal (see P 3.58). Thus, it suffices to prove the desired inequality for  $a = b$ ; that is, to show that  $a^2c = 1$  involves

$$a^2 + 2ac + \frac{50}{2a+c+5} \geq \frac{37}{4}.$$

This is equivalent to

$$a^2 + \frac{2}{a} + \frac{50a^2}{2a^3 + 5a^2 + 1} \geq \frac{37}{4},$$

which can be written in the obvious form

$$(a-1)^2(2a-1)^2(2a^2+11a+8) \geq 0.$$

The equality holds for  $a = b = c = 1$ , and for  $a = b = 1/2$  and  $c = 4$  (or any cyclic permutation).

□

**P 3.84.** If  $a, b, c$  are positive real numbers, then

$$(a+b+c-3) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 3 \right) + abc + \frac{1}{abc} \geq 2.$$

(Vasile Cîrtoaje, 2004)

**Solution.** Since the inequality does not exchange by substituting  $a, b, c$  with  $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ , respectively, we may consider only the case  $abc \geq 1$ . Using the notation

$$p = a + b + c, \quad r = abc, \quad r \geq 1,$$

we can write the inequality as

$$(p-3) \left( \frac{ab+bc+ca}{r} - 3 \right) + r + \frac{1}{r} \geq 2.$$

By P 3.58, for fixed  $p$  and  $r$ , the sum  $q = ab + bc + ca$  has the minimum value when two of  $a, b, c$  are equal. Since  $p \geq 3\sqrt[3]{r} \geq 3$  (by the AM-GM inequality), it suffices to prove the desired inequality for  $b = c$ , when it becomes as follows

$$a \left( b^2 + \frac{2}{b} - 3 \right) + \frac{1}{a} \left( \frac{1}{b^2} + 2b - 3 \right) \geq 6 \left( b + \frac{1}{b} - 2 \right),$$

$$(b-1)^2 \left[ \left( ab + \frac{1}{ab} - 2 \right) + 2 \left( a + \frac{1}{a} - 2 \right) \right] \geq 0.$$

Since  $ab + \frac{1}{ab} \geq 2$  and  $a + \frac{1}{a} \geq 2$ , the conclusion follows. The equality holds for  $a = b = 1$ , or  $b = c = 1$ , or  $c = a = 1$ .

□

**P 3.85.** If  $a, b, c$  are positive real numbers such that

$$abc = 1,$$

then

$$(a) \quad \frac{3}{7} \left( ab + bc + ca - \frac{2}{3} \right) \geq \sqrt{\frac{2}{3}(a + b + c) - 1};$$

$$(b) \quad ab + bc + ca - 3 \geq \frac{46}{27}(\sqrt{a + b + c - 2} - 1).$$

(Vasile Cîrtoaje, 2009)

**Solution.** Let

$$p = a + b + c, \quad p \geq 3.$$

For  $abc = 1$  and fixed  $p$ , the sum  $ab + bc + ca$  has the minimum value when two of  $a, b, c$  are equal (see P 3.58). Thus, it suffices to consider the case  $a = b$ .

(a) For  $a = b$ , the desired inequality is equivalent to

$$3a^3 - 2a + 6 \geq 7\sqrt{\frac{4a^3 - 3a^2 + 2}{3}}.$$

By squaring, we get

$$(a - 1)^2(3a - 1)^2(3a^2 + 8a + 10) \geq 0,$$

which is true. The equality holds for  $a = b = c = 1$ , and also for  $(a, b, c) = \left(\frac{1}{3}, \frac{1}{3}, 9\right)$  or any cyclic permutation.

(b) For  $a = b$ , the desired inequality becomes

$$27a^3 - 35a + 54 \geq 46\sqrt{2a^3 - 2a^2 + 1}.$$

By squaring, we get the obvious inequality

$$(a - 1)^2(9a - 5)^2(9a^2 + 28a + 32) \geq 0.$$

The equality holds for  $a = b = c = 1$ , and also for  $(a, b, c) = \left(\frac{5}{9}, \frac{5}{9}, \frac{81}{25}\right)$  or any cyclic permutation.

□

**P 3.86.** Let  $a, b, c$  be positive real numbers.

(a) If  $abc = 2$ , then

$$(a + b + c - 3)^2 + 1 \geq \frac{a^2 + b^2 + c^2}{3}.$$

(b) If  $abc = \frac{1}{2}$ , then

$$a^2 + b^2 + c^2 + 3(3 - a - b - c)^2 \geq 3.$$

(Vasile Cîrtoaje, 2007)

**Solution.** Let

$$p = a + b + c.$$

(a) Write the inequality as

$$(p - 3)^2 + 1 \geq \frac{p^2 - 2(ab + bc + ca)}{3}.$$

For  $abc = 2$  and fixed  $p$ , the sum  $ab + bc + ca$  has the minimum value when two of  $a, b, c$  are equal (see P 3.58). Thus, it suffices to consider the case  $a = b$ , when the inequality becomes as follows

$$\begin{aligned} \left(2a + \frac{2}{a^2} - 3\right)^2 + 1 &\geq \frac{2a^2}{3} + \frac{4}{3a^4}, \\ 5a^6 - 18a^5 + 15a^4 + 12a^3 - 18a^2 + 4 &\geq 0, \\ (a - 1)^2(5a^4 - 8a^3 - 6a^2 + 8a + 4) &\geq 0. \end{aligned}$$

Since

$$5a^4 - 8a^3 - 6a^2 + 8a + 4 = 4(a - 1)^4 + a(a^3 + 8a^3 - 30a + 24),$$

it suffices to prove that  $a^3 + 8a^3 - 30a + 24 \geq 0$ . Indeed, for  $a \geq 1$ , we have

$$\begin{aligned} a^3 + 8a^3 - 30a + 24 &= (a - 1)^3 + 11a^2 - 33a + 25 \\ &= (a - 1)^3 + 11\left(a - \frac{3}{2}\right)^2 + \frac{1}{4} > 0, \end{aligned}$$

and for  $a < 1$ , we have

$$a^3 + 8a^3 - 30a + 24 = a(1 - a)^2 + 2(2 - a)(6 - 5a) > 0.$$

The equality holds for  $(a, b, c) = (1, 1, 2)$  or any cyclic permutation.

(b) Write the inequality as

$$p^2 - 2(ab + bc + ca) + 3(3 - p)^2 \geq 3.$$

For  $abc = 1/2$  and fixed  $p$ , the sum  $ab + bc + ca$  has the maximum value when two of  $a, b, c$  are equal (see P 3.58). Thus, it suffices to consider the case  $a = b$ , when the inequality becomes in succession

$$\begin{aligned} 2a^2 + \frac{1}{4a^2} + 3 \left( 3 - 2a - \frac{1}{2a^2} \right)^2 &\geq 3, \\ 14a^6 - 36a^5 + 24a^4 + 6a^3 - 9a^2 + 1 &\geq 0, \\ (a-1)^2(14a^4 - 8a^3 - 6a^2 + 2a + 1) &\geq 0. \end{aligned}$$

Since

$$14a^4 - 8a^3 - 6a^2 + 2a + 1 = (a-1)^4 + a(13a^3 - 4a^2 - 12a + 6),$$

it suffices to prove that  $13a^3 - 4a^2 - 12a + 6 \geq 0$ . Indeed,

$$9(13a^3 - 4a^2 - 12a + 6) = 13(a+1)(3a-2)^2 + 2(a-1)^2 + a^2 > 0.$$

The equality holds for  $(a, b, c) = \left(1, 1, \frac{1}{2}\right)$  or any cyclic permutation.

□

**P 3.87.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = 3,$$

then

$$4 \left( \frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \right) + 9abc \geq 21.$$

**Solution.** Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

We write the required inequality in the homogeneous form

$$\frac{4p^2q^2}{9r} + 9r \geq \frac{5p^3}{3}.$$

For fixed  $p$  and  $r$ , it suffices to consider the case when  $q$  has the minimum value; that is, when two of  $a, b, c$  are equal (see P 3.58). Due to symmetry and homogeneity, we can set  $b = c = 1$ , when

$$p = a + 2, \quad q = 2a + 1, \quad r = a,$$

the inequality becomes

$$4(a+2)^2(2a+1)^2 + 81a^2 \geq 15(a+2)^3,$$

which is equivalent to

$$(a-1)^2(a-4)^2 \geq 0.$$

The equality holds for  $a = b = c = 1$ , and for  $a = 2$  and  $b = c = 1/2$  (or any cyclic permutation).

□



**P 3.88.** If  $a, b, c$  are nonnegative real numbers such that

$$ab + bc + ca = abc + 2,$$

then

$$a^2 + b^2 + c^2 + abc \geq 4.$$

(Vasile Cîrtoaje, 2011)

**First Solution.** Among the numbers  $1 - a$ ,  $1 - b$  and  $1 - c$  there are always two with the same sign; let us say  $(1 - b)(1 - c) \geq 0$ . Thus, we have

$$a(1 - b)(1 - c) \geq 0,$$

$$a + abc \geq ab + ac,$$

$$a + (ab + bc + ca - 2) \geq ab + ac,$$

$$a + bc \geq 2,$$

and hence

$$\begin{aligned} a^2 + b^2 + c^2 + abc - 4 &\geq a^2 + 2bc + abc - 4 \\ &= (a + 2)(a + bc - 2) \geq 0. \end{aligned}$$

The equality holds for  $a = b = c = 1$ , and for  $a = 0$  and  $b = c = \sqrt{2}$  (or any cyclic permutation).

**Second Solution.** For  $a = 0$ , we need to show that  $bc = 2$  involves  $b^2 + c^2 \geq 4$ . This is true since

$$b^2 + c^2 \geq 2bc = 4.$$

Consider further that  $a, b, c$  are positive, and write the required inequality as

$$a^2 + b^2 + c^2 + abc \geq 2(ab + bc + ca - abc),$$

$$3abc \geq 2(ab + bc + ca) - a^2 - b^2 - c^2.$$

Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

We need to show that  $q = r + 2$  implies  $3r \geq 4q - p^2$ . For fixed  $q$  and  $r$ , the sum  $p = a + b + c$  has the minimum value when two of  $a, b, c$  are equal (see Remark 2 from P 3.58). Thus, it suffices to consider the case  $b = c$ , when  $p = a + 2b$ ,  $q = 2ab + b^2$ ,  $r = ab^2$ . We need to prove that

$$2ab + b^2 = ab^2 + 2$$

implies

$$3ab^2 \geq 4(2ab + b^2) - (a + 2b)^2,$$

which is equivalent to

$$a(a + 3b^2 - 4b) \geq 0.$$

For the nontrivial case  $b < 4/3$ , we have

$$a + 3b^2 - 4b = \frac{2 - b^2}{b(2 - b)} + 3b^2 - 4b = \frac{(1 - b)^2(2 + 4b - 3b^2)}{b(2 - b)} \geq 0.$$

□

**P 3.89.** If  $a, b, c$  are nonnegative real numbers such that  $a + b + c = 3$ , then

$$(a + b)(b + c)(c + a) \geq (a + bc)(b + ca)(c + ab).$$

**Solution.** Write the inequality in the homogeneous form  $f_6(a, b, c) \geq 0$ , where

$$f_6(a, b, c) = p^3(a + b)(b + c)(c + a) - (pa + 3bc)(pb + 3ca)(pc + 3ab), \quad p = a + b + c.$$

Clearly,  $f_6(a, b, c)$  has the highest coefficient  $A = -27$ . Thus, according to P 3.76-(a), it suffices to prove that  $f_6(a, 1, 1) \geq 0$  and  $f_6(0, b, c) \geq 0$  for  $a, b, c \geq 0$ . We have

$$\begin{aligned} f_6(a, 1, 1) &= 2(a + 2)^3(a + 1)^2 - 4(a^2 + 2a + 3)(2a + 1)^2 \\ &= 2(a^5 + a^3 - 4a^2 + 2) = 2(a - 1)^2(a^3 + 2a^2 + 4a + 2) \geq 0. \end{aligned}$$

Also,

$$f_6(0, b, c) = bc(b + c)^4 - 3b^2c^2(b + c)^2 = bc(b + c)^2(b^2 - bc + c^2) \geq 0.$$

The equality holds for  $a = b = c = 1$ , and also for  $a = b = 0$  and  $c = 3$  (or any cyclic permutation).

□

**P 3.90.** Let  $a, b, c$  be positive numbers such that

$$a + b + c \leq 3\sqrt[4]{abc}.$$

Prove that

$$a^2 + b^2 + c^2 \leq 3.$$

(Vasile Cîrtoaje, 2018)

**Solution.** By the AM-GM inequality, we have

$$a + b + c \geq 3\sqrt[3]{abc},$$

and from

$$3\sqrt[3]{abc} \leq a + b + c \leq 3\sqrt[4]{abc},$$

we get  $abc \leq 1$ . Denote

$$x = \sqrt[6]{abc}, \quad x \leq 1.$$

Since

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) \leq 9\sqrt{abc} - 6\sqrt[3]{a^2b^2c^2} = 9x^3 - 6x^4,$$

we only need to show that

$$9x^3 - 6x^4 \leq 3,$$

which is equivalent to

$$(1 - x)(1 + x + x^2 - 2x^3) \geq 0.$$

This is true since

$$1 + x + x^2 - 2x^3 = 1 + x(1 - x^2) + x^2(1 - x) > 0.$$

The equality occurs for  $a = b = c = 1$ .

□

**P 3.91.** If  $a, b, c$  are positive real numbers, then

$$\left(\frac{b+c}{a} - 2 - \sqrt{2}\right)^2 + \left(\frac{c+a}{b} - 2 - \sqrt{2}\right)^2 + \left(\frac{a+b}{c} - 2 - \sqrt{2}\right)^2 \geq 6.$$

(Vasile Cîrtoaje, 2012)

**Solution.** Without loss of generality, we can assume that  $a = \max\{a, b, c\}$ . Let

$$m = 2 + \sqrt{2}, \quad t = \frac{b+c}{2}.$$

We will show that

$$E(a, b, c) \geq E(a, t, t) \geq 6,$$

where

$$E = \left(\frac{b+c}{a} - m\right)^2 + \left(\frac{c+a}{b} - m\right)^2 + \left(\frac{a+b}{c} - m\right)^2.$$

Write the left inequality as

$$\left(\frac{c+a}{b} - m\right)^2 + \left(\frac{a+b}{c} - m\right)^2 \geq 2\left(\frac{2a}{b+c} - m + 1\right)^2. \quad (*)$$

According to the identity

$$2x^2 + 2y^2 = (x - y)^2 + (x + y)^2,$$

we have

$$2\left(\frac{c+a}{b} - m\right)^2 + 2\left(\frac{a+b}{c} - m\right)^2 = \left(\frac{c+a}{b} - \frac{a+b}{c}\right)^2 + \left(\frac{c+a}{b} + \frac{a+b}{c} - 2m\right)^2.$$

Thus, we can rewrite the inequality (\*) as

$$\left(\frac{c+a}{b} - \frac{a+b}{c}\right)^2 \geq 4\left(\frac{2a}{b+c} - m + 1\right)^2 - \left(\frac{c+a}{b} + \frac{a+b}{c} - 2m\right)^2,$$

$$\frac{(a+b+c)^2(b-c)^2}{b^2c^2} + \left(\frac{4a}{b+c} + \frac{c+a}{b} + \frac{a+b}{c} - 4m + 2\right) \frac{(a+b+c)(b-c)^2}{bc(b+c)} \geq 0.$$

This is true if  $f(a) \geq 0$ , where

$$f(a) = \frac{(a+b+c)(b+c)}{bc} + \frac{4a}{b+c} + \frac{c+a}{b} + \frac{a+b}{c} - 4m + 2.$$

Since  $f(a)$  is increasing and  $a = \max\{a, b, c\}$ , it suffices to show that  $f\left(\frac{b+c}{2}\right) \geq 0$ . Indeed,

$$f\left(\frac{b+c}{2}\right) = \frac{3(b-c)^2}{bc} + 6 - 4\sqrt{2} \geq 6 - 4\sqrt{2} > 0.$$

Write now the right inequality  $E(a, t, t) \geq 6$  as

$$\left(\frac{b+c}{a} - m\right)^2 + 2\left(\frac{2a}{b+c} - m + 1\right)^2 \geq 6.$$

Setting  $\frac{b+c}{a} = x$ , this inequality becomes

$$(x-m)^2 + 2\left(\frac{2}{x} - m + 1\right)^2 \geq 6,$$

$$\frac{(x-2)^2(x-\sqrt{2})^2}{x^2} \geq 0.$$

The proof is completed. The equality holds for  $a = b = c$ , and for  $\frac{a}{\sqrt{2}} = b = c$  (or any cyclic permutation). □

**P 3.92.** If  $a, b, c$  are positive real numbers, then

$$2(a^3 + b^3 + c^3) + 9(ab + bc + ca) + 39 \geq 24(a + b + c).$$

(Vasile Cîrtoaje, 2010)

**Solution.** Let  $p = a + b + c$  and  $q = ab + bc + ca$ . Since  $a^3 + b^3 + c^3 = 3abc + p^3 - 3pq$ , we can write the inequality as

$$6abc + 2p^3 + 3(3 - 2p)q + 39 \geq 24p.$$

By Schur's inequality of degree three, we have

$$9abc \geq 4pq - p^3.$$

Therefore, it suffices to show that

$$\frac{2}{3}(4pq - p^3) + 2p^3 + 3(3 - 2p)q + 39 \geq 24p,$$

which is equivalent to

$$4p^3 + 117 \geq 72p + (10p - 27)q.$$

*Case 1:*  $10p - 27 \geq 0$ . Since  $3q \leq p^2$ , we have

$$\begin{aligned} 4p^3 + 117 - 72p - (10p - 27)q &\geq 4p^3 + 117 - 72p - \frac{(10p - 27)p^2}{3} \\ &= \frac{1}{3}(p - 3)^2(2p + 39) \geq 0. \end{aligned}$$

*Case 2:*  $10p - 27 < 0$ . From  $(3p - 8)^2 \geq 0$ , we get

$$9p^2 - 48p + 64 \geq 0,$$

$$18q \geq -9 \sum a^2 + 48p - 64.$$

Using this inequality and

$$\sum (10a - 9) = 10p - 27 < 0,$$

we get

$$\begin{aligned} 2 \left( 2 \sum a^3 + 9q + 39 - 24p \right) &\geq 4 \sum a^3 + \left( -9 \sum a^2 + 48p - 64 \right) + 78 - 48p \\ &= \sum \left( 4a^3 - 9a^2 + \frac{14}{3} \right) > \sum \left( 4a^3 - 9a^2 + \frac{14}{3} \right) + \frac{14}{27} \sum (10a - 9) \\ &= \sum a \left( 4a^2 - 9a + \frac{140}{27} \right) > \sum a \left( 4a^2 - 9a + \frac{81}{16} \right) = \sum a \left( 2a - \frac{9}{4} \right)^2 \geq 0. \end{aligned}$$

The equality holds for  $a = b = c = 1$ .

□

**P 3.93.** If  $a, b, c$  are nonnegative real numbers such that  $a^2 + b^2 + c^2 = 3$ , then

$$a^3 + b^3 + c^3 - 3 \geq |(a-b)(b-c)(c-a)|.$$

**Solution.** Assume that  $a \leq b \leq c$ , write the inequality in the homogeneous form

$$a^3 + b^3 + c^3 - 3 \left( \frac{a^2 + b^2 + c^2}{3} \right)^{3/2} \geq (b-a)(c-b)(c-a),$$

and use the substitution

$$b = a + p, \quad c = a + q, \quad 0 \leq p \leq q.$$

For fixed  $p$  and  $q$ , we need to show that

$$f(a) \geq pq(q-p),$$

where

$$f(a) = a^3 + b^3 + c^3 - 3 \left( \frac{a^2 + b^2 + c^2}{3} \right)^{3/2}.$$

Since  $a' = b' = c' = 1$ , we have

$$\begin{aligned} f'(a) &= 3(a^2 + b^2 + c^2) - 3(a+b+c) \sqrt{\frac{a^2 + b^2 + c^2}{3}} \\ &= 9 \sqrt{\frac{a^2 + b^2 + c^2}{3}} \left( \sqrt{\frac{a^2 + b^2 + c^2}{3}} - \frac{a+b+c}{3} \right) \geq 0. \end{aligned}$$

Thus,  $f(a)$  is increasing,  $f(a) \geq f(0)$ , and it suffices to show that  $f(0) \geq pq(q-p)$ , that is

$$p^3 + q^3 - 3 \left( \frac{p^2 + q^2}{3} \right)^{3/2} \geq pq(q-p).$$

Consider the non-trivial case  $p > 0$ . Due to homogeneity, we may assume that  $p = 1$  and  $q \geq 1$ , when the inequality becomes as follows:

$$q^3 - q^2 + q + 1 \geq 3 \left( \frac{1 + q^2}{3} \right)^{3/2},$$

$$3(q^3 - q^2 + q + 1)^2 \geq (q^2 + 1)^3,$$

$$q^6 - 3q^5 + 3q^4 - 3q^2 + 3q + 1 \geq 0,$$

$$q^3(q-1)^3 + q^3 - 3q^2 + 3q + 1 \geq 0,$$

$$(q^3 + 1)(q-1)^3 + 2 \geq 0.$$

The last inequality is clearly true. The equality holds for  $a = b = c = 1$ .

□

**P 3.94.** Let  $a, b, c$  be nonnegative real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove that

$$1 - abc \geq \frac{5}{3} \min\{(a-b)^2, (b-c)^2, (c-a)^2\}.$$

(Vasile Cîrtoaje, 2019)

**Solution.** Assume that  $a \geq b \geq c$ . For  $a = c$ , the inequality is a trivial equality. Next assume  $a > c$ . There are two cases to consider:  $a - b \geq b - c$  and  $a - b \leq b - c$ .

Case 1:  $a - b \geq b - c$ . Write the inequality in the homogeneous form  $f(a) \leq 0$ , where

$$f(a) = abc + \frac{5}{3}(b-c)^2 g^{1/2}(a) - g^{3/2}(a), \quad g(a) = \frac{a^2 + b^2 + c^2}{3}.$$

We will show that

$$f(a) \leq f(2b - c) \leq 0.$$

The left inequality is true if  $f'(a) \leq 0$ . Since

$$g'(a) = \frac{2a}{3}$$

and

$$f'(a) = bc + \frac{5a(b-c)^2}{9g^{1/2}} - ag^{1/2} = bc + \frac{5a(b-c)^2}{9} - a,$$

we need to show that

$$a \left[ 1 - \frac{5(b-c)^2}{9} \right] \geq bc.$$

Since  $a \geq b$ , this is true if

$$1 - \frac{5(b-c)^2}{9} \geq c,$$

that is

$$3(1-c) \geq \frac{5(b-c)^2}{3}.$$

It is enough to show that

$$3(1-c) \geq 2(b-c)^2.$$

From

$$3 = a^2 + b^2 + c^2 \geq 2b^2 + c^2,$$

we get

$$b \leq \sqrt{\frac{3-c^2}{2}}.$$

Thus, it suffices to show that

$$3(1-c) \geq 2 \left( \sqrt{\frac{3-c^2}{2}} - c \right)^2,$$

which is equivalent to

$$c(2\sqrt{6-2c^2}-3-c) \geq 0,$$

$$\frac{3c(1-c)(5+3c)}{2\sqrt{6-2c^2}+3+c} \geq 0.$$

Since  $f$  is decreasing and  $a \geq 2b-c$ , we have  $f(a) \leq f(2b-c)$ .

The inequality  $f(2b-c) \geq 0$  is true if the original inequality holds for  $a = 2b-c$ . Thus, we need to show that

$$3bc(2b-c) + 5(b-c)^2 \leq 3$$

for

$$(2b-c)^2 + b^2 + c^2 = 3,$$

which involves

$$2(b-c)^2 + 3b^2 = 3, \quad b \leq 1.$$

Indeed, we have

$$3bc(2b-c) + 5(b-c)^2 - 3 \leq 3c(2b-c) + 5(b-c)^2 - 3 = 0.$$

*Case 2:*  $a-b \leq b-c$ . We have

$$c \leq 2b-a, \quad 2b-a \geq 0.$$

Write the inequality in the homogeneous form  $f(c) \leq 0$ , where

$$f(c) = abc + \frac{5}{3}(a-b)^2 g^{1/2}(c) - g^{3/2}(c), \quad g(c) = \frac{a^2 + b^2 + c^2}{3}.$$

We will show that

$$f(c) \leq f(2b-a) \leq 0.$$

The left inequality is true if  $f'(c) \geq 0$ . We have

$$g'(c) = \frac{2c}{3}$$

and

$$f'(c) = ab + \frac{5c(a-b)^2}{9g^{1/2}} - cg^{1/2} = ab + \frac{5c(b-c)^2}{9} - c \geq ab - c \geq b - c \geq 0.$$

The inequality  $f(2b-a) \geq 0$  is true if the original inequality holds for  $c = 2b-a$ . Thus, we need to show that

$$3ab(2b-a) + 5(a-b)^2 \leq 3$$

for

$$a^2 + b^2 + (2b-a)^2 = 3,$$

which involves

$$2(a-b)^2 + 3b^2 = 3, \quad b \leq 1.$$



Indeed, we have

$$3ab(2b - a) + 5(a - b)^2 - 3 \leq 3a(2b - a) + 5(a - b)^2 - 3 = 0.$$

The proof is completed. The equality occurs for  $a = b = c = 1$ , and also for  $a = 2\sqrt{\frac{3}{5}}$ ,  $b = \sqrt{\frac{3}{5}}$  and  $c = 0$  (or any permutation). □

**P 3.95.** If  $a, b, c$  are nonnegative real numbers, then

$$a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 \geq 2|a^3b + b^3c + c^3a - ab^3 - bc^3 - ca^3|.$$

**Solution.** Assume that  $a \leq b \leq c$  and write the inequality as

$$(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 \geq 4(a + b + c)(a - b)(b - c)(c - a).$$

Using the substitution  $b = a + p$  and  $c = a + q$ , where  $q \geq p \geq 0$ , the inequality can be restated as

$$4Aa^2 + 4Ba + C \geq 0,$$

where

$$\begin{aligned} A &= p^2 - pq + q^2, \quad B = p^3 + q(p - q)^2, \\ C &= p^4 + 2p^3q - p^2q^2 - 2pq^3 + q^4 = (p^2 + pq - q^2)^2. \end{aligned}$$

Since  $A \geq 0$ ,  $B \geq 0$  and  $C \geq 0$ , the inequality is obviously true. For  $a \leq b \leq c$ , the equality occurs when  $a = b = c$ , and also when  $a = 0$  and  $\frac{c}{b} = \frac{1 + \sqrt{5}}{2}$ . □

**P 3.96.** If  $a, b, c$  are nonnegative real numbers, then

$$a^4 + b^4 + c^4 - abc(a + b + c) \geq 2\sqrt{2} |a^3b + b^3c + c^3a - ab^3 - bc^3 - ca^3|.$$

(Pham Kim Hung, 2006)

**Solution.** Assume that  $a \leq b \leq c$  and write the inequality as

$$a^2(a^2 - bc) + b^2(b^2 - ca) + c^2(c^2 - ab) \geq 2\sqrt{2}(a + b + c)(a - b)(b - c)(c - a).$$

Using the substitution  $b = a + p$  and  $c = a + q$ , where  $q \geq p \geq 0$ , the inequality becomes

$$Aa^2 + Ba + C \geq 0,$$

where

$$A = 5(p^2 - pq + q^2), \quad B = 4p^3 + (6\sqrt{2} - 1)p^2q - (6\sqrt{2} + 1)pq^2 + 4q^3, \\ C = p^4 + q^4 + 2\sqrt{2}pq(p^2 - q^2).$$

Since

$$A \geq 0,$$

$$B \geq \frac{25}{4}p^2q - 10pq^2 + 4q^3 = q \left( \frac{5p}{2} - 2q \right)^2 \geq 0$$

and

$$C = (p^2 + \sqrt{2}pq - q^2)^2 \geq 0,$$

the conclusion follows. For  $a \leq b \leq c$ , the equality occurs when  $a = b = c$ , and also when  $a = 0$  and  $\frac{c}{b} = \frac{\sqrt{2} + \sqrt{6}}{2}$ . □

**P 3.97.** If  $a, b, c$  are nonnegative real numbers such that  $a + b + c = 3$ , then

$$(a^3b + b^3c + c^3a - 3abc)(ab^3 + bc^3 + ca^3 - 3abc) \geq (a^2b^2 + b^2c^2 + c^2a^2 - 3abc)^2.$$

(Vasile Cîrtoaje, 2008)

**Solution.** Write the inequality in the homogeneous form

$$AB \geq C^2,$$

where

$$A = a^3b + b^3c + c^3a - abc(a + b + c), \\ B = ab^3 + bc^3 + ca^3 - abc(a + b + c), \\ C = a^2b^2 + b^2c^2 + c^2a^2 - abc(a + b + c).$$

From the Cauchy-Schwarz inequality

$$(c + a + b)(a^3b + b^3c + c^3a) \geq abc(a + b + c)^2,$$

it follows that  $A \geq 0$ , with equality for  $a = b = c$ . Thus, the desired inequality  $AB \geq C^2$  is true if

$$At^2 - 2Ct + B \geq 0$$

for all real  $t$ . This inequality is equivalent to

$$(a^3b + b^3c + c^3a)t^2 - 2(a^2b^2 + b^2c^2 + c^2a^2)t + ab^3 + bc^3 + ca^3 \geq abc(a + b + c)(t - 1)^2, \\ ab(at - b)^2 + bc(bt - c)^2 + ca(ct - a)^2 \geq abc(a + b + c)(t - 1)^2.$$

Clearly, the last inequality follows immediately from the Cauchy-Schwarz inequality

$$(c + a + b)[ab(at - b)^2 + bc(bt - c)^2 + ca(ct - a)^2] \geq abc(a + b + c)^2(t - 1)^2.$$

The equality holds for  $a = b = c$ , and also for  $a = 0$  or  $b = 0$  or  $c = 0$ .

**Remark.** Actually, the following identity holds:

$$AB - C^2 = abc(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)^2.$$

□

**P 3.98.** If  $a, b, c \geq -5$  such that

$$a + b + c = 3,$$

then

$$\frac{1 - a}{1 + a + a^2} + \frac{1 - b}{1 + b + b^2} + \frac{1 - c}{1 + c + c^2} \geq 0.$$

(Vasile Cîrtoaje, 2014)

**First Solution.** Using the substitution

$$a = x - 5, \quad b = y - 5, \quad c = z - 5,$$

we need to prove that if  $x, y, z \geq 0$  such that  $x + y + z = 18$ , then

$$\frac{6 - x}{x^2 - 9x + 21} + \frac{6 - y}{y^2 - 9y + 21} + \frac{6 - z}{z^2 - 9z + 21} \geq 0.$$

Denoting

$$p = \frac{x + y + z}{18},$$

we can write this inequality as  $f_5(x, y, z) \geq 0$ , where

$$f_5(x, y, z) = \sum (6p - x)(y^2 - 9yp + 21p^2)(z^2 - 9zp + 21p^2)$$

is a symmetric homogeneous polynomial of degree 5. According to Remark from P 3.68, it suffices to prove this inequality for  $y = z$  and for  $x = 0$ . Therefore, we only need to prove the original inequality for  $b = c$  and for  $a = -5$ .

*Case 1:*  $b = c = \frac{3 - a}{2}$ . Since

$$\frac{1 - b}{1 + b + b^2} = \frac{1 - c}{1 + c + c^2} = \frac{2(a - 1)}{a^2 - 8a + 19},$$

we need to show that

$$\frac{1 - a}{1 + a + a^2} + \frac{4(a - 1)}{a^2 - 8a + 19} \geq 0,$$

which is equivalent to

$$(a-1)^2(a+5) \geq 0.$$

Case 2:  $a = -5$ ,  $b + c = 8$ . We can write the desired inequality as follows:

$$\left(\frac{1}{7} + \frac{1-b}{1+b+b^2}\right) + \left(\frac{1}{7} + \frac{1-c}{1+c+c^2}\right) \geq 0,$$

$$\frac{(b-4)(b-2)}{1+b+b^2} + \frac{(c-4)(c-2)}{1+c+c^2} \geq 0,$$

$$\frac{b-c}{2} \left( \frac{b-2}{1+b+b^2} - \frac{c-2}{1+c+c^2} \right) \geq 0,$$

$$\frac{(b-c)^2[3+2(b+c)-bc]}{2(1+b+b^2)(1+c+c^2)} \geq 0.$$

The last inequality is true since

$$2(b+c) - bc = \left(\frac{b+c}{2}\right)^2 - bc = \left(\frac{b-c}{2}\right)^2 \geq 0.$$

The proof is completed. The equality occurs for  $a = b = c = 1$ , and also for  $a = -5$  and  $b = c = 4$  (or any cyclic permutation).

**Second Solution.** Assume that  $a \leq b \leq c$  and denote

$$E(a, b, c) = \frac{1-a}{1+a+a^2} + \frac{1-b}{1+b+b^2} + \frac{1-c}{1+c+c^2}.$$

We will show that

$$E(a, b, c) \geq E(a, t, t) \geq 0,$$

where

$$t = \frac{b+c}{2} = \frac{3-a}{2}.$$

From  $-5 \leq a \leq 1$  it follows that

$$t \in [1, 4].$$

Write the left inequality as follows:

$$\left(\frac{1-b}{1+b+b^2} - \frac{1-t}{1+t+t^2}\right) + \left(\frac{1-c}{1+c+c^2} - \frac{1-t}{1+t+t^2}\right) \geq 0,$$

$$(b-c) \left[ \frac{(b-1)t - b - 2}{1+b+b^2} - \frac{(c-1)t - c - 2}{1+c+c^2} \right] \geq 0,$$

$$(b-c)^2[(2+b+c-bc)t + 1 + 2(b+c) + bc] \geq 0,$$

$$(b-c)^2[2t^2 + 6t + 1 - bc(t-1)] \geq 0.$$

The last inequality is true since

$$\begin{aligned} 2t^2 + 6t + 1 - bc(t-1) &\geq 2t^2 + 6t + 1 - t^2(t-1) \\ &= t(4-t)(1+t) + 2t + 1 > 0. \end{aligned}$$

Also, we have

$$\begin{aligned} E(a, t, t) &= \frac{1-a}{1+a+a^2} + \frac{2(1-t)}{1+t+t^2} \\ &= \frac{2(t-1)}{4t^2-14t+13} + \frac{2(1-t)}{1+t+t^2} \\ &= \frac{6(1-t)^2(4-t)}{(4t^2-14t+13)(1+t+t^2)} \geq 0. \end{aligned}$$

□

**P 3.99.** Let  $a, b, c \neq \frac{1}{k}$  be nonnegative real numbers such that

$$a + b + c = 3.$$

If  $k \geq \frac{4}{3}$ , then

$$\frac{1-a}{(1-ka)^2} + \frac{1-b}{(1-kb)^2} + \frac{1-c}{(1-kc)^2} \geq 0.$$

(Vasile Cîrtoaje, 2012)

**Solution.** Denoting  $p = (a+b+c)/3$ , we may write the inequality as  $f_5(x, y, z) \geq 0$ , where

$$f_5(x, y, z) = \sum (p-a)(p-kb)^2(p-kc)^2$$

is a symmetric homogeneous polynomial of degree 5. According to P 3.68, it suffices to prove this inequality for  $b = c$  and for  $a = 0$ .

*Case 1:*  $b = c$ . Since  $a = 3 - 2b$ , the original inequality is equivalent to the following sequence of inequalities:

$$\begin{aligned} \frac{1-a}{(1-ka)^2} + \frac{2(1-b)}{(1-kb)^2} &\geq 0, \\ \frac{2(b-1)}{(1-3k+2kb)^2} + \frac{2(1-b)}{(1-kb)^2} &\geq 0, \\ k(b-1)^2[k(3-b)-2] &\geq 0. \end{aligned}$$

The last inequality holds since

$$k(3-b)-2 \geq \frac{4}{3}(3-b)-2 = \frac{2(3-2b)}{3} = \frac{2a}{3} \geq 0.$$

Case 2:  $a = 0$ . Since  $b + c = 3$ , the original inequality becomes as follows:

$$\begin{aligned} 1 &\geq \frac{b-1}{(1-kb)^2} + \frac{c-1}{(1-kc)^2}, \\ (1-kb)^2(1-kc)^2 &\geq (b-1)(1-kc)^2 + (c-1)(1-kb)^2, \\ (k^2bc - 3k + 1)^2 &\geq 1 + 6k - 9k^2 + (5k^2 - 4k)bc, \\ k^3b^2c^2 + 18k - 12 &\geq (6k^2 + 3k - 4)bc. \end{aligned}$$

Since

$$k^3b^2c^2 + 18k - 12 \geq 2\sqrt{k^3(18k - 12)} bc,$$

it suffices to show that

$$4k^3(18k - 12) \geq (6k^2 + 3k - 4)^2,$$

which is equivalent to

$$\begin{aligned} 36k^4 - 84k^3 + 39k^2 + 24k - 16 &\geq 0, \\ (3k - 4)(12k^3 - 12k^2 - 3k + 4) &\geq 0. \end{aligned}$$

The last inequality holds since

$$12k^3 - 12k^2 - 3k + 4 > 12k^3 - 12k^2 - 4k + 4 = 4(k-1)(3k^2 - 1) > 0.$$

The proof is completed. The equality occurs for  $a = b = c = 1$ . If  $k = 4/3$ , then the equality holds also for  $a = 0$  and  $b = c = 3/2$  (or any cyclic permutation). □

**P 3.100.** If  $a, b, c$  are positive real numbers such that

$$abc = 1,$$

then

$$3(2a^2 + 1)(2b^2 + 1)(2c^2 + 1) \leq (a + b + c)^4.$$

(Vasile Cîrtoaje, 2015)

**Solution.** Since

$$2a^2 + 1 = 2a^2 + (abc)^{2/3} = a^{2/3}[2a^{4/3} + (bc)^{2/3}],$$

we can write the inequality in the homogeneous form

$$3[2a^{4/3} + (bc)^{2/3}][2b^{4/3} + (ca)^{2/3}][2c^{4/3} + (ab)^{2/3}] \leq (a + b + c)^4.$$

Since the condition  $abc = 1$  becomes superfluous, we may now assume that  $a + b + c = 3$ , when the inequality can be written as

$$[2a^{4/3} + (bc)^{2/3}][2b^{4/3} + (ca)^{2/3}][2c^{4/3} + (ab)^{2/3}] \leq 27,$$

$$4 \sum a^2 b^2 + 2(abc)^{2/3} \sum a^2 + 9(abc)^{4/3} \leq 27.$$

By the AM-GM inequality, we have  $abc \leq 1$  and

$$2abc + 1 \geq 3(abc)^{2/3}.$$

Therefore, it suffices to show that

$$4 \sum a^2 b^2 + \frac{2}{3}(2abc + 1) \sum a^2 + 9abc \leq 27,$$

which is equivalent to

$$(9 + 8q)r + 63 + 4q - 12q^2 \geq 0,$$

where

$$q = ab + bc + ca, \quad r = abc, \quad 0 < q \leq 3, \quad 0 < r \leq 1.$$

By Schur's inequality

$$(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca),$$

we get

$$3r \geq 4q - 9,$$

hence

$$\begin{aligned} (9 + 8q)r + 63 + 4q - 12q^2 &\geq (6 + 9q)r + 63 + 4q - 12q^2 \\ &\geq (2 + 3q)(4q - 9) + 63 + 4q - 12q^2 \\ &= 15(3 - q) \geq 0. \end{aligned}$$

The equality occurs for  $a = b = c = 1$ .

□

**P 3.101.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = \sqrt{3},$$

then

$$(3\sqrt{3} - 5) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq a^2 + b^2 + c^2.$$

(Vasile Cîrtoaje, 2014)

**Solution.** Assume that

$$a \geq b \geq c, \quad a \geq \frac{\sqrt{3}}{3},$$

and show that

$$F(a, b, c) \geq F(a, x, x) \geq 0,$$

where

$$x = \frac{b+c}{2} = \frac{\sqrt{3}-a}{2}, \quad F(a, b, c) = (3\sqrt{3}-5) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - a^2 - b^2 - c^2.$$

The left inequality is equivalent to

$$(b-c)^2 \left[ 2(3\sqrt{3}-5) - bc(b+c) \right] \geq 0.$$

This is true since

$$\begin{aligned} bc(b+c) &\leq \frac{(b+c)^3}{4} = \frac{(\sqrt{3}-a)^3}{4} \\ &\leq \frac{(\sqrt{3}-\sqrt{3}/3)^3}{4} = \frac{2}{3\sqrt{3}} < 2(3\sqrt{3}-5). \end{aligned}$$

The right inequality,  $F(a, x, x) \geq 0$ , is equivalent to

$$\begin{aligned} \sqrt{3} a^4 - 5a^3 + 3\sqrt{3} a^2 + 5(3-2\sqrt{3})a + 6\sqrt{3} - 10 &\geq 0, \\ (a-1)^2 \left[ \sqrt{3} a^2 - (5-2\sqrt{3})a + 6\sqrt{3} - 10 \right] &\geq 0. \end{aligned}$$

Thus, it suffices to show that

$$\sqrt{3} a^2 - (5-2\sqrt{3})a + 6\sqrt{3} - 10 \geq 0.$$

This inequality is strict because

$$D = (5-2\sqrt{3})^2 - 4\sqrt{3}(6\sqrt{3}-10) = 20\sqrt{3} - 35 < 0.$$

The equality occurs for  $a = 1$  and  $b = c = \frac{\sqrt{3}-1}{2}$  (or any cyclic permutation).

□

**P 3.102.** If  $a, b, c \geq 1$  such that

$$a + b + c = 4,$$

then

$$12 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 5(a^2 + b^2 + c^2).$$

(Vasile Cîrtoaje, 2014)

**Solution.** Assume that

$$a \geq b \geq c, \quad a \geq \frac{4}{3},$$

and show that

$$F(a, b, c) \geq F(a, x, x) \geq 0,$$



where

$$F(a, b, c) = 12 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - 5(a^2 + b^2 + c^2)$$

and

$$x = \frac{b+c}{2}.$$

From

$$x \geq \frac{1+1}{2} = 1, \quad x = \frac{4-a}{2} \leq \frac{4-4/3}{2} = \frac{4}{3},$$

we get

$$1 \leq x \leq \frac{4}{3}.$$

The left inequality is equivalent to

$$(b-c)^2[24 - 5bc(b+c)] \geq 0.$$

This is true since

$$5bc(b+c) \leq \frac{5(b+c)^3}{4} = 10x^3 \leq \frac{640}{27} < 24.$$

The right inequality,  $F(a, x, x) \geq 0$ , is equivalent to

$$F(4-2x, x, x) \geq 0,$$

$$15x^4 - 70x^3 + 120x^2 - 89x + 24 \geq 0,$$

$$(x-1)(15x^3 - 55x^2 + 65x - 24) \geq 0,$$

$$(x-1)[1 - 5(x-1)^2(5-3x)] \geq 0.$$

We only need to show that

$$1 \geq 15(x-1)^2 \left( \frac{5}{3} - x \right).$$

Since

$$(x-1) \left( \frac{5}{3} - x \right) \leq \frac{1}{4} \left[ (x-1) + \left( \frac{5}{3} - x \right) \right]^2 = \frac{1}{9},$$

it suffices to show that

$$1 \geq \frac{15}{9}(x-1),$$

which is equivalent to

$$x \leq \frac{8}{5}.$$

The proof is completed. The equality occurs for  $a = 2$  and  $b = c = 1$  (or any cyclic permutation).

□

**P 3.103.** If  $a, b, c$  are positive real numbers such that

$$a + b + c = 3, \quad c \leq \frac{15}{32},$$

then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq a^2 + b^2 + c^2.$$

(Vasile Cîrtoaje, 2018)

**Solution.** Write the inequality as

$$\frac{a+b}{ab} + 2ab \geq (a+b)^2 + c^2 - \frac{1}{c}.$$

Since

$$\frac{a+b}{ab} + 2ab \geq 2\sqrt{2(a+b)},$$

it suffices to show that

$$2\sqrt{2(a+b)} \geq (a+b)^2 + c^2 - \frac{1}{c},$$

which is equivalent to

$$\sqrt{8(3-c)} + \frac{1}{c} + 6c - 2c^2 - 9 \geq 0.$$

We can get this inequality by summing the inequalities

$$\sqrt{8(3-c)} - \frac{9}{2} \geq 0$$

and

$$\frac{1}{c} + 6c - 2c^2 - \frac{9}{2} \geq 0.$$

The first inequality reduces to  $c \leq \frac{15}{32}$ , and the second inequality to

$$(2-c)(1-2c)^2 \geq 0.$$

The inequality is an equality for  $a = b = c = 1$ .

□

**P 3.104.** If  $a \geq b \geq c \geq 0$  and  $ab + bc + ca = 3$ , then

$$(a) \quad b + c \leq 2;$$

$$(b) \quad b^2 + bc + c^2 \leq 3.$$

(Vasile Cîrtoaje, 2018)

**Solution.** From

$$3a^2 \geq ab + bc + ca = 3,$$

it follows that  $a \geq 1$ , and from  $(a - b)(a - c) \geq 0$ , we get

$$a^2 + bc \geq a(b + c),$$

hence

$$3 = a(b + c) + bc \geq 2a(b + c) - a^2,$$

$$b + c \leq \frac{a^2 + 3}{2a}.$$

(a) It suffices to show that

$$\frac{a^2 + 3}{2a} \leq 2,$$

that is

$$(a - 1)(a - 3) \leq 0.$$

Since this is true for  $a \leq 3$ , it remains to consider the case  $a \geq 3$ , when

$$b + c = \frac{3 - bc}{a} \leq \frac{3}{a} \leq 1 < 2.$$

The equality occurs for  $a = b = c = 1$ .

(b) From

$$\begin{aligned} b^2 + bc + c^2 - 3 &= (b + c)^2 - bc = (b + c)^2 + a(b + c) - 6 \leq \frac{(a^2 + 3)^2}{4a^2} + \frac{a^2 + 3}{2} - 6 \\ &= \frac{3(a^4 - 4a^2 + 3)}{4a^2} = \frac{3(a^2 - 1)(a^2 - 3)}{4a^2}, \end{aligned}$$

it follows that  $b^2 + bc + c^2 \leq 3$  for  $a^2 \leq 3$ . Consider further the case  $a^2 \geq 3$ , when

$$b^2 + bc + c^2 \leq (b + c)^2 = \left(\frac{3 - bc}{a}\right)^2 \leq \frac{9}{a^2} \leq 3.$$

The equality occurs for  $a = b = c = 1$ , and also for  $a = b = \sqrt{3}$  and  $c = 0$ .

□

**P 3.105.** If  $a, b, c \in \left[0, 1 + \frac{1}{\sqrt{2}}\right]$  and  $a^2 + b^2 + c^2 = 3$ , then

$$a + b + c \geq abc + 2.$$

(Vasile Cîrtoaje, 2019)

**Solution.** Assume that  $a \geq b \geq c$  and denote  $x = b + c$ . From

$$b + c \leq \sqrt{2(b^2 + c^2)} = \sqrt{6 - 2a^2}$$

and

$$b + c \geq \sqrt{b^2 + c^2} = \sqrt{3 - a^2}.$$

we get

$$\sqrt{3 - a^2} \leq x \leq \sqrt{6 - 2a^2}, \quad 1 \leq a \leq 1 + \frac{1}{\sqrt{2}}.$$

Since

$$2bc = (b + c)^2 - b^2 - c^2 = (b + c)^2 + a^2 - 3,$$

the required inequality can be written as

$$2a + 2(b + c) \geq a[(b + c)^2 + a^2 - 3] + 4,$$

that is  $f(x) \leq 0$ , where

$$f(x) = ax^2 - 2x + a^3 - 5a + 4.$$

Since  $f$  is a quadratic convex function, it suffices to show that  $f(\sqrt{3 - a^2}) \leq 0$  and  $f(\sqrt{6 - 2a^2}) \leq 0$ . We have

$$f(\sqrt{3 - a^2}) = 2(2 - a - \sqrt{3 - a^2}) = \frac{2(2a^2 - 4a + 1)}{2 - a + \sqrt{3 - a^2}} \leq 0.$$

Also,

$$\begin{aligned} f(\sqrt{6 - 2a^2}) &= 4 + a - a^3 - 2\sqrt{6 - 2a^2} = \frac{a^6 - 2a^4 - 8a^3 + 9a^2 + 8a - 8}{4 + a - a^3 + 2\sqrt{6 - 2a^2}} \\ &= \frac{(a + 1)(a - 1)^2(a^3 + a^2 - 8)}{4 + a - a^3 + 2\sqrt{6 - 2a^2}}. \end{aligned}$$

Because

$$a^3 + a^2 - 8 \leq \left(1 + \frac{1}{\sqrt{2}}\right)^3 + \left(1 + \frac{1}{\sqrt{2}}\right)^2 - 8 = \frac{11\sqrt{2} - 16}{4} < 0,$$

we get  $f(\sqrt{6 - 2a^2}) \leq 0$ .

The inequality is an equality for  $a = b = c = 1$ , and also for

$$a = 1 + \frac{1}{\sqrt{2}}, \quad b = 1 - \frac{1}{\sqrt{2}}, \quad c = 0$$

(or any permutation).

□

**P 3.106.** Let  $a, b, c \geq \frac{1}{6}$  be real numbers such that  $a^2 + b^2 + c^2 = 3$ . Then,

$$a + b + c \geq abc + 2.$$

(Vasile Cîrtoaje, 2019)

**Solution.** Assume that  $a = \min\{a, b, c\}$ , hence  $\frac{1}{6} \leq a \leq 1$ , and denote  $s = a + b + c$  and  $x = b + c$ . From

$$\left(a - \frac{1}{6}\right) \left(b - \frac{1}{6}\right) + \left(b - \frac{1}{6}\right) \left(c - \frac{1}{6}\right) + \left(c - \frac{1}{6}\right) \left(a - \frac{1}{6}\right) \geq 0,$$

we get

$$\begin{aligned} ab + bc + ca + \frac{1}{12} &\geq \frac{1}{3}(a + b + c), \\ (a + b + c)^2 - (a^2 + b^2 + c^2) + \frac{1}{6} &\geq \frac{2}{3}(a + b + c), \\ 6s^2 - 4s - 17 &\geq 0, \end{aligned}$$

hence

$$s \geq k, \quad k = \frac{1}{3} + \sqrt{\frac{53}{18}} \approx 2.0493. \quad 6k^2 - 4k - 17 = 0.$$

From

$$k - a \leq b + c \leq \sqrt{2(b^2 + c^2)},$$

we get

$$k - a \leq x \leq \sqrt{6 - 2a^2}.$$

Since

$$\begin{aligned} 2bc &= (b + c)^2 - (b^2 + c^2) = x^2 - 3 + a^2, \\ 2abc + 4 - 2(a + b + c) &= a(x^2 - 3 + a^2) + 4 - 2a - 2x, \end{aligned}$$

we may write the inequality as  $f(x) \leq 0$ , where

$$f(x) = ax^2 - 2x + a^3 - 5a + 4.$$

Since  $f$  is a quadratic convex function, it is enough to show that  $f(\sqrt{6 - 2a^2}) \leq 0$  and  $f(k - a) \geq 0$ .

The first inequality is equivalent to

$$-a^3 + a + 4 \leq 2\sqrt{6 - 2a^2}.$$

This is true if

$$(-a^3 + a + 4)^2 \leq 4(6 - 2a^2),$$

that is

$$\begin{aligned} a^6 - 2a^4 - 8a^3 + 9a^2 + 8a - 8 &\leq 0, \\ (a - 1)^2(a^4 + 2a^3 + a^2 - 8a - 8) &\leq 0. \end{aligned}$$

Since  $a < 1$ , the inequality is clearly true.

The second inequality is equivalent to

$$2a^3 - 2ka^2 + (k^2 - 3)a + 4 - 2k \leq 0,$$

which can be written as  $g(a) \leq 0$ , where

$$g(a) = 12a^3 - 12ka^2 + (4k - 1)a + 24 - 12k,$$

with

$$g'(a) = 36a^2 - 24ka + 4k - 1 = (6a - 1)(6a - 4k + 1).$$

Since  $6a - 1 \geq 0$  and  $6a - 4k + 1 \leq 7 - 4k < 0$ , we have  $g'(a) \leq 0$ ,  $g(a)$  is strictly decreasing, hence  $g(a) \leq g(1/6)$ . So, it suffices to show that  $g(1/6) \leq 0$ . Indeed,

$$g(1/6) = \frac{5(43 - 105k)}{9} \approx -0.01928 < 0.$$

The equality occurs for  $a = b = c = 1$ .

□

**P 3.107.** If  $a, b, c$  are nonnegative real numbers such that

$$ab + bc + ca + 6abc = 9,$$

then

$$2(a + b + c) \geq ab + bc + ca + 3.$$

(Vasile Cîrtoaje, 2019)

**Solution.** For  $a = 0$ , we need to show that  $2(b + c) \geq bc + 3$  for  $bc = 9$ . Indeed,

$$2(b + c) - bc - 3 \geq 4\sqrt{bc} - bc - 3 = 0.$$

Assume now that  $a \geq b \geq c > 0$ . According to Remark 2 from P 3.58, for fixed  $ab + bc + ca$  and  $abc$ , the sum  $a + b + c$  has the minimum value when  $a \geq b = c$ . Therefore, it suffices to consider this case, when we need to show that

$$2a + 4b \geq 2ab + b^2 + 3$$

for

$$a = \frac{9 - b^2}{2b(3b + 1)}, \quad 0 < b < 3.$$

Substituting  $a$ , the required inequality becomes

$$(1 - b)^2(1 + b)(3 - b) \geq 0.$$

The inequality is an equality for  $a = b = c = 1$ , and also for  $a = 0$  and  $b = c = 3$  (or any cyclic permutation).

**Remark.** Similarly, we can prove the following generalization:

- If  $a, b, c$  are nonnegative real numbers such that

$$ab + bc + ca + kabc = 3 + k, \quad k > 0,$$

then

$$\frac{ab + bc + ca - 3}{a + b + c - 3} \leq \frac{k}{2\sqrt{k+3} - 3},$$

with equality for  $a = 0$  and  $b = c = k$  (or any cyclic permutation).

For  $k = 1$ , we get that  $a + b + c + abc = 4$  involves the nice inequality

$$a + b + c \geq ab + bc + ca.$$

□

**P 3.108.** If  $a, b, c$  are nonnegative real numbers such that

$$ab + bc + ca + abc = 4,$$

then

$$4(a + b + c) + 3a^2b^2c^2 \geq 15.$$

(Vasile Cîrtoaje, 2019)

**Solution.** For  $a = 0$ , we need to show that  $4(b + c) \geq 15$  for  $bc = 4$ . Indeed,

$$4(b + c) - 15 \geq 8\sqrt{bc} - 15 = 1.$$

Assume now that  $a \geq b \geq c > 0$ . According to Remark 2 from P 3.58, for fixed  $ab + bc + ca$  and  $abc$ , the sum  $a + b + c$  has the minimum value when  $a \geq b = c$ . Therefore, it suffices to consider this case, when we need to show that

$$4a + 8b + 3a^2b^4 \geq 15$$

for

$$a = \frac{2-b}{b}, \quad 0 < b < 2.$$

Substituting  $a$ , the required inequality becomes

$$3b^5 - 12b^4 + 12b^3 + 8b^2 - 19b + 8 \geq 0,$$

$$(b-1)^2(3b^3 - 6b^2 - 3b + 8) \geq 0.$$

This is true since

$$3b^3 - 6b^2 - 3b + 8 = (2-b)^3 + b(2b-3)^2 > 0.$$

The inequality is an equality for  $a = b = c = 1$ .

□

**P 3.109.** If  $a, b, c \in \left[0, \frac{5}{3}\right]$  such that  $a + b + c = 3$ , then

$$(a + b)(b + c)(c + a) \geq 8\sqrt[3]{abc}.$$

(Vasile Cîrtoaje, 2018)

**Solution.** Assume that  $a \geq b \geq c$  and write the inequality as

$$(a + b + c)(ab + bc + ca) \geq abc + 8\sqrt[3]{abc}.$$

According to Remark 2 from P 3.57, for  $a + b + c = 3$  and fixed  $ab + bc + ca$ , the product  $abc$  has the maximum value when  $a \geq b = c$  or  $a = \frac{5}{3}$ . Therefore, it suffices to consider these cases.

Case 1:  $a \geq b = c$ . For

$$b = c = \frac{3 - a}{2}, \quad a \in \left[1, \frac{5}{3}\right],$$

the inequality is equivalent to

$$\left(\frac{a + 3}{4}\right)^6 (3 - a) \geq 2a.$$

Substituting

$$\frac{a + 3}{4} = x, \quad 1 \leq x \leq \frac{7}{6},$$

we need to show that

$$x^6(3 - 2x) \geq 4x - 3,$$

which is equivalent to

$$\begin{aligned} 2x^7 - 3x^6 + 4x - 3 &\leq 0, \\ (x - 1)^2(2x^5 + x^4 - x^2 - 2x - 3) &\leq 0. \end{aligned}$$

Since

$$2x^5 + x^4 - x^2 - 2x - 3 = x(2x^4 + x^3 + 2x - 8) - 3(x - 1)^2 \leq xf(x),$$

where

$$f(x) = 2x^4 + x^3 + 2x - 8,$$

it is enough to show that  $f(x) \leq 0$ . Indeed, we have

$$f(x) \leq f\left(\frac{7}{6}\right) < 0.$$

Case 2:  $a = \frac{5}{3}$ . Since  $b + c = \frac{4}{3}$ , we can write the inequality as

$$5 + bc \geq 5\sqrt[3]{\frac{5bc}{3}}.$$



From

$$\frac{4}{3} = b + c \geq 2\sqrt{bc},$$

we get  $bc \leq \frac{4}{9}$ . Substituting

$$\sqrt[3]{\frac{5bc}{3}} = x, \quad x \leq \frac{\sqrt[3]{20}}{3},$$

the inequality becomes  $f(x) \geq 0$ , where

$$f(x) = 3x^3 - 30x + 25.$$

Since

$$f'(x) = 9x^2 - 30 \leq \sqrt[3]{400} - 30 < 0,$$

$f$  is decreasing, hence

$$f(x) \geq f\left(\frac{\sqrt[3]{20}}{3}\right) = 5\left(\frac{49}{9} - 2\sqrt[3]{20}\right) > 0.$$

The equality holds for  $a = b = c = 1$ .

**Remark.** In the absence of the condition  $a, b, c \leq \frac{5}{3}$ , the following inequality holds:

$$(a+b)(b+c)(c+a) \geq 8\sqrt{abc}.$$

According to P 3.57, it suffices to consider the case  $b = c = \frac{3-a}{2}$ , when the inequality becomes

$$\begin{aligned} (a+3)^2 &\geq 16\sqrt{a}, \\ a+3 &\geq 4\sqrt[4]{a}. \end{aligned}$$

Indeed, by the AM-GM inequality, we have

$$a+3 = a+1+1+1 \geq 4\sqrt[4]{a \cdot 1 \cdot 1 \cdot 1}.$$

□

**P 3.110.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$a+b+c + \left(1 - \frac{1}{\sqrt{3}}\right)(a-c)^2 \geq 3.$$

(Vasile Cîrtoaje, 2020)

**Proof.** Denote

$$k = 1 - \frac{1}{\sqrt{3}},$$

and write the inequality as

$$a + b + c + k(a - c)^2 \geq 3.$$

**First Solution.** Denote  $x = a + c$ . Since

$$(a - c)^2 = 2(a^2 + c^2) - (a + c)^2 = 2(3 - b^2) - x^2,$$

we may write the inequality as  $f(x) \geq 0$ , where

$$f(x) = x + b + k(6 - 2b^2 - x^2) - 3.$$

For fixed  $b$ , we have  $x \in [m, M]$ , where

$$m = \sqrt{3 - b^2}, \quad M = b + \sqrt{3 - 2b^2}.$$

Indeed, from  $(a + c)^2 \geq a^2 + c^2$ , we get

$$x \geq \sqrt{3 - b^2} = m,$$

and from  $(b^2 - a^2)(b^2 - c^2) \leq 0$ , we get

$$a^2 c^2 \leq b^2(a^2 + c^2) - b^4 = 3b^2 - 2b^4, \quad ac \leq b\sqrt{3 - 2b^2},$$

hence

$$x = \sqrt{3 - b^2 + 2ac} \leq \sqrt{3 - b^2 + 2b\sqrt{3 - 2b^2}} = b + \sqrt{3 - 2b^2} = M.$$

We have  $x = m$  for  $c = 0$ , and  $x = M$  for  $a = b$  or  $b = c$ . Since  $f(x)$  is a quadratic concave function, it suffices to show that  $f(m) \geq 0$  and  $f(M) \geq 0$ , that means to prove the required inequality for  $c = 0$ , for  $a = b$  and for  $b = c$ .

*Case 1:*  $c = 0$ . We need to show that

$$a + b + ka^2 \geq 3$$

for  $a \geq b \geq 0$  and  $a^2 + b^2 = 3$ . Write the inequality as follows:

$$\sqrt{3 - b^2} + b + k(3 - b^2) \geq 3,$$

$$(1 - kb)b \geq \sqrt{3} - \sqrt{3 - b^2},$$

$$(1 - kb)b \geq \frac{b^2}{\sqrt{3} + \sqrt{3 - b^2}},$$

$$(1 - kb) \left( \sqrt{3} + \sqrt{3 - b^2} \right) \geq b.$$

Since the left hand side is decreasing and the right hand side is increasing with respect to  $b$ , it is enough to prove the inequality for  $b = \sqrt{\frac{3}{2}}$ , when it becomes

$$1 - \frac{\sqrt{6} - \sqrt{2}}{2} \geq \sqrt{2} - 1,$$

$$4 \geq \sqrt{6} + \sqrt{2}.$$

*Case 2:*  $a = b \geq c > 0$ . We need to show that

$$(2a + c)\sqrt{x} + k(a - c)^2 - 3x \geq 0$$

for

$$x = \frac{2a^2 + c^2}{3}.$$

Due to homogeneity, we may set  $c = 1$  (which involves  $a \geq 1$ ), when the inequality becomes

$$(2a + 1)\sqrt{x} + k(a - 1)^2 - 3x \geq 0,$$

where

$$x = \frac{2a^2 + 1}{3}.$$

Write the inequality as follows:

$$k(a - 1)^2 \geq \sqrt{x}(3\sqrt{x} - 2a - 1),$$

$$k(a - 1)^2 \geq \frac{2(a - 1)^2\sqrt{x}}{3\sqrt{x} + 2a + 1}.$$

It is true if

$$k \geq \frac{2\sqrt{x}}{3\sqrt{x} + 2a + 1},$$

which is equivalent to

$$k(2a + 1) \geq (2 - 3k)\sqrt{x},$$

$$2a + 1 \geq \sqrt{2a^2 + 1}.$$

*Case 3:*  $a \geq b = c > 0$ . We need to show that

$$(a + 2c)\sqrt{x} + k(a - c)^2 - 3x \geq 0$$

for

$$x = \frac{a^2 + 2c^2}{3}.$$

Due to homogeneity, we may set  $c = 1$  (which involves  $a \geq 1$ ), when the inequality becomes

$$(a + 2)\sqrt{x} + k(a - 1)^2 - 3x \geq 0,$$

where

$$x = \frac{a^2 + 2}{3}.$$

Write the inequality as follows:

$$\begin{aligned} k(a-1)^2 &\geq \sqrt{x}(3\sqrt{x} - a - 2), \\ k(a-1)^2 &\geq \frac{2(a-1)^2\sqrt{x}}{3\sqrt{x} + a + 2}. \end{aligned}$$

It is true if

$$k \geq \frac{2\sqrt{x}}{3\sqrt{x} + a + 2},$$

which is equivalent to

$$\begin{aligned} k(a+2) &\geq (2-3k)\sqrt{x}, \\ a+2 &\geq \sqrt{a^2+2}. \end{aligned}$$

The proof is completed. The equality occurs for  $a = b = c = 1$ , and also for  $a = \sqrt{3}$  and  $b = c = 0$  (or any cyclic permutation).

**Second Solution.** Consider the nontrivial case  $a > c$  and write the inequality in the homogeneous form  $f(b) \geq 0$ , where

$$f(b) = (a+b+c)\sqrt{g(b)} + k(a-c)^2 - 3g(b), \quad g = \frac{a^2 + b^2 + c^2}{3}.$$

Since

$$g'(b) = \frac{2b}{3},$$

we have

$$f'(b) = \sqrt{g} + \frac{(a+b+c)b}{3\sqrt{g}} - 2b = bh(b),$$

where

$$\begin{aligned} h(b) &= \frac{\sqrt{g}}{b} + \frac{a+b+c}{3\sqrt{g}} - 2, \\ h'(b) &= \frac{-\sqrt{g}}{b^2} + \frac{1}{3\sqrt{g}} + \frac{1}{3\sqrt{g}} - \frac{b(a+b+c)}{9g^{3/2}} \\ &\leq \frac{-\sqrt{g}}{b^2} + \frac{2}{3\sqrt{g}} = \frac{-3g + 2b^2}{2b^2\sqrt{g}} \leq 0. \end{aligned}$$

Since  $h(b)$  is a decreasing function, there are three possible cases:  $g(b) \geq 0$  for  $b \in [c, a]$ , hence  $f(b)$  is increasing on  $[c, a]$ ;  $g(b) \geq 0$  for  $b \in [c, c_1]$  and  $g(b) \leq 0$  for  $b \in [c_1, a]$ , hence  $f(b)$  is increasing on  $[c, c_1]$  and decreasing on  $[c_1, a]$ ;  $g(b) \leq 0$  for  $b \in [c, a]$ , hence  $f(b)$  is decreasing on  $[c, a]$ . In all these cases  $f(b)$  has the minimum value when  $b \in \{c, a\}$ . As a consequence, we only need to prove the required inequality for  $b = a$  and  $b = c$ . Both cases were presented in the first solution.

**Remark.** The inequality is symmetric because it can be written in the form

$$a + b + c + \left(1 - \frac{1}{\sqrt{3}}\right) \max\{(a-b)^2, (b-c)^2, (c-a)^2\} \geq 3,$$

without the condition  $a \geq b \geq c$ .

□

**P 3.111.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and  $a^2 + b^2 + c^2 = 3$ , then

$$1 - abc \leq \sqrt{\frac{2}{3}} (a - c).$$

(Vasile Cîrtoaje, 2018)

**First Solution.** Denoting  $x = ac$ , we need to show that  $f(x) \geq 0$ , where

$$f(x) = bx + \sqrt{\frac{2(3 - b^2 - 2x)}{3}} - 1.$$

For fixed  $b$ ,  $b \in \left[0, \sqrt{\frac{3}{2}}\right]$ , we have  $x \in [0, M]$ , where

$$M = b\sqrt{3 - 2b^2}.$$

Indeed, from  $(b^2 - a^2)(b^2 - c^2) \leq 0$ , we get

$$a^2 c^2 \leq b^2(a^2 + c^2) - b^4 = 3b^2 - 2b^4, \quad x \leq b\sqrt{3 - 2b^2} = M.$$

From

$$f''(x) = -\sqrt{\frac{2}{3}} \cdot \frac{1}{(3 - b^2 - 2x)^{3/2}} \leq 0,$$

it follows that  $f$  is a concave function, therefore it suffices to show that  $f(0) \geq 0$  and  $f(M) \geq 0$ . We have

$$f(0) = \sqrt{\frac{2(3 - b^2)}{3}} - 1 \geq 0$$

because

$$3 = a^2 + b^2 + c^2 \geq 2b^2.$$

Since

$$\sqrt{3 - b^2 - 2M} = \left| \sqrt{3 - 2b^2} - b \right|,$$

we may write the inequality  $f(M) \geq 0$  as follows:

$$\sqrt{\frac{2}{3}} \left| \sqrt{3 - 2b^2} - b \right| \geq 1 - b^2 \sqrt{3 - 2b^2},$$

$$\frac{\sqrt{6}|1-b^2|}{\sqrt{3-2b^2}+b} \geq \frac{(1-b^2)^2(1+2b^2)}{1+b^2\sqrt{3-2b^2}}.$$

This is true if

$$\frac{\sqrt{6}}{\sqrt{3-2b^2}+b} \geq \frac{|1-b^2|(1+2b^2)}{1+b^2\sqrt{3-2b^2}}.$$

*Case 1:*  $b \in [0, 1]$ . We claim that

$$\frac{\sqrt{6}}{\sqrt{3-2b^2}+b} > 1 \geq \frac{(1-b^2)(1+2b^2)}{1+b^2\sqrt{3-2b^2}}.$$

Indeed, the left inequality is equivalent to

$$\sqrt{6}-b > \sqrt{3-2b^2},$$

which, by squaring, becomes

$$3b^2 - 2\sqrt{6}b + 3 \geq 0,$$

$$3(b-1)^2 + 2(3-\sqrt{6})b \geq 0,$$

while the right inequality is equivalent to

$$2b^4 + b^2(\sqrt{3-2b^2}-1) \geq 0.$$

*Case 2:*  $b \in \left[1, \sqrt{\frac{3}{2}}\right]$ . We need to show that

$$\frac{\sqrt{6}}{\sqrt{3-2b^2}+b} \geq \frac{(b^2-1)(1+2b^2)}{1+b^2\sqrt{3-2b^2}},$$

which can be written as

$$A(b)\sqrt{3-2b^2} \geq B(b),$$

where

$$A(b) = \sqrt{6}b^2 - (b^2-1)(1+2b^2), \quad B(b) = b(b^2-1)(1+2b^2) - \sqrt{6}.$$

This inequality is true because

$$A(b) = 1 + (\sqrt{6}+1)b^2 - 2b^4 > 3b^2 - 2b^4 = b^2(3-2b^2) \geq 0$$

and

$$B(b) \leq B\left(\sqrt{\frac{2}{3}}\right) = 0.$$

The equality occurs for  $a = b = c = 1$ , and also for  $a = b = \sqrt{\frac{3}{2}}$  and  $c = 0$  (or any cyclic permutation).

**Second Solution** (by Anhduy98). Denoting

$$p = a + b + c, \quad q = ab + bc + ca,$$

we have

$$q = \frac{p^2 - 3}{2}, \quad p \leq 3.$$

Since

$$(a - c)^2 - (a^2 + b^2 + c^2 - ab - bc - ca) = (a - b)(b - c) \geq 0,$$

it is enough to show that

$$abc + \sqrt{\frac{2(a^2 + b^2 + c^2 - ab - bc - ca)}{3}} \geq 1,$$

that is

$$abc \geq 1 - \sqrt{\frac{2(p^2 - 3q)}{3}}$$

or

$$abc \geq 1 - x,$$

where

$$x = \sqrt{\frac{2(p^2 - 3q)}{3}}.$$

From

$$p^2 - 3q = \frac{3x^2}{2}, \quad p^2 - 2q = 3,$$

we get

$$p = \sqrt{3(3 - x^2)}, \quad q = \frac{3(2 - x^2)}{2}, \quad x \in [0, \sqrt{2}].$$

For  $x \geq 1$ , the inequality  $abc \geq 1 - x$  is true because  $abc \geq 0 \geq 1 - x$ . Consider now that  $1 \leq x \leq \sqrt{2}$ . By Schur's inequality

$$p^3 + 9abc \geq 4pq,$$

we get

$$3abc \geq (1 - x^2)\sqrt{3(3 - x^2)}.$$

Thus, the inequality  $abc \geq 1 - x$  holds if

$$(1 - x^2)\sqrt{3(3 - x^2)} \geq 3(1 - x),$$

which is true if

$$(1 + x)^2(3 - x^2) \geq 3.$$

Indeed,

$$3 - (1 + x)^2(3 - x^2) = x[x(x^2 - 2) + 2(x^2 - 3)] \leq 0.$$

**Third Solution.** Denoting

$$F(a, b, c) = abc + \sqrt{\frac{2}{3}} (a - c)$$

and

$$x = \sqrt{\frac{a^2 + b^2}{2}},$$

we can show that

$$F(a, b, c) \geq F(x, x, c) \geq 1.$$

**Remark.** The inequality is symmetric because it can be written in the form

$$1 - abc \leq \sqrt{\frac{2}{3}} \max\{|a - b|, |b - c|, |c - a|\},$$

without the condition  $a \geq b \geq c$ .

□

**P 3.112.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \leq \frac{7}{10} (a - c)^2.$$

(Vasile Cîrtoaje, 2018)

**First Solution.** Write the inequality as follows:

$$10abc + 7(a - c)^2 \geq 10,$$

$$10abc + 7(3 - b^2 - 2ac) \geq 10,$$

$$11 - 7b^2 \geq 2(7 - 5b)ac.$$

From

$$3 = a^2 + b^2 + c^2 \geq 2b^2,$$

we get  $b \leq \sqrt{\frac{3}{2}}$ , hence

$$11 - 7b^2 > 0, \quad 7 - 2b > 0.$$

On the other hand, from  $(b^2 - a^2)(b^2 - c^2) \leq 0$ , we get

$$a^2 c^2 \leq b^2(a^2 + c^2) - b^4 = 3b^2 - 2b^4, \quad ac \leq b\sqrt{3 - 2b^2}.$$



Thus, it is enough to show that

$$11 - 7b^2 \geq 2(7 - 5b)b\sqrt{3 - 2b^2}.$$

By squaring, this inequality becomes

$$200b^6 - 560b^5 + 141b^4 + 840b^3 - 742b^2 + 121 \geq 0,$$

or

$$(b - 1)^2 f(b) \geq 0,$$

where

$$\begin{aligned} f(b) &= 200b^4 - 160b^3 - 379b^2 + 242b + 121 = 8b(5b - 6)^2 \left(b + \frac{8}{5}\right) \\ &\quad + 101b^2 - \frac{1094}{5}b + 121 > 100b^2 + 120 - 219b \geq (40\sqrt{30} - 219)b \geq 0. \end{aligned}$$

The proof is completed. The equality occurs for  $a = b = c = 1$ .

**Second Solution** (by Nguyen Van Huyen). Since

$$(a - c)^2 - (a^2 + b^2 + c^2 - ab - bc - ca) = (a - b)(b - c) \geq 0,$$

it is enough to show that

$$abc + \frac{7}{10}(a^2 + b^2 + c^2 - ab - bc - ca) \geq 1,$$

which is equivalent to

$$10abc + 11 \geq 7(ab + bc + ca).$$

According to P 3.57, for  $a^2 + b^2 + c^2 = 3$  and fixed  $ab + bc + ca$  (which involve fixed  $a + b + c$  and  $ab + bc + ca$ ), the product  $abc$  has the minimum value when  $c = 0$  or  $a = b \geq c$ . Thus, we only need to consider these cases.

*Case 1:*  $c = 0$ . We need to show that  $a^2 + b^2 = 3$  involves  $11 \geq 7ab$ . Indeed, we have

$$2(11 - 7ab) \geq 22 - 7(a^2 + b^2) = 1 > 0.$$

*Case 2:*  $a = b \geq c$ . We need to show that  $2a^2 + c^2 = 3$  involves

$$10a^2c + 11 \geq 7(a^2 + 2ac),$$

which is equivalent to

$$\begin{aligned} (10c - 7)a^2 + 11 &\geq 14ac, \\ 1 + 30c + 7c^2 - 10c^3 &\geq 14c\sqrt{6 - 2c^2}. \end{aligned}$$

Since  $c \leq 1$ , the left side is positive. By squaring, the inequality becomes

$$100c^6 - 140c^5 - 159c^4 + 400c^3 - 262c^2 + 60c + 1 \geq 0,$$

$$(c-1)^2(100c^4 + 60c^3 - 139c^2 + 62c + 1) \geq 0.$$

This is true because

$$\begin{aligned} 100c^4 + 60c^3 - 139c^2 + 62c + 1 &\geq 100c^4 + 60c^3 - 140c^2 + 60c + 1 \\ &= (10c^2 - 1)^2 + 60c(c-1)^2 > 0. \end{aligned}$$

□

**P 3.113.** If  $a \geq b \geq c \geq \frac{1}{3}$  and  $a^2 + b^2 + c^2 = 3$ , then

$$1 - abc \leq \frac{11}{18} (a - c).$$

(Vasile Cîrtoaje, 2018)

**Solution.** Denoting  $x = ac$ , we need to show that  $f(x) \geq 0$ , where

$$f(x) = bx - 1 + \frac{11}{18} \sqrt{3 - b^2 - 2x}.$$

From

$$(9a^2 - 1)(9c^2 - 1) \geq 0$$

and

$$(b^2 - a^2)(b^2 - c^2) \leq 0,$$

we get  $x \in [m, M]$ , where

$$m = \frac{1}{9} \sqrt{26 - 9b^2}, \quad M = b\sqrt{3 - 2b^2}.$$

We have  $x = m$  for  $c = \frac{1}{3}$ , and  $x = M$  for  $a = b$  or  $b = c$ . Since

$$f''(x) = \frac{-11}{18(3 - b^2 - 2x)^{3/2}} < 0,$$

$f$  is concave. Therefore, it suffices to show that  $f(m) \geq 0$  and  $f(M) \geq 0$ , that means to prove the required inequality for  $c = \frac{1}{3}$ , for  $a = b$  and for  $b = c$ .

*Case 1:*  $c = \frac{1}{3}$ . We need to prove that

$$18ab + 33a \geq 65$$

for  $a^2 + b^2 = \frac{26}{9}$  and  $a \geq b \geq \frac{1}{3}$ . From

$$\frac{26}{9} = a^2 + b^2 \leq 2a^2$$

and

$$\frac{26}{9} = a^2 + b^2 \geq a^2 + \frac{1}{9},$$

we get

$$\frac{6}{5} < \frac{\sqrt{13}}{3} \leq a \leq \frac{5}{3}.$$

Write the required inequality as follows:

$$6a\sqrt{26 - 9a^2} \geq 65 - 33a,$$

$$36a^2(26 - 9a^2) \geq (65 - 33a)^2,$$

$$324a^4 + 153a^2 - 4290a + 4225 \leq 0,$$

$$(3a - 5)(108a^3 + 180a^2 + 351a - 845) \leq 0,$$

$$(3a - 5)[(5a - 6)(20a^2 + 60a + 141) + 8a^3 + 6a + 1] \leq 0.$$

It is true since  $3a - 5 \leq 0$  and  $5a - 6 > 0$ .

*Case 2:*  $a = b$ . Consider the nontrivial case  $a = b > c$ , that is  $c < 1$ . We need to prove that

$$11(a - c) \geq 18(1 - a^2c)$$

for  $2a^2 + c^2 = 3$ ,  $\frac{1}{3} \leq c < 1$  and  $3 = 2a^2 + c^2 \geq 2a^2 + \frac{1}{9}$ , hence

$$a \leq \frac{\sqrt{13}}{3}.$$

Write the required inequality as follows:

$$\frac{11(a^2 - c^2)}{a + c} \geq 9(2 - 3c + c^3),$$

$$\frac{33(1 - c)(1 + c)}{a + c} \geq 18(1 - c)^2(2 + c),$$

$$\frac{11(1 + c)}{a + c} \geq 6(1 - c)(2 + c),$$

$$\frac{11(1 + c)}{6(1 - c)(2 + c)} \geq a + c.$$

We will show that

$$\frac{11(1 + c)}{6(1 - c)(2 + c)} - c \geq \frac{26}{21} > \frac{\sqrt{13}}{3} \geq a.$$

The left inequality is equivalent to

$$42c^3 + 94c^2 + 45c - 27 \geq 0,$$

$$(3c - 1)(14c^2 + 36c + 27) \geq 0.$$

*Case 3:*  $b = c$ . Consider the nontrivial case  $a > b = c$ , that is  $a > 1$ . We need to prove that

$$11(a - c) \geq 18(1 - ac^2)$$

for  $a^2 + 2c^2 = 3$ , and  $3 = a^2 + 2c^2 \geq a^2 + \frac{1}{9}$ , hence

$$1 < a \leq \frac{5}{3}.$$

Write the required inequality as follows:

$$\frac{11(a^2 - c^2)}{a + c} \geq 9(a^3 - 3a + 2),$$

$$\frac{33(a - 1)(a + 1)}{a + c} \geq 18(a - 1)^2(a + 2),$$

$$\frac{11(a + 1)}{a + c} \geq 6(a - 1)(a + 2),$$

$$\frac{11(a + 1)}{6(a - 1)(a + 2)} \geq a + c.$$

We will show that

$$\frac{11(a + 1)}{6(a - 1)(a + 2)} - a \geq \frac{16}{3} - 3a \geq c.$$

The left inequality is equivalent to

$$12a^3 - 20a^2 - 45a + 75 \leq 0,$$

$$(5 - 3a)(15 - 4a^2) \geq 0.$$

The right inequality is equivalent to

$$16 - 9a \geq 3\sqrt{\frac{3 - a^2}{2}},$$

$$2(16 - 9a)^2 \geq 9(3 - a^2),$$

$$171a^2 - 576a + 485 \geq 0,$$

$$(5 - 3a)(97 - 57a) \geq 0.$$

The inequality is an equality for  $a = b = c = 1$ , and also for  $a = \frac{5}{3}$ ,  $b = c = \frac{1}{3}$  (or any cyclic permutation).

□

**P 3.114.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - \sqrt{abc} \leq \frac{2}{3} (a - c)^2.$$

(Vasile Cîrtoaje, 2019)

**First Solution.** Denoting  $x = ac$ , we need to show that  $f(x) \geq 0$ , where

$$f(x) = 3\sqrt{bx} + 2(3 - b^2 - 2x) - 3 = 3\sqrt{bx} - 4x + 3 - 2b^2.$$

For fixed  $b$ ,  $b \in \left[0, \sqrt{\frac{3}{2}}\right]$ , we have  $x \in [0, M]$ , where

$$M = b\sqrt{3 - 2b^2}.$$

Indeed, from  $(b^2 - a^2)(b^2 - c^2) \leq 0$ , we get

$$a^2c^2 \leq b^2(a^2 + c^2) - b^4 = 3b^2 - 2b^4, \quad x \leq b\sqrt{3 - 2b^2} = M.$$

From

$$f''(x) = -\frac{3}{4}\sqrt{bx}^{-3/2} < 0,$$

it follows that  $f$  is a concave function, therefore it suffices to show that  $f(0) \geq 0$  and  $f(M) \geq 0$ . We have

$$f(0) = 3 - 2b^2 \geq 0$$

because

$$3 = a^2 + b^2 + c^2 \geq 2b^2.$$

Write the inequality  $f(M) \geq 0$  as follows:

$$3\sqrt{bM} - 4M + 3 - 2b^2 \geq 0.$$

Using the substitution

$$x = \sqrt[4]{3 - 2b^2}, \quad 0 \leq x \leq \sqrt[4]{3},$$

which yields  $M = bx^2$ , the inequality becomes

$$3bx - 4bx^2 + x^4 \geq 0,$$

$$x^3 \geq b(4x - 3).$$

Consider the nontrivial case  $x \geq 4/3$ . By squaring, the inequality becomes

$$2x^6 \geq (3 - x^4)(4x - 3)^2,$$

$$6x^6 - 8x^5 + 3x^4 - 16x^2 + 24x - 9 \geq 0,$$

$$(x-1)^2(6x^4 + 4x^3 + 5x^2 + 6x - 9) \geq 0.$$

It is true because

$$4x^3 + 5x^2 + 6x - 9 = (4x-3)(x^2 + 2x + 3) \geq 0.$$

The equality occurs for  $a = b = c = 1$ , and also for  $a = b = \sqrt{\frac{3}{2}}$  and  $c = 0$  (or any cyclic permutation).

**Second Solution.** Denoting

$$p = a + b + c, \quad q = ab + bc + ca,$$

we have

$$q = \frac{p^2 - 3}{2},$$

and from  $3(a^2 + b^2 + c^2) \geq p^2$ , we get  $p \leq 3$ . Since

$$(a-c)^2 - (a^2 + b^2 + c^2 - ab - bc - ca) = (a-b)(b-c) \geq 0,$$

it is enough to show that

$$1 - \sqrt{abc} \leq \frac{2(a^2 + b^2 + c^2 - ab - bc - ca)}{3}.$$

that is

$$1 - \sqrt{abc} \leq \frac{6-2q}{3}, \quad \sqrt{abc} \geq \frac{p^2-6}{3}.$$

For  $p \leq \sqrt{6}$ , the inequality is true, and for  $\sqrt{6} \leq p \leq 3$ , it is equivalent to

$$9abc \geq (p^2 - 6)^2.$$

By Schur's inequality of third degree, we have

$$9abc \geq 4pq - p^3 = p^3 - 6p.$$

Therefore,

$$9abc - (p^2 - 6)^2 \geq p^3 - 6p - (p^2 - 6)^2 = (p^2 - 6)(2 + p)(3 - p) \geq 0.$$

□

**P 3.115.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \leq \frac{2}{3} a(a - c)^2.$$

(Vasile Cîrtoaje, 2020)

**Solution.** Let us denote

$$g = \sqrt{\frac{a^2 + b^2 + c^2}{3}},$$

and write the inequality in the homogeneous form  $f \leq 0$ , where

$$f = 3g^3 - 3abc - 2a(a - c)^2.$$

For fixed  $a$  and  $b$ , since

$$g'(c) = \frac{c}{3g},$$

we have

$$f'(c) = 3cg - 3ab + 4a(a - c) = c\sqrt{3(a^2 + b^2 + c^2)} - 3ab + 4a(a - c).$$

We will show that

$$c\sqrt{3(a^2 + b^2 + c^2)} - 3ab \geq c\sqrt{3(2a^2 + c^2)} - 3a^2,$$

which is equivalent to

$$3a(a - b) \geq c\sqrt{3(2a^2 + c^2)} - c\sqrt{3(a^2 + b^2 + c^2)},$$

$$3a(a - b) \geq \frac{3c(a - b)(a + b)}{\sqrt{3(2a^2 + c^2)} + \sqrt{3(a^2 + b^2 + c^2)}}.$$

This is true if

$$1 \geq \frac{a + b}{2\sqrt{3(a^2 + b^2 + c^2)}},$$

which is obvious. So, we have

$$\begin{aligned} f'(c) &\geq c\sqrt{3(2a^2 + c^2)} - 3a^2 + 4a(a - c) = c\sqrt{3(2a^2 + c^2)} + a^2 - 4ac \\ &\geq c(2a + c) + a^2 - 4ac = (a - c)^2 \geq 0. \end{aligned}$$

Since  $f(c)$  is increasing, we have  $f(c) \leq f(b)$ , and it remains to prove that  $f(b) \leq 0$ , that is

$$3 \left( \frac{a^2 + 2b^2}{3} \right)^{3/2} - 3ab^2 - 2a(a - b)^2 \leq 0.$$

For  $b = 0$ , the inequality is trivial. For  $b > 0$ , due to homogeneity, we may consider  $b = 1$ , when  $a \geq 1$ . The inequality becomes

$$3 \left( \frac{a^2 + 2}{3} \right)^{3/2} - 3a - 2a(a - 1)^2 \leq 0,$$

$$2a^3 - 4a^2 + 5a \geq 3 \left( \frac{a^2 + 2}{3} \right)^{3/2},$$

and, by squaring,

$$11a^6 - 48a^5 + 102a^4 - 120a^3 + 63a^2 - 8 \geq 0,$$

$$(a - 1)^3(11a^3 - 15a^2 + 24a + 8) \geq 0.$$

The last inequality is clearly true for  $a \geq 1$ .

The equality occurs for  $a = b = c = 1$ .

**Remark.** Denoting

$$x = \max\{a, b, c\}, \quad y = \min\{a, b, c\},$$

we may remove the condition  $a \geq b \geq c$  to write the inequality in the symmetric form

$$1 - abc \geq \frac{2}{3} x(x - y)^2.$$

□

**P 3.116.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \leq \frac{1}{9}(5a + c)(a - c)^2.$$

(Vasile Cîrtoaje, 2020)

**Solution.** Let us denote

$$g = \sqrt{\frac{a^2 + b^2 + c^2}{3}},$$

and write the inequality in the homogeneous form  $f \leq 0$ , where

$$f = 3g^3 - 3abc - \frac{1}{3}(5a + c)(a - c)^2.$$

For fixed  $a$  and  $b$ , since

$$g'(c) = \frac{c}{3g},$$



we have

$$f'(c) = 3cg - 3ab + (a - c)(3a + c) = c\sqrt{3(a^2 + b^2 + c^2)} - 3ab + (a - c)(3a + c).$$

We will show that

$$c\sqrt{3(a^2 + b^2 + c^2)} - 3ab \geq c\sqrt{3(2a^2 + c^2)} - 3a^2,$$

which is equivalent to

$$\begin{aligned} 3a(a - b) &\geq c\sqrt{3(2a^2 + c^2)} - c\sqrt{3(a^2 + b^2 + c^2)}, \\ 3a(a - b) &\geq \frac{3c(a - b)(a + b)}{\sqrt{3(2a^2 + c^2)} + \sqrt{3(a^2 + b^2 + c^2)}}. \end{aligned}$$

This is true if

$$1 \geq \frac{a + b}{2\sqrt{3(a^2 + b^2 + c^2)}},$$

which is obvious. So, we have

$$\begin{aligned} f'(c) &\geq c\sqrt{3(2a^2 + c^2)} - 3a^2 + (a - c)(3a + c) = c\sqrt{3(2a^2 + c^2)} - 2ac - c^2 \\ &= c \left[ \sqrt{3(2a^2 + c^2)} - 2a - c \right] = \frac{2c(a - c)^2}{\sqrt{3(2a^2 + c^2)} + 2a + c} \geq 0. \end{aligned}$$

Since  $f(c)$  is increasing, we have  $f(c) \leq f(b)$ , and it remains to prove that  $f(b) \leq 0$ , that is

$$9 \left( \frac{a^2 + 2b^2}{3} \right)^{3/2} - 9ab^2 - (5a + b)(a - b)^2 \leq 0.$$

For  $b = 0$ , the inequality is trivial. For  $b > 0$ , due to homogeneity, we may consider  $b = 1$ , when  $a \geq 1$ . The inequality becomes

$$\begin{aligned} 9 \left( \frac{a^2 + 2}{3} \right)^{3/2} - 9a - (5a + 1)(a - 1)^2 &\leq 0, \\ 5a^3 - 9a^2 + 12a + 1 &\geq 9 \left( \frac{a^2 + 2}{3} \right)^{3/2}, \end{aligned}$$

and, by squaring,

$$\begin{aligned} (5a^3 - 9a^2 + 12a + 1)^2 &\geq 3(a^2 + 2)^3, \\ 22a^6 - 90a^5 + 183a^4 - 206a^3 + 90a^2 + 24a - 23 &\geq 0, \\ (a - 1)^3(22a^3 - 24a^2 + 45a + 23) &\geq 0. \end{aligned}$$

The last inequality is clearly true for  $a \geq 1$ .

The equality occurs for  $a = b = c = 1$ .

□

**P 3.117.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \geq \frac{2}{3}(b - c)^2.$$

(Vasile Cîrtoaje, 2019)

**First Solution.** Write the inequality as follows:

$$3 - 3abc \geq 2(3 - a^2 - 2bc),$$

$$(4 - 3a)bc \geq 3 - 2a^2.$$

We have

$$3a^2 \geq a^2 + b^2 + c^2 = 3, \quad a \geq 1.$$

Case 1:  $1 \leq a \leq \sqrt{\frac{3}{2}}$ . From  $(a^2 - b^2)(a^2 - c^2) \geq 0$ , we get

$$b^2 c^2 \geq a^2(b^2 + c^2) - a^4 = a^2(3 - a^2) - a^4, \quad bc \geq a\sqrt{3 - 2a^2}.$$

Thus, it is enough to show that

$$a(4 - 3a)\sqrt{3 - 2a^2} \geq 3 - 2a^2.$$

This is true if

$$a(4 - 3a) \geq \sqrt{3 - 2a^2},$$

which, by squaring, becomes

$$3a^4 - 8a^3 + 6a^2 - 1 \geq 0,$$

$$(a - 1)^3(3a + 1) \geq 0.$$

Case 2:  $\sqrt{\frac{3}{2}} \leq a \leq \frac{4}{3}$ . We have

$$3 - 2a^2 \leq 0 \leq (4 - 3a)bc.$$

Case 3:  $\frac{4}{3} \leq a \leq \sqrt{3}$ . Since

$$2bc \leq b^2 + c^2 = 3 - a^2,$$

it is enough to show that

$$(3a - 4)(3 - a^2) \leq 2(2a^2 - 3).$$

which is equivalent to

$$a^3 - 3a + 2 \geq 0,$$

$$(a - 1)^2(a + 2) \geq 0.$$

The equality occurs for  $a = b = c = 1$ , and also for  $a = b = \sqrt{\frac{3}{2}}$  and  $c = 0$ .

**Second Solution** (by Ali3985). From

$$3 = a^2 + b^2 + c^2 \geq 2ab + c^2 \geq 2b^2 + c^2,$$

we get

$$ab \leq \frac{1}{2}(3 - c^2), \quad b \leq \frac{3 - c^2}{2}.$$

Thus,

$$abc + \frac{2}{3}(b - c)^2 \leq \frac{1}{2}(3 - c^2)c + \frac{2}{3}\left(\sqrt{\frac{3 - c^2}{2}} - c\right)^2,$$

and it remains to show that

$$\frac{1}{2}(3 - c^2)c + \frac{2}{3}\left(\sqrt{\frac{3 - c^2}{2}} - c\right)^2 \leq 1,$$

which is equivalent to

$$8c\sqrt{\frac{3 - c^2}{2}} \geq -3c^3 + 2c^2 + 9c.$$

By squaring, the inequality becomes

$$c^2(1 - c)^3(3c + 5) \geq 0.$$

This is true because

$$3c^2 \leq a^2 + b^2 + c^2 = 3, \quad c \leq 1.$$

□

**P 3.118.** Let  $a, b, c$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 = 3.$$

If  $a \geq b \geq c$ , then

$$1 - abc \geq \frac{2}{3}(1 + \sqrt{2})(a - b)(b - c).$$

(Vasile Cîrtoaje, 2019)

**Solution.** Write the inequality in the homogeneous form  $F \geq 0$ , where

$$F = g^{3/2} - abc - k(a - b)(b - c)g^{1/2}, \quad k = \frac{2}{3}(1 + \sqrt{2}), \quad g = \frac{a^2 + b^2 + c^2}{3}.$$

Denote

$$x = a - c, \quad y = b - c, \quad x \geq y \geq 0,$$

and consider that  $x$  and  $y$  are fixed. We need to show that  $F(c) \geq 0$ . Since  $a' = b' = c' = 1$  and

$$g' = \frac{2}{3}(a + b + c),$$

we have

$$F'(c) = (a + b + c)g^{1/2} - (bc + ca + ab) - \frac{k}{3}(a - b)(b - c)(a + b + c)g^{-1/2} \geq E(c),$$

where

$$E(c) = (a + b + c)g^{1/2} - (ab + bc + ca) - k(a - b)(b - c).$$

By the AM-GM inequality, we have

$$E'(c) = 3g^{1/2} + \frac{1}{3}(a + b + c)^2g^{-1/2} - 2(a + b + c) \geq 2(a + b + c) - 2(a + b + c) = 0.$$

As a consequence,  $E(c)$  is increasing, hence  $E(c) \geq E(0)$ , where

$$E(0) = (a + b)\sqrt{\frac{a^2 + b^2}{3}} - ab - k(a - b)b.$$

We will show that  $E(0) \geq 0$ . Since  $a^2 + b^2 \geq \frac{1}{2}(a + b)^2$ , it is enough to show that

$$\frac{1}{\sqrt{6}}(a + b)^2 \geq ab + k(a - b)b.$$

Since  $\sqrt{6} < 32/13$  and  $k < 13/8$ , it suffices to prove that

$$\frac{13}{32}(a + b)^2 \geq ab + \frac{13}{8}(a - b)b,$$

which is equivalent to

$$13(a^2 + 5b^2) \geq 58ab.$$

Indeed,

$$13(a^2 + 5b^2) \geq 26\sqrt{5} ab \geq 58ab.$$

From  $E(c) \geq E(0) \geq 0$ , it follows that  $F'(c) \geq 0$ ,  $F(c)$  is increasing, therefore  $F(c) \geq F(0)$ . So, we only need to show that  $F(0) \geq 0$ , that is

$$\begin{aligned} \frac{a^2 + b^2}{3} - k(a - b)b &\geq 0, \\ a^2 + (3k + 1)b^2 &\geq 3kab. \end{aligned}$$

Indeed, we have

$$a^2 + (3k + 1)b^2 \geq 2\sqrt{3k + 1} ab \geq 3kab.$$

The proof is completed. The equality occurs for  $a = b = c = 1$ , and also for  $a = \sqrt{3k + 1} b$  and  $c = 0$ , that is

$$a = \frac{1}{2}\sqrt{6 + 3\sqrt{2}}, \quad b = \frac{1}{2}\sqrt{6 - 3\sqrt{2}}, \quad c = 0.$$

□

**P 3.119.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$\begin{aligned} (a) \quad & 1 - abc \geq 2b(a - b)(b - c); \\ (b) \quad & 1 - abc \geq (a - c)(a - b)(b - c); \\ (c) \quad & 1 - abc \geq a(a - b)(b - c); \\ (d) \quad & 1 - abc \geq (a + c)(a - b)(b - c). \end{aligned}$$

(Vasile Cîrtoaje, 2020)

**Solution.** (a) Since

$$4(a - b)(b - c) \leq (a - b + b - c)^2 = (a - c)^2,$$

it is enough to show that

$$1 - abc \geq \frac{1}{2}b(a - c)^2,$$

which is equivalent to

$$\begin{aligned} 2 &\geq b(a^2 + c^2), \\ 2 &\geq b(3 - b^2), \\ (b - 1)^2(b + 2) &\geq 0. \end{aligned}$$

The equality occurs for  $a = b = c = 1$ .

(b) Write the inequality in the homogeneous form

$$\left(\frac{a^2 + b^2 + c^2}{3}\right)^{3/2} - abc - (a - b)(b - c)(a - c) \geq 0.$$

Use the substitution

$$a = x + c, \quad b = y + c, \quad x \geq y \geq 0,$$

and, for fixed  $x$  and  $y$ , write the inequality as  $f(c) \geq 0$ , where

$$f(c) = \left(\frac{a^2 + b^2 + c^2}{3}\right)^{3/2} - abc - (a - b)(b - c)(a - c).$$

Since

$$a' = b' = c' = 1,$$

we have

$$f'(c) = (a + b + c)\frac{a^2 + b^2 + c^2}{3} - ab - bc - ca$$

$$\geq \frac{1}{3}(a+b+c)^2 - ab - bc - ca \geq 0.$$

Since  $f(c)$  is increasing, we have  $f(c) \geq f(0)$ . Therefore, it is enough to show that  $f(0) \geq 0$ , that is

$$\left(\frac{a^2+b^2}{3}\right)^{3/2} \geq ab(a-b).$$

By squaring, we need to show that

$$(a^2+b^2)^3 \geq 27(a^2+b^2-2ab)a^2b^2,$$

$$(a^2+b^2-3ab)^2(a^2+b^2+6ab) \geq 0.$$

The equality occurs for  $a=b=c=1$ , and also for  $a^2+b^2-3ab=0$  and  $c=0$ , that is

$$a = \frac{\sqrt{5}+1}{2}, \quad b = \frac{\sqrt{5}-1}{2}, \quad c = 0.$$

(c) According to (a), the inequality is true if  $a \leq 2b$ . Consider further the case  $a \geq 2b$ , denote

$$g = \sqrt{\frac{a^2+b^2+c^2}{3}}, \quad g \leq a,$$

and write the inequality in the homogeneous form  $f \geq 0$ , where

$$f = g^3 - abc - a(a-b)(b-c).$$

For fixed  $a$  and  $b$  ( $a \geq 2b$ ), since

$$g'(c) = \frac{c}{3g},$$

we have

$$f'(c) = cg - ab + a(a-b) = cg + a(a-2b) \geq 0.$$

Thus,  $f(c)$  is increasing, hence

$$f(c) \geq f(0) = \left(\frac{a^2+b^2}{3}\right)^{3/2} - ab(a-b).$$

We only need to show that

$$(a^2+b^2)^3 \geq 27a^2b^2(a-b)^2,$$

which is equivalent to

$$(a^2+b^2)^3 \geq 27a^2b^2(a^2+b^2) - 54a^3b^3 \geq 0,$$

$$(a^2+b^2-3ab)^2(a^2+b^2+6ab) \geq 0.$$

The equality occurs for  $a=b=c=1$ , and also for

$$a = \frac{\sqrt{5}+1}{2}, \quad b = \frac{\sqrt{5}-1}{2}, \quad c = 0.$$

(d) Write the required inequality as follows:

$$\begin{aligned}
 1 &\geq abc + (a+c)(-b^2 + ab + bc - ca), \\
 \left(\frac{a^2 + b^2 + c^2}{3}\right)^{3/2} &\geq (a+c-b)(ab+bc-ca), \\
 (a^2 + b^2 + c^2)^3 &\geq 27(a-b+c)^2(ab+bc-ca)^2, \\
 (a^2 + b^2 + c^2)^3 &\geq 27[a^2 + b^2 + c^2 - 2(ab+bc-ca)](ab+bc-ca)^2, \\
 [a^2 + b^2 + c^2 - 3(ab+bc-ca)]^2 [a^2 + b^2 + c^2 + 6(ab+bc-ca)] &\geq 0.
 \end{aligned}$$

The equality occurs for  $a = b = c = 1$ , and also for  $a^2 + b^2 + c^2 = 3$  and  $(a+c)b = 1+ac$ .

**Remark** Denoting

$$x = \max\{a, b, c\}, \quad y = \min\{a, b, c\}, \quad s = a + b + c,$$

we may remove the condition  $a \geq b \geq c$  to write the inequalities in the symmetric forms:

$$\begin{aligned}
 (a) \quad 1 - abc &\geq 2(s-x-y)(2x+y-s)(s-x-2y); \\
 (b) \quad 1 - abc &\geq (x-y)(2x+y-s)(s-x-2y); \\
 (c) \quad 1 - abc &\geq x(2x+y-s)(s-x-2y); \\
 (d) \quad 1 - abc &\geq (x+y)(2x+y-s)(s-x-2y).
 \end{aligned}$$

□

**P 3.120.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$\begin{aligned}
 (a) \quad 1 - abc &\geq \frac{2}{3} b(a-b)^2; \\
 (b) \quad 1 - abc &\geq \frac{2}{27} (2a+7b)(a-b)^2.
 \end{aligned}$$

(Vasile Cîrtoaje, 2020)

**Solution.** Let us denote

$$g = \sqrt{\frac{a^2 + b^2 + c^2}{3}}$$

and

$$f = g^3 - abc.$$

For fixed  $a$  and  $b$ , since

$$g'(c) = \frac{c}{3g},$$

we have

$$f'(c) = cg - ab \leq bg - ab = b(g - a) \leq 0.$$

Since  $f(c)$  is decreasing, we have  $f(c) \geq f(b)$ .

(a) We need to prove the homogeneous inequality

$$3f(b) \geq 2b(a - b)^2,$$

that is

$$3 \left( \frac{a^2 + 2b^2}{3} \right)^{3/2} - 3ab^2 - 2b(a - b)^2 \geq 0.$$

For  $b = 0$ , the inequality is trivial. For  $b > 0$ , due to homogeneity, we may consider  $b = 1$ , when  $a \geq 1$ . The inequality becomes

$$3 \left( \frac{a^2 + 2}{3} \right)^{3/2} - 3a - 2(a - 1)^2 \geq 0,$$

$$3 \left( \frac{a^2 + 2}{3} \right)^{3/2} \geq 2a^2 - a + 2,$$

and, by squaring,

$$\begin{aligned} (a^2 + 2)^3 &\geq 3(2a^2 - a + 2)^2, \\ a^6 - 6a^4 + 12a^3 - 15a^2 + 12a - 4 &\geq 0, \\ (a - 1)^3(a^3 + 3a^2 + 4) &\geq 0. \end{aligned}$$

The equality occurs for  $a = b = c = 1$ .

(b) We need to prove the homogeneous inequality

$$27f(b) \geq 2b(2a + 7b)(a - b)^2,$$

that is

$$27 \left( \frac{a^2 + 2b^2}{3} \right)^{3/2} - 27ab^2 - 2(2a + 7b)(a - b)^2 \geq 0.$$

For  $b = 0$ , the inequality is trivial. For  $b > 0$ , due to homogeneity, we may consider  $b = 1$ , when  $a \geq 1$ . The inequality becomes

$$27 \left( \frac{a^2 + 2}{3} \right)^{3/2} - 27a - 2(2a + 7)(a - 1)^2 \geq 0,$$

$$27 \left( \frac{a^2 + 2}{3} \right)^{3/2} \geq 4a^3 + 6a^2 + 3a + 14,$$

and, by squaring,

$$27(a^2 + 2)^3 \geq (4a^3 + 6a^2 + 3a + 14)^2,$$



$$11a^6 - 48a^5 + 102a^4 - 148a^3 + 147a^2 - 84a + 20 \geq 0,$$

$$(a-1)^4(11a^2 - 4a + 20) \geq 0.$$

The equality occurs for  $a = b = c = 1$ .

□

**P 3.121.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$(a) \quad 1 - abc \geq \frac{1}{3} (b+c)(b-c)^2;$$

$$(b) \quad 1 - abc \geq \frac{2}{27} (7b+2c)(b-c)^2.$$

(Vasile Cîrtoaje, 2020)

**Solution.** Let us denote

$$g = \sqrt{\frac{a^2 + b^2 + c^2}{3}}$$

and

$$f = g^3 - abc.$$

For fixed  $b$  and  $c$ , since

$$g'(a) = \frac{a}{3g},$$

we have

$$f'(a) = ag - bc \leq bg - bc = b(g - c) \geq 0.$$

Since  $f(a)$  is increasing, we have  $f(a) \geq f(b)$ .

(a) We need to prove the homogeneous inequality

$$3f(b) \geq (b+c)(b-c)^2,$$

that is

$$3 \left( \frac{2b^2 + c^2}{3} \right)^{3/2} - 3b^2c - (b+c)(b-c)^2 \geq 0.$$

For  $c = 0$ , the inequality is trivial. For  $c > 0$ , due to homogeneity, we may consider  $c = 1$ , when  $b \geq 1$ . The inequality becomes

$$3 \left( \frac{2b^2 + 1}{3} \right)^{3/2} - 3b^2 - (b+1)(b-1)^2 \geq 0,$$

$$3 \left( \frac{2b^2 + 1}{3} \right)^{3/2} \geq b^3 + 2b^2 - b + 1,$$

and, by squaring,

$$\begin{aligned} (2b^2 + 1)^3 &\geq 3(b^3 + 2b^2 - b + 1)^2, \\ 5b^6 - 12b^5 + 6b^4 + 6b^3 - 9b^2 + 6b - 2 &\geq 0, \\ (b - 1)^3(b + 1)(5b^2 - 2b + 2) &\geq 0. \end{aligned}$$

The equality occurs for  $a = b = c = 1$ .

(b) We need to prove the homogeneous inequality

$$27f(b) \geq 2(7b + 2c)(b - c)^2,$$

that is

$$27 \left( \frac{2b^2 + c^2}{3} \right)^{3/2} - 27b^2c - 2(7b + 2c)(b - c)^2 \geq 0.$$

For  $c = 0$ , the inequality is trivial. For  $c > 0$ , due to homogeneity, we may consider  $c = 1$ , when  $b \geq 1$ . The inequality becomes

$$\begin{aligned} 27 \left( \frac{2b^2 + 1}{3} \right)^{3/2} - 27b^2 - 2(7b + 2)(b - 1)^2 &\geq 0, \\ 27 \left( \frac{2b^2 + 1}{3} \right)^{3/2} &\geq 14b^3 + 3b^2 + 6b + 4, \end{aligned}$$

and, by squaring,

$$\begin{aligned} 27(2b^2 + 1)^3 &\geq (14b^3 + 3b^2 + 6b + 4)^2, \\ 20b^6 - 84b^5 + 147b^4 - 148b^3 + 102b^2 - 48b + 11 &\geq 0, \\ (b - 1)^4(20b^2 - 4b + 11) &\geq 0. \end{aligned}$$

The equality occurs for  $a = b = c = 1$ .

□

**P 3.122.** If  $a, b, c$  are nonnegative real numbers such that  $a^2 + b^2 + c^2 = 3$ , then

$$\begin{aligned} (a) \quad &1 - \sqrt{abc} \geq (a - b)(b - c); \\ (b) \quad &1 - \sqrt[3]{a^2b^2c^2} \geq \frac{4}{3}(a - b)(b - c). \end{aligned}$$

(Vasile Cîrtoaje, 2020)

**Solution.** (a) Write the inequality as follows:

$$\begin{aligned} 4 - 4\sqrt{abc} &\geq 4(ab + bc - b^2 - ac), \\ a^2 + b^2 + c^2 + 1 - 4\sqrt{abc} &\geq 4(ab + bc - b^2 - ac), \\ (a - 2b + c)^2 + 1 + b^2 + 2ac &\geq 4\sqrt{abc}, \\ (a - 2b + c)^2 + (1 - b)^2 + 2(\sqrt{b} - \sqrt{ac})^2 &\geq 0. \end{aligned}$$

The equality occurs for  $a = b = c = 1$ .

(b) Write the inequality as follows (*Kiyoras, 2001*):

$$\begin{aligned} a^2 + b^2 + c^2 - 3\sqrt[3]{a^2b^2c^2} - 4(a - b)(b - c) &\geq 0, \\ (a - 2b + c)^2 + b^2 + 2ac - 3\sqrt[3]{a^2b^2c^2} &\geq 0. \end{aligned}$$

Since

$$b^2 + ac + ac \geq 3\sqrt[3]{a^2b^2c^2}.$$

(by AM-GM), the proof is completed. The equality occurs for  $a = b = c = 1$ .

**Remark.** The inequality (b) is stronger than (a) because

$$1 - \sqrt{abc} \geq \frac{3}{4} \left( 1 - \sqrt[3]{a^2b^2c^2} \right),$$

which is equivalent to

$$(x - 1)^2(3x^2 + 2x + 1) \geq 0,$$

where  $x = \sqrt[6]{abc}$ .

□

**P 3.123.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \geq \frac{2}{3}b(\sqrt{a} - \sqrt{c})^2.$$

(*Vasile Cîrtoaje, 2020*)

**Solution.** Let us denote

$$g = \sqrt{\frac{a^2 + b^2 + c^2}{3}}, \quad \sqrt{bc} \leq g \leq a,$$

and write the inequality in the homogeneous form  $f \geq 0$ , where

$$f = a^2 + b^2 + c^2 - \frac{3abc}{g} - 2b(\sqrt{a} - \sqrt{c})^2.$$

For fixed  $b$  and  $c$ , since

$$g'(a) = \frac{a}{3g},$$

we have

$$\begin{aligned} f'(a) &= 2a - \frac{3bc}{g} + \frac{a^2bc}{g^3} - \frac{2b(\sqrt{a} - \sqrt{c})}{\sqrt{a}} \\ &\geq 2a - \frac{2bc}{g} - 2b \left(1 - \sqrt{\frac{c}{a}}\right) \geq 2a - \frac{2bc}{\sqrt{bc}} - 2b \left(1 - \sqrt{\frac{c}{a}}\right) \\ &= 2a - 2\sqrt{bc} - 2b \left(1 - \sqrt{\frac{c}{a}}\right) \geq 2a - 2\sqrt{ac} - 2a \left(1 - \sqrt{\frac{c}{a}}\right) = 0. \end{aligned}$$

Since  $f(a)$  is increasing, we have  $f(a) \geq f(b)$ , and it remains to prove that  $f(b) \geq 0$ , that is

$$2b^2 + c^2 - \frac{3b^2c}{g} \geq 2b(\sqrt{b} - \sqrt{c})^2,$$

where

$$g = \sqrt{\frac{2b^2 + c^2}{3}}.$$

For  $c = 0$ , the inequality is an equality. For  $c > 0$ , due to homogeneity, we may consider  $c = 1$ , when  $b \geq 1$ . The inequality becomes

$$2b^2 + 1 - \frac{3b^2}{g} \geq 2b(\sqrt{b} - 1)^2,$$

where

$$g = \sqrt{\frac{2b^2 + 1}{3}}.$$

Since

$$g \geq \frac{2b + 1}{3},$$

it is enough to show that

$$2b^2 + 1 - \frac{9b^2}{2b + 1} \geq 2b(\sqrt{b} - 1)^2,$$

which is equivalent to

$$\frac{(b - 1)^2(4b + 1)}{2b + 1} \geq 2b(\sqrt{b} - 1)^2.$$

This is true if

$$\frac{(\sqrt{b} + 1)^2(4b + 1)}{2b + 1} \geq 2b,$$

which is equivalent to

$$8b(\sqrt{b} - 1) + 3(b - 1) + 2\sqrt{b} \geq 0.$$

The equality occurs for  $a = b = c = 1$ , and also for  $a = b = \sqrt{\frac{3}{2}}$  and  $c = 0$ .

**Remark.** We claim that, in the same conditions, the following stronger inequality holds:

$$1 - abc \geq \frac{2}{3}(b + 2c)(\sqrt{a} - \sqrt{c})^2.$$

□

**P 3.124.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \geq \frac{8}{3}b(\sqrt{a} - \sqrt{b})^2.$$

(Vasile Cîrtoaje, 2020)

**Solution.** Let us denote

$$g = \sqrt{\frac{a^2 + b^2 + c^2}{3}}, \quad g \leq a,$$

and write the inequality in the homogeneous form  $f \geq 0$ , where

$$f = a^2 + b^2 + c^2 - \frac{3abc}{g} - 8b(\sqrt{a} - \sqrt{b})^2.$$

For fixed  $a$  and  $b$ , since

$$g'(c) = \frac{c}{3g},$$

we have

$$\begin{aligned} f'(c) &= 2c - \frac{3ab}{g} + \frac{abc^2}{g^3} = 2c - \frac{3ab}{g} + \frac{3abc^2}{g(a^2 + b^2 + c^2)} \\ &= 2c - \frac{3ab(a^2 + b^2)}{g(a^2 + b^2 + c^2)} \leq 2c - \frac{2ab}{g} \\ &\leq 2b - \frac{2ab}{g} = 2b \left(1 - \frac{a}{g}\right) \leq 0. \end{aligned}$$

Since  $f(c)$  is decreasing, we have  $f(c) \geq f(b)$ , and it remains to prove that  $f(b) \geq 0$ , that is

$$a^2 + 2b^2 - \frac{3ab^2}{g} \geq 8b(\sqrt{a} - \sqrt{b})^2,$$

where

$$g = \sqrt{\frac{a^2 + 2b^2}{3}}.$$

For  $b = 0$ , the inequality is true. For  $b > 0$ , due to homogeneity, we may consider  $b = 1$ , when  $a \geq 1$ . The inequality becomes

$$a^2 + 2 - \frac{3a}{g} \geq 8(\sqrt{a} - 1)^2,$$

where

$$g = \sqrt{\frac{a^2 + 2}{3}}.$$

Since

$$\begin{aligned} a^2 + 2 - \frac{3a}{g} &= \frac{(a^2 + 2)^2 - \frac{9a^2}{g^2}}{a^2 + 2 + \frac{3a}{g}} \geq \frac{(a^2 + 2)^2 - \frac{9a^2}{g^2}}{2(a^2 + 2)} \\ &= \frac{(a^2 + 2)^3 - 27a^2}{2(a^2 + 2)^2} = \frac{(a^2 - 1)^2(a^2 + 8)}{2(a^2 + 2)^2}, \end{aligned}$$

it suffices to show that

$$\frac{(a^2 - 1)^2(a^2 + 8)}{2(a^2 + 2)^2} \geq 8(\sqrt{a} - 1)^2,$$

which is true if

$$(\sqrt{a} + 1)^2(a + 1)^2(a^2 + 8) \geq 16(a^2 + 2)^2.$$

Denoting  $x = \sqrt{a}$ ,  $x \geq 1$ , we can write this inequality as follows:

$$\begin{aligned} (x + 1)^2(x^2 + 1)^2(x^4 + 8) &\geq 16(x^4 + 2)^2, \\ x^{10} + 2x^9 - 13x^8 + 4x^7 + 11x^6 + 18x^5 - 39x^4 + 32x^3 + 24x^2 + 16x - 56 &\geq 0, \\ (x - 1)h(x) &\geq 0, \end{aligned}$$

where

$$\begin{aligned} h(x) &= x^9 + 3x^8 - 10x^7 - 6x^6 + 5x^5 + 23x^4 - 16x^3 + 16x^2 + 40x + 56 \\ &= x^4 h_1(x) + 4x^2(x - 2)^2 + 40x + 56, \\ h_1(x) &= x^5 + 3x^4 - 10x^3 - 6x^2 + 5x + 19. \end{aligned}$$

To complete the proof, we need to show that  $h_1(x) \geq 0$ . Indeed, we have

$$54h_1(x) = (3x - 5)^2(6x^3 + 38x^2 + 41) + xh_2(x) + 1,$$

where

$$h_2(x) = 450x^2 + 1500 - 1643x \geq (300\sqrt{30} - 1643)x > 0.$$

The equality occurs for  $a = b = c = 1$ .

□

**P 3.125.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \geq \frac{1}{3}(a + 3c)(\sqrt{a} - \sqrt{c})^2.$$

(Vasile Cîrtoaje, 2020)

**Solution.** Let us denote

$$g = \sqrt{\frac{a^2 + b^2 + c^2}{3}}, \quad g \leq a,$$

and write the inequality in the homogeneous form  $f \geq 0$ , where

$$f = a^2 + b^2 + c^2 - \frac{3abc}{g} - (a + 3c)(\sqrt{a} - \sqrt{c})^2.$$

For fixed  $b$  and  $c$ , since

$$g'(a) = \frac{a}{3g},$$

we have

$$\begin{aligned} f'(a) &= 2a - \frac{3bc}{g} + \frac{a^2bc}{g^3} - (\sqrt{a} - \sqrt{c})^2 - (a + 3c) \left(1 - \sqrt{\frac{c}{a}}\right) \\ &= -\frac{3bc}{g} + \frac{a^2bc}{g^3} - 4c + 3 \left(\sqrt{ac} + c\sqrt{\frac{c}{a}}\right) \\ &\geq -\frac{3bc}{g} + \frac{a^2bc}{g^3} + 2c. \end{aligned}$$

Due to homogeneity, we may consider  $g = 1$ , when

$$\begin{aligned} f'(a) &\geq -3bc + a^2bc + 2c = 2c - bc(3 - a^2) \geq 2c - ac(3 - a^2) \\ &= c(2 - 3a + a^3) = c(1 - a)^2(2 + a) \geq 0. \end{aligned}$$

Since  $f(a)$  is increasing, we have  $f(a) \geq f(b)$ , and it remains to prove that  $f(b) \geq 0$ , that is

$$2b^2 + c^2 - \frac{3b^2c}{g} \geq (b + 3c)(\sqrt{b} - \sqrt{c})^2,$$

where

$$g = \sqrt{\frac{2b^2 + c^2}{3}}.$$

For  $c = 0$ , the inequality is an identity. For  $c > 0$ , due to homogeneity, we may consider  $c = 1$ , when  $b \geq 1$ . The inequality becomes

$$2b^2 + 1 - \frac{3b^2}{g} \geq (b + 3)(\sqrt{b} - 1)^2,$$

where

$$g = \sqrt{\frac{2b^2 + 1}{3}}.$$

Since

$$g \geq \frac{2b + 1}{3},$$

it suffices to show that

$$2b^2 + 1 - \frac{9b^2}{2b + 1} \geq (b + 3)(\sqrt{b} - 1)^2,$$

which is equivalent to

$$\frac{(b-1)^2(4b+1)}{2b+1} \geq (b+3)(\sqrt{b}-1)^2.$$

This is true if

$$(\sqrt{b}+1)^2(4b+1) \geq (b+3)(2b+1).$$

Since

$$(\sqrt{b}+1)^2 > b+2,$$

we have

$$\begin{aligned} (\sqrt{b}+1)^2(4b+1) - (b+3)(2b+1) &> (b+2)(4b+1) - (b+3)(2b+1) \\ &= 2b^2 + 2b - 1 > 0. \end{aligned}$$

The equality occurs for  $a = b = c = 1$ , and also for  $a = \sqrt{3}$  and  $b = c = 0$ .

**Remark.** We claim that, in the same conditions, the following stronger inequality holds:

$$1 - abc \geq \frac{1}{3}(a+5c)(\sqrt{a}-\sqrt{c})^2.$$

□

**P 3.126.** If  $a, b, c$  are nonnegative real numbers such that  $a \geq b \geq c$  and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \geq \frac{2}{3}(a+3c)(\sqrt{b}-\sqrt{c})^2.$$

(Vasile Cîrtoaje, 2020)

**Solution.** Let us denote

$$g = \sqrt{\frac{a^2 + b^2 + c^2}{3}}, \quad g \leq a,$$

and write the inequality in the homogeneous form  $f \geq 0$ , where

$$f = a^2 + b^2 + c^2 - \frac{3abc}{g} - 2(a+3c)(\sqrt{b}-\sqrt{c})^2.$$

For fixed  $b$  and  $c$ , since

$$g'(a) = \frac{a}{3g},$$

we have

$$f'(a) = 2a - \frac{3bc}{g} + \frac{a^2bc}{g^3} - 2(\sqrt{b}-\sqrt{c})^2$$



$$\geq 2a - \frac{2bc}{g} - 2(\sqrt{a} - \sqrt{c})^2 = 2(a - b) + 2c \left( 2\sqrt{\frac{b}{c}} - \frac{b}{g} - 1 \right).$$

To prove that  $f'(a) \geq 0$ , it is sufficient to show that

$$2\sqrt{\frac{b}{c}} \geq \frac{b}{g} + 1.$$

Since

$$\frac{b}{g} = \frac{3b}{\sqrt{3(a^2 + b^2 + c^2)}} \leq \frac{3b}{a + b + c} \leq \frac{3b}{2b + c} = \frac{3}{2 + \frac{c}{b}},$$

it suffices to show that

$$2\sqrt{\frac{b}{c}} \geq \frac{3}{2 + \frac{c}{b}} + 1.$$

Denoting

$$x = \sqrt{\frac{c}{b}}, \quad 0 \leq x \leq 1,$$

the inequality becomes

$$\frac{2}{x} \geq \frac{3}{2 + x^2} + 1,$$

that is

$$\begin{aligned} 4 - 5x + 2x^2 - x^3 &\geq 0, \\ (1 - x)(4 - x + x^2) &\geq 0. \end{aligned}$$

Since  $f(a)$  is increasing, we have  $f(a) \geq f(b)$ , and it remains to prove that  $f(b) \geq 0$ , that is

$$2b^2 + c^2 - \frac{3b^2c}{g} \geq 2(b + 3c)(\sqrt{b} - \sqrt{c})^2 \geq 0,$$

where

$$g = \sqrt{\frac{2b^2 + c^2}{3}}.$$

For  $c = 0$ , the inequality is an identity. For  $c > 0$ , due to homogeneity, we may consider  $c = 1$ , when  $b \geq 1$ . The inequality becomes

$$2b^2 + 1 - \frac{3b^2}{g} \geq 2(b + 3)(\sqrt{b} - 1)^2,$$

where

$$g = \sqrt{\frac{2b^2 + 1}{3}}.$$

Since

$$2b^2 + 1 - \frac{3b^2}{g} = \frac{(2b^2 + 1)^2 - \frac{9b^4}{g^2}}{2b^2 + 1 + \frac{3b^2}{g}}$$

$$= \frac{(2b^2 + 1)^3 - 27b^4}{(2b^2 + 1)(2b^2 + 1 + \frac{3b^2}{g})} = \frac{(b^2 - 1)^2(8b^2 + 1)}{(2b^2 + 1)(2b^2 + 1 + \frac{3b^2}{g})}$$

and

$$g \geq 2b + 1,$$

we have

$$\begin{aligned} 2b^2 + 1 - \frac{3b^2}{g} &\geq \frac{(b^2 - 1)^2(8b^2 + 1)}{(2b^2 + 1)(2b^2 + 1 + \frac{3b^2}{2b+1})} \\ &= \frac{(b^2 - 1)^2(8b^2 + 1)(2b + 1)}{(2b^2 + 1)((4b^3 + 11b^2 + 2b + 1))}. \end{aligned}$$

Thus, it suffices to show that

$$\frac{(b^2 - 1)^2(8b^2 + 1)(2b + 1)}{(2b^2 + 1)((4b^3 + 11b^2 + 2b + 1))} \geq 2(b + 3)(\sqrt{b} - 1)^2,$$

which is true if

$$(\sqrt{b} + 1)^2(b + 1)^2(8b^2 + 1)(2b + 1) \geq 2(b + 3)(2b^2 + 1)(4b^3 + 11b^2 + 2b + 1).$$

Consider two cases:  $1 \leq b \leq 3$  and  $b \geq 3$ .

*Case 1:*  $1 \leq b \leq 3$ . Since

$$(\sqrt{b} + 1)^2 - (b + 3) = 2(\sqrt{b} - 1) \geq 0,$$

it suffices to show that

$$(b + 1)^2(8b^2 + 1)(2b + 1) \geq 2(2b^2 + 1)(4b^3 + 11b^2 + 2b + 1),$$

which is equivalent to

$$\begin{aligned} -b^4 + 18b^3 - 13b^2 - 1 &\geq 0, \\ (b - 1)(-4b^3 + 14b^2 + b + 1) &\geq 0. \end{aligned}$$

This is true because

$$-4b^3 + 14b^2 + b + 1 = 4b^2(3 - b) + 2b^2 + b + 1 > 0.$$

*Case 2:*  $b \geq 3$ . Since

$$(\sqrt{b} + 1)^2 - (b + 4) = 2\sqrt{b} - 3 \geq 2\sqrt{3} - 3 > 0,$$

it suffices to show that

$$(b + 4)(b + 1)^2(8b^2 + 1)(2b + 1) \geq 2(b + 3)(2b^2 + 1)(4b^3 + 11b^2 + 2b + 1).$$

Moreover, since

$$(8b^2 + 1)(2b + 1) > 8b^2(2b + 1) = 8b(2b^2 + b) \geq 8b(2b^2 + 1),$$

it suffices to show that

$$4b(b+4)(b+1)^2 \geq (b+3)(4b^3 + 11b^2 + 2b + 1),$$

which reduces to the obvious inequality

$$b^3 + b^2 + 9b - 3 \geq 0.$$

The equality occurs for  $a = b = c = 1$ , and also for  $a = \sqrt{3}$  and  $b = c = 0$ .

□

**P 3.127.** If  $a \geq b \geq c > 0$ , then

$$(a) \quad a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq 12a^2c^2(b-c)^2;$$

$$(b) \quad a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq 10a^3c(b-c)^2.$$

(Vasile Cîrtoaje, 2014)

**Solution.** (a) Let us denote

$$E(a, b, c) = a^6 + b^6 + c^6 - 3a^2b^2c^2 - 12a^2c^2(b-c)^2.$$

We will show that

$$E(a, b, c) \geq E(b, b, c) \geq 0.$$

We have

$$\begin{aligned} E(a, b, c) - E(b, b, c) &= (a^2 - b^2)[a^4 + a^2b^2 + b^4 - 3b^2c^2 - 12c^2(b-c)^2] \\ &\geq (a^2 - b^2)[3b^2(b^2 - c^2) - 12c^2(b-c)^2] \\ &= 3(a^2 - b^2)(b-c)[b^3 + c(b-2c)^2] \geq 0. \end{aligned}$$

Also,

$$\begin{aligned} E(b, b, c) &= 2b^6 + c^6 - 3b^4c^2 - 12b^2c^2(b-c)^2 \\ &= (b^2 - c^2)^2(2b^2 + c^2) - 12b^2c^2(b-c)^2 \\ &= (b-c)^2(2b^4 + 4b^3c - 9b^2c^2 + 2bc^3 + c^4) \\ &= (b-c)^3(2b^3 + 6b^2c^2 - 3bc^2 - c^3) \geq 0. \end{aligned}$$

The equality holds for  $a = b = c$ .

(b) Let

$$E(a, b, c) = a^6 + b^6 + c^6 - 3a^2b^2c^2 - 12a^2c^2(b-c)^2.$$

We will show that

$$E(a, b, c) \geq E(b, b, c) \geq 0.$$

To prove the left inequality, it suffices to show that for fixed  $b$  and  $c$ , the function

$$f(a) = E(a, b, c)$$

is increasing on  $[b, \infty)$ ; that is,  $f'(a) \geq 0$ . Indeed, we have the derivative

$$\begin{aligned} f'(a) &= 6a[a^4 - b^2c^2 - 5ac(b-c)^2] \geq 6a[a^4 - a^2c^2 - 5ac(a-c)^2] \\ &= 6a^2(a-c)[a(a+c) - 5c(a-c)] = 6a^2(a-c)[(a-2c)^2 + c^2] \geq 0. \end{aligned}$$

With regard to the right inequality, we have

$$\begin{aligned} E(b, b, c) &= 2b^6 + c^6 - 3b^4c^2 - 10b^3c(b-c)^2 \\ &= (b^2 - c^2)^2(2b^2 + c^2) - 10b^3c(b-c)^2 = (b-c)^2g(b, c), \end{aligned}$$

where

$$g(b, c) = 2b^4 - 6b^3c + 3b^2c^2 + 2bc^3 + c^4.$$

Since

$$g(b, c) = 2b(b-c)(b-2c)^2 + c \cdot h(b, c), \quad h(b, c) = 4b^3 - 13b^2c + 10bc^2 + c^3,$$

it suffices to show that  $h(b, c) \geq 0$ . For  $b \geq 2c$ , we have

$$h(b, c) = b(b-2c)(4b-5c) + c^3 > 0.$$

Also, for  $c \leq b \leq 2c$ , we have

$$2h(b, c) = (2c-b)(b-c)^2 + b(3b-5c)^2 \geq 0.$$

Thus, the proof is completed. The equality holds for  $a = b = c$ .

□

**P 3.128.** If  $a \geq b \geq c > 0$ , then

$$a + b + c - 3\sqrt[3]{abc} \geq \frac{3(a-c)^2}{4(a+b+c)}.$$

(Vasile Cîrtoaje, 2009)

**Solution.** Due to homogeneity, assume that  $a + b + c = 3$ . Let

$$x = \left(\frac{a+c}{2}\right)^2, \quad y = ac, \quad x \geq y.$$

We have

$$x = \left(\frac{3-b}{2}\right)^2, \quad x - y = \left(\frac{a-c}{2}\right)^2.$$

The desired inequality is equivalent to

$$3 - 3\sqrt[3]{by} \geq x - y.$$

There are two cases to consider.

*Case 1:*  $b \leq 1$ . By the AM-GM inequality, we have

$$y + 2\sqrt{b} \geq 3\sqrt[3]{by}.$$

Thus, it suffices to show that

$$3 - 2\sqrt{b} \geq x.$$

Indeed,

$$3 - 2\sqrt{b} - x = 3 - 2\sqrt{b} - \left(\frac{3-b}{2}\right)^2 = \frac{1}{4}(1 - \sqrt{b})^3(3 + \sqrt{b}) \geq 0.$$

*Case 2:*  $b \geq 1$ . From

$$a + b + c = b + \frac{a+c}{2} + \frac{a+c}{2} \geq 3\sqrt[3]{b\left(\frac{a+c}{2}\right)^2},$$

we get

$$3 \geq 3\sqrt[3]{bx}.$$

Therefore, it suffices to prove that

$$3\sqrt[3]{bx} - 3\sqrt[3]{by} \geq x - y,$$

which is equivalent to

$$(\sqrt[3]{x} - \sqrt[3]{y}) \left( 3\sqrt[3]{b} - \sqrt[3]{x^2} - \sqrt[3]{xy} - \sqrt[3]{y^2} \right) \geq 0.$$

Since

$$y \leq x = \left(\frac{3-b}{2}\right)^2 \leq 1 \leq b,$$

the inequality is clearly true. The equality holds for  $a = b = c$ .

□

**P 3.129.** If  $a \geq b \geq c > 0$ , then

$$(a) \quad a + b + c - 3\sqrt[3]{abc} \geq \frac{3(b-c)^2}{4b+5c};$$

$$(b) \quad a + b + c - 3\sqrt[3]{abc} \geq \frac{25(b-c)^2}{7(3b+11c)}.$$

(Vasile Cîrtoaje, 2009)

**Solution.** We use the inequality

$$a + b + c - 3\sqrt[3]{abc} \geq 2b + c - 3\sqrt[3]{b^2c},$$

which is equivalent to

$$\begin{aligned} a - b &\geq 3\sqrt[3]{bc} \left( \sqrt[3]{a} - \sqrt[3]{b} \right), \\ \left( \sqrt[3]{a} - \sqrt[3]{b} \right) \left( \sqrt[3]{a^2} + \sqrt[3]{ab} + \sqrt[3]{b^2} - 3\sqrt[3]{bc} \right) &\geq 0. \end{aligned}$$

Since  $a \geq b \geq c$ , the inequality is obvious.

(a) It suffices to show that

$$2b + c - 3\sqrt[3]{b^2c} \geq \frac{3(b-c)^2}{4b+5c}.$$

Setting  $c = 1$  and  $b = x^3$ ,  $x \geq 1$ , this inequality becomes as follows:

$$\begin{aligned} (4x^3 + 5)(2x^3 - 3x^2 + 1) &\geq 3(x^3 - 1)^2, \\ (x-1)^2(5x^4 - 2x^3 - 9x^2 + 4x + 2) &\geq 0, \\ (x-1)^4(5x^2 + 8x + 2) &\geq 0. \end{aligned}$$

The equality holds for  $a = b = c$ .

(b) It suffices to show that

$$2b + c - 3\sqrt[3]{b^2c} \geq \frac{25(b-c)^2}{7(3b+11c)}.$$

Setting  $c = 1$  and  $b = x^3$ ,  $x \geq 1$ , this inequality becomes in succession:

$$\begin{aligned} 7(3x^3 + 11)(2x^3 - 3x^2 + 1) &\geq 25(x^3 - 1)^2, \\ (x-1)^2(17x^4 - 29x^3 - 75x^2 + 104x + 52) &\geq 0, \\ (x-1)^2(x-2)^2(17x^3 + 39x + 13) &\geq 0. \end{aligned}$$

The equality holds for  $a = b = c$ , and for  $a = b = 8c$ .

**Remark.** The following generalization holds:

- If  $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ , then

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{3i(n-j+1)(a_i - a_j)^2}{2(2n+i-2j+2)a_i + 2(n+2i-j+1)a_j}$$

for all  $i < j$ .

□

**P 3.130.** If  $a \geq b \geq c > 0$ , then

$$(a) \quad a + b + c - 3\sqrt[3]{abc} \geq \frac{3(a-b)^2}{5a+4b};$$

$$(b) \quad a + b + c - 3\sqrt[3]{abc} \geq \frac{64(a-b)^2}{7(11a+24b)}.$$

(Vasile Cîrtoaje, 2009)

**Solution.** We use the inequality

$$a + b + c - 3\sqrt[3]{abc} \geq a + 2b - 3\sqrt[3]{ab^2},$$

which is equivalent to

$$3\sqrt[3]{ab} \left( \sqrt[3]{b} - \sqrt[3]{c} \right) \geq b - c,$$

$$\left( \sqrt[3]{b} - \sqrt[3]{c} \right) \left( 3\sqrt[3]{ab} - \sqrt[3]{b^2} - \sqrt[3]{bc} - \sqrt[3]{c^2} \right) \geq 0.$$

Since  $a \geq b \geq c$ , the inequality is obvious.

(a) It suffices to show that

$$a + 2b - 3\sqrt[3]{ab^2} \geq \frac{3(a-b)^2}{5a+4b}.$$

Setting  $b = 1$  (due to homogeneity) and  $a = x^3$ ,  $x \geq 1$ , this inequality becomes as follows:

$$(5x^3 + 4)(x^3 - 3x + 2) \geq 3(x^3 - 1)^2,$$

$$(x - 1)^2(2x^4 + 4x^3 - 9x^2 - 2x + 5) \geq 0,$$

$$(x - 1)^4(2x^2 + 8x + 5) \geq 0.$$

The equality holds for  $a = b = c$ .

(b) It suffices to show that

$$a + 2b - 3\sqrt[3]{ab^2} \geq \frac{64(a-b)^2}{7(11a+24b)}.$$

Setting  $b = 1$  and  $a = x^3$ ,  $x \geq 1$ , this inequality becomes in succession:

$$7(11x^3 + 24)(x^3 - 3x + 2) \geq 64(x^3 - 1)^2,$$

$$(x - 1)^2(13x^4 + 26x^3 - 192x^2 + 40x + 272) \geq 0,$$

$$(x - 1)^2(x - 2)^2(13x^3 + 78x + 68) \geq 0.$$

The equality holds for  $a = b = c$ , and for  $\frac{a}{8} = b = c$ .

□

**P 3.131.** If  $a \geq b \geq c \geq 0$ , then

$$\frac{(a-c)^2}{2(a+c)} \leq a+b+c-3\sqrt[3]{abc} \leq \frac{2(a-c)^2}{a+5c}.$$

(Vasile Cîrtoaje, 2007)

**Solution.** (a) To prove the inequality

$$a+b+c-3\sqrt[3]{abc} \geq \frac{(a-c)^2}{2(a+c)},$$

we will show that

$$a+b+c-3\sqrt[3]{abc} \geq a+c-2\sqrt{ac} \geq \frac{(a-c)^2}{2(a+c)}. \quad (*)$$

The left inequality is equivalent to

$$b+2\sqrt{ac} \geq 3\sqrt[3]{abc},$$

which is a consequence of the AM-GM inequality. The right inequality in (\*) can be written as follows:

$$\begin{aligned} a^2+c^2+6ac &\geq 4(a+c)\sqrt{ac}, \\ (\sqrt{a}-\sqrt{c})^4 &\geq 0. \end{aligned}$$

The equality holds for  $a=b=c$ .

(b) To prove the inequality

$$a+b+c-3\sqrt[3]{abc} \leq \frac{2(a-c)^2}{a+5c},$$

we will show that

$$a+b+c-3\sqrt[3]{abc} \leq 2a+c-3\sqrt[3]{a^2c} \leq \frac{2(a-c)^2}{a+5c}. \quad (**)$$

Write the left inequality as

$$\begin{aligned} a-b-3\sqrt[3]{ac}(\sqrt[3]{a}-\sqrt[3]{b}) &\geq 0, \\ (\sqrt[3]{a}-\sqrt[3]{b})\left(\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}-3\sqrt[3]{ac}\right) &\geq 0. \end{aligned}$$

This is true since

$$\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2} \geq 3\sqrt[3]{ab} \geq 3\sqrt[3]{ac}.$$

The right inequality in (\*\*) is an equality for  $c=0$ . For  $c>0$ , due to homogeneity, we may assume that  $c=1$ . In addition, making the substitution  $a=x^3$ ,  $x \geq 1$ , the right inequality in (\*\*) becomes in succession

$$(x^3+5)(2x^3-3x^2+1) \leq 2(x^3-1)^2,$$



$$(x-1)^2(x^3+2x^2-2x-1) \geq 0,$$

$$(x-1)^3(x^2+3x+1) \geq 0.$$

The equality holds for  $a = b = c$ , and also for  $a = b$  and  $c = 0$ .

□

**P 3.132.** Let  $a, b, c$  be nonnegative real numbers such that  $a \geq b \geq c$ . Prove that

$$(a) \quad a^3 + b^3 + c^3 - 3abc \geq 2(2b - a - c)^3;$$

$$(b) \quad a^3 + b^3 + c^3 - 3abc \geq (a - 2b + c)^3.$$

**Solution.** (a) Write the inequality as

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ca) \geq 2(2b-a-c)^3.$$

For the non-trivial case  $2b - a - c \geq 0$ , since

$$a+b+c \geq 2(2b-a-c),$$

it suffices to show that

$$a^2 + b^2 + c^2 - ab - bc - ca \geq (2b - a - c)^2.$$

This is equivalent to the obvious inequality

$$3(a-b)(b-c) \geq 0.$$

The equality holds for  $a = b = c$ , and also for  $a = b$  and  $c = 0$ .

(b) Write the inequality as

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ca) \geq (a-2b+c)^3.$$

For the non-trivial case  $a - 2b + c \geq 0$ , since

$$a+b+c \geq a-2b+c,$$

it suffices to show that

$$a^2 + b^2 + c^2 - ab - bc - ca \geq (a - 2b + c)^2,$$

which is equivalent to

$$3(a-b)(b-c) \geq 0.$$

The equality holds for  $a = b = c$ , and also for  $b = c = 0$ .

□

**P 3.133.** Let  $a, b, c$  be nonnegative real numbers such that  $a \geq b \geq c$ . Prove that

$$(a) \quad a^3 + b^3 + c^3 - 3abc \geq 3(a^2 - b^2)(b - c);$$

$$(b) \quad a^3 + b^3 + c^3 - 3abc \geq \frac{9}{2}(a - b)(b^2 - c^2).$$

**Solution.** (a) Write the inequality as

$$(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \geq 3(a + b)(a - b)(b - c).$$

Since

$$a + b + c \geq a + b,$$

it suffices to show that

$$a^2 + b^2 + c^2 - ab - bc - ca \geq 3(a - b)(b - c).$$

Indeed,

$$a^2 + b^2 + c^2 - ab - bc - ca - 3(a - b)(b - c) = (a - 2b + c)^2 \geq 0.$$

The equality holds for  $a = b = c$ , and also for  $a = 2b$  and  $c = 0$ .

(b) Write the inequality as

$$(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \geq \frac{9}{2}(a - b)(b - c)(b + c).$$

Since

$$a + b + c \geq \frac{3}{2}(b + c),$$

it suffices to show that

$$a^2 + b^2 + c^2 - ab - bc - ca \geq 3(a - b)(b - c).$$

This is equivalent to the obvious inequality

$$(a - 2b + c)^2 \geq 0.$$

The equality holds for  $a = b = c$ .

□

**P 3.134.** Let  $a, b, c$  be nonnegative real numbers such that  $a = \max\{a, b, c\}$ . Prove that

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq 2(b^4 + c^4 + 4b^2c^2)(b - c)^2.$$

**Solution.** Because the inequality is symmetric in  $b$  and  $c$ , we may assume that  $b \geq c$ ; that is,

$$a \geq b \geq c.$$

We will show that

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq 2b^6 + c^6 - 3b^4c^2 \geq 2(b^4 + c^4 + 4b^2c^2)(b - c)^2.$$

The left inequality is equivalent to the obvious inequality

$$(a^2 - b^2)(a^4 + a^2b^2 + b^4 - 3b^2c^2) \geq 0.$$

The right inequality is equivalent to

$$\begin{aligned} (b^2 - c^2)^2(2b^2 + c^2) &\geq 2(b^4 + c^4 + 4b^2c^2)(b - c)^2, \\ (b - c)^2[(b + c)^2(2b^2 + c^2) - 2(b^4 + c^4 + 4b^2c^2)] &\geq 0, \\ c(b - c)^3(4b^2 - bc + c^2) &\geq 0. \end{aligned}$$

The equality holds for  $a = b = c$ , for  $a = b$  and  $c = 0$ , and for  $a = c$  and  $b = 0$ .

□

**P 3.135.** Let  $a, b, c$  be nonnegative real numbers such that  $a = \max\{a, b, c\}$ . Prove that

$$a^2 + b^2 + c^2 \geq \frac{9abc}{a + b + c} + \frac{5}{3}(b - c)^2.$$

**Solution.** Because the inequality is symmetric in  $b$  and  $c$ , we may assume that  $b \geq c$ , hence

$$a \geq b \geq c.$$

Write the inequality as follows:

$$\begin{aligned} (a^2 + b^2 + c^2)(a + b + c) - 9abc &\geq \frac{5}{3}(a + b + c)(b - c)^2; \\ a^3 + b^3 + c^3 - 3abc + \sum a(b - c)^2 &\geq \frac{5}{3}(a + b + c)(b - c)^2; \\ (a + b + c) \sum (b - c)^2 + 2 \sum a(b - c)^2 &\geq \frac{10}{3}(a + b + c)(b - c)^2. \end{aligned}$$

It suffices to show that

$$(a + b + c)[(a - c)^2 + (b - c)^2] + 2a(b - c)^2 + 2b(a - c)^2 \geq \frac{10}{3}(a + b + c)(b - c)^2.$$

This inequality is true if

$$(a + b + c)[(b - c)^2 + (b - c)^2] + 2a(b - c)^2 + 2b(b - c)^2 \geq \frac{10}{3}(a + b + c)(b - c)^2.$$

Thus, we only need to show that

$$2(a + b + c) + 2a + 2b \geq \frac{10}{3}(a + b + c),$$

which reduces to  $a + b - 2c \geq 0$ . The equality holds for  $a = b = c$ .

□

**P 3.136.** Let  $a, b, c$  be nonnegative real numbers such that  $a + b > 0$ . Prove that

$$abc \geq (b + c - a)(c + a - b)(a + b - c) + \frac{ab(a - b)^2}{a + b}.$$

(Vasile Cîrtoaje, 2011)

**Solution.** Since

$$(b + c - a)(c + a - b)(a + b - c) = \frac{2(a^2b^2 + b^2c^2 + c^2a^2) - a^4 - b^4 - c^4}{a + b + c},$$

we can rewrite the inequality as

$$a^4 + b^4 + c^4 + abc(a + b + c) \geq 2(a^2b^2 + b^2c^2 + c^2a^2) + \frac{ab(a + b + c)(a - b)^2}{a + b}.$$

By Schur's inequality of fourth degree, we have

$$a^4 + b^4 + c^4 + abc(a + b + c) \geq \sum ab(a^2 + b^2).$$

Therefore, it suffices to prove that

$$\sum ab(a^2 + b^2) \geq 2(a^2b^2 + b^2c^2 + c^2a^2) + \frac{ab(a + b + c)(a - b)^2}{a + b},$$

which is equivalent to

$$\sum ab(a - b)^2 \geq \frac{ab(a + b + c)(a - b)^2}{a + b},$$

or

$$bc(b - c)^2 + ca(c - a)^2 \geq \frac{abc(a - b)^2}{a + b}.$$

This inequality follows immediately from the Cauchy-Schwarz inequality

$$(a + b)[bc(b - c)^2 + ca(c - a)^2] \geq [\sqrt{abc}(b - c) + \sqrt{abc}(c - a)]^2.$$

The equality holds for  $a = b = c$ , and for  $a = 0$  and  $b = c$  (or any cyclic permutation).

□

**P 3.137.** Let  $a, b, c$  be nonnegative real numbers such that  $a \geq b \geq c$ . Prove that

$$(a) \quad abc \geq (b+c-a)(c+a-b)(a+b-c) + \frac{2ab(a-b)^2}{a+b};$$

$$(b) \quad abc \geq (b+c-a)(c+a-b)(a+b-c) + \frac{27b(a-b)^4}{4a^2}.$$

(Vasile Cîrtoaje, 2011)

**Solution.** (a) Write the inequality as

$$\sum a(a-b)(a-c) \geq \frac{2ab(a-b)^2}{a+b}.$$

Since

$$c(c-a)(c-b) \geq 0,$$

it suffices to show that

$$a(a-b)(a-c) + b(b-c)(c-a) \geq \frac{2ab(a-b)^2}{a+b}.$$

Since

$$\begin{aligned} a(a-b)(a-c) &= a(a-b)[(a-b) + (b-c)] = a(a-b)^2 + a(a-b)(b-c) \\ &\geq \frac{2ab(a-b)^2}{a+b} + a(a-b)(b-c), \end{aligned}$$

it suffices to show that

$$a(a-b)(b-c) + b(b-c)(b-a) \geq 0.$$

This inequality is equivalent to

$$(a-b)^2(b-c) \geq 0.$$

The equality holds for  $a = b = c$ , and for  $a = b$  and  $c = 0$ .

(b) Write the inequality as

$$\sum a(a-b)(a-c) \geq \frac{27b(a-b)^4}{4a^2}.$$

Since

$$c(c-a)(c-b) \geq 0,$$

it suffices to show that

$$a(a-b)(a-c) + b(b-c)(c-a) \geq \frac{27b(a-b)^4}{4a^2},$$

which is equivalent to

$$a(a-b)^2 + a(a-b)(b-c) + b(b-c)(c-a) \geq \frac{27b(a-b)^4}{4a^2}.$$

Since

$$a(a-b)^2 - \frac{27b(a-b)^4}{4a^2} = \frac{(a-b)^2(a-3b)^2}{4a^2},$$

it suffices to show that

$$a(a-b)(b-c) + b(b-c)(b-a) \geq 0.$$

This inequality is equivalent to

$$(a-b)^2(b-c) \geq 0.$$

The equality holds for  $a = b = c$ , and for  $a/3 = b = c$ .

□

**P 3.138.** Let  $a, b, c$  be nonnegative real numbers such that  $a + b > 0$ . Prove that

$$\sum a^2(a-b)(a-c) \geq a^2b^2 \left( \frac{a-b}{a+b} \right)^2.$$

(Vasile Cîrtoaje, 2011)

**Solution.** Without loss of generality, assume that  $a \geq b$ . There three cases to consider.

*Case 1.*  $c \geq a \geq b$ . Since

$$\begin{aligned} a^2(a-b)(a-c) + c^2(c-a)(c-b) &\geq a^2(a-b)(a-c) + c^2(c-a)(a-b) \\ &= (a-b)(c-a)^2(c+a) \geq 0, \end{aligned}$$

it suffices to show that

$$b^2(a-b)(c-b) \geq a^2b^2 \left( \frac{a-b}{a+b} \right)^2.$$

Since  $c-b \geq a-b$ , this is true if

$$1 \geq \left( \frac{a}{a+b} \right)^2,$$

which is true.

*Case 2.*  $a \geq b \geq c$ . Since

$$c^2(c-a)(c-b) \geq 0$$

and

$$\begin{aligned} a^2(a-b)(a-c) + b^2(b-c)(b-a) &= (a-b)^2[a^2 + ab + b^2 - c(a+b)] \\ &\geq (a-b)^2[a^2 + ab + b^2 - b(a+b)] = a^2(a-b)^2, \end{aligned}$$

it suffices to show that

$$1 \geq \left( \frac{b}{a+b} \right)^2,$$

which is true.

Case 3.  $a \geq c \geq b$ . Since

$$b^2(b-c)(b-a) \geq b^2(c-b)^2$$

and

$$\begin{aligned} a^2(a-b)(a-c) + c^2(c-a)(c-b) &= (a-c)^2[a^2 + ac + c^2 - b(a+c)] \\ &\geq (a-c)^2[a^2 + ac + c^2 - c(a+c)] = a^2(a-c)^2, \end{aligned}$$

it suffices to show that

$$b^2(c-b)^2 + a^2(a-c)^2 \geq a^2b^2 \left( \frac{a-b}{a+b} \right)^2.$$

By the Cauchy-Schwarz inequality, we have

$$\left( \frac{1}{b^2} + \frac{1}{a^2} \right) [b^2(c-b)^2 + a^2(a-c)^2] \geq [(c-b) + (a-c)]^2 = (a-b)^2.$$

Therefore, it suffices to prove that

$$\frac{a^2b^2(a-b)^2}{a^2 + b^2} \geq a^2b^2 \left( \frac{a-b}{a+b} \right)^2,$$

which is clearly true.

This completes the proof. The equality holds for  $a = b = c$ , and for  $a = 0$  and  $b = c$  (or any cyclic permutation).

**Remark.** Similarly, we can prove the following generalization.

• Let  $a, b, c$  be nonnegative real numbers such that  $a + b > 0$ . If  $k$  is a positive natural number, then

$$\sum a^k(a-b)(a-c) \geq \left( \frac{ab}{a+b} \right)^k (a-b)^2.$$

□

**P 3.139.** Let  $a, b, c$  be nonnegative real numbers such that  $a \geq b \geq c$ . Prove that

$$(a) \quad \sum a^2(a-b)(a-c) \geq 4a^2b^2 \left( \frac{a-b}{a+b} \right)^2;$$

$$(b) \quad \sum a^2(a-b)(a-c) \geq \frac{27b(a-b)^4}{4a}.$$

(Vasile Cîrtoaje, 2011)

**Solution.** (a) Since  $c^2(c-a)(c-b) \geq 0$ , it suffices to show that

$$a^2(a-b)(a-c) + b^2(b-c)(b-a) \geq 4a^2b^2 \left( \frac{a-b}{a+b} \right)^2.$$

Since

$$\begin{aligned} a^2(a-b)(a-c) &= a^2(a-b)[(a-b) + (b-c)] \\ &= a^2(a-b)^2 + a^2(a-b)(b-c) \geq 4a^2b^2 \left( \frac{a-b}{a+b} \right)^2 + a^2(a-b)(b-c), \end{aligned}$$

it suffices to show that

$$a^2(a-b)(b-c) + b^2(b-c)(b-a) \geq 0.$$

This inequality is equivalent to

$$(a-b)^2(a+b)(b-c) \geq 0.$$

The equality holds for  $a = b = c$ , and for  $a = b$  and  $c = 0$ .

(b) Since  $c^2(c-a)(c-b) \geq 0$ , it suffices to show that

$$a^2(a-b)(a-c) + b^2(b-c)(c-a) \geq \frac{27b(a-b)^4}{4a},$$

which is equivalent to

$$a^2(a-b)^2 + a(a-b)(b-c) + b^2(b-c)(c-a) \geq \frac{27b(a-b)^4}{4a}.$$

Since

$$a^2(a-b)^2 - \frac{27b(a-b)^4}{4a} = \frac{(a-b)^2(a-3b)^2(4a-3b)}{4a} \geq 0,$$

it suffices to show that

$$a^2(a-b)(b-c) + b^2(b-c)(b-a) \geq 0.$$

This inequality is equivalent to

$$(a-b)^2(a+b)(b-c) \geq 0.$$

The equality holds for  $a = b = c$ , and for  $a/3 = b = c$ .

□

**P 3.140.** If  $a, b, c$  are nonnegative real numbers such that  $ab + bc + ca = 3$ , then

$$(a) \quad a^2 + b^2 + c^2 - a - b - c \geq \frac{5}{8}(a-c)^2;$$

$$(b) \quad a^2 + b^2 + c^2 - a - b - c \geq \frac{5}{2} \min\{(a-b)^2, (b-c)^2, (c-a)^2\}.$$

(Vasile Cîrtoaje, 2014)



**Solution.** Denote

$$E = a^2 + b^2 + c^2 - a - b - c, \quad S = a^2 + b^2 + c^2 - ab - bc - ca.$$

From

$$(a + b + c)^2 \geq 3(ab + bc + ca),$$

it follows that

$$a + b + c \geq 3.$$

We have

$$a + b + c - \sqrt{3(ab + bc + ca)} = \frac{S}{a + b + c + \sqrt{3(ab + bc + ca)}},$$

$$a + b + c - 3 = \frac{S}{a + b + c + 3},$$

$$(a + b + c)^2 - 3(a + b + c) = \frac{(a + b + c)S}{a + b + c + 3},$$

$$-3(a + b + c) = -(a + b + c)^2 + \frac{S}{1 + \frac{3}{a + b + c}},$$

$$-3(a + b + c) \geq -(a + b + c)^2 + \frac{S}{2},$$

therefore

$$E \geq a^2 + b^2 + c^2 - \frac{1}{3}(a + b + c)^2 + \frac{S}{6},$$

which is equivalent to

$$E \geq \frac{5S}{6}.$$

(a) It suffices to show that

$$\frac{5S}{6} \geq \frac{5}{8}(a - c)^2,$$

which is equivalent to

$$\frac{S}{3} \geq \frac{(a - c)^2}{4},$$

$$\frac{(a - b)^2 + (b - c)^2 + (c - a)^2}{3} \geq \frac{(a - c)^2}{2},$$

$$2(a - b)^2 + 2(b - c)^2 \geq (a - c)^2,$$

$$2(a - b)^2 + 2(b - c)^2 \geq [(a - b) + (b - c)]^2,$$

$$(a - b)^2 + (b - c)^2 \geq 2(a - b)(b - c),$$

$$(a - 2b + c)^2 \geq 0.$$

The equality holds for  $a = b = c = 1$ .

(b) Due to symmetry, without loss of generality, assume that

$$a \geq b \geq c.$$

It suffices to show that

$$\frac{5S}{6} \geq \frac{5}{2} \min\{(a-b)^2, (b-c)^2\},$$

which is equivalent to

$$\begin{aligned} S &\geq 3 \min\{(a-b)^2, (b-c)^2\}, \\ (a-b)^2 + (b-c)^2 + (a-c)^2 &\geq 6 \min\{(a-b)^2, (b-c)^2\}, \\ (a-b)^2 + (b-c)^2 + [(a-b) + (b-c)]^2 &\geq 6 \min\{(a-b)^2, (b-c)^2\}, \\ (a-b)^2 + (b-c)^2 + (a-b)(b-c) &\geq 3 \min\{(a-b)^2, (b-c)^2\}. \end{aligned}$$

The last inequality is clearly true. The equality holds for  $a = b = c = 1$ .

□

**P 3.141.** If  $a, b, c$  are nonnegative real numbers such that  $ab + bc + ca = 3$ , then

$$\frac{a^3 + b^3 + c^3}{a + b + c} \geq 1 + \frac{5}{9}(a - c)^2.$$

(Vasile Cîrtoaje, 2014)

**Solution.** It suffices to consider the case

$$a \geq b \geq c.$$

Write the inequality as

$$E \geq \frac{5}{9}(a - c)^2,$$

where

$$E = \frac{a^3 + b^3 + c^3}{a + b + c} - 1.$$

We have

$$\begin{aligned} E &= \frac{a^3 + b^3 + c^3}{a + b + c} - \frac{ab + bc + ca}{3} \\ &= \frac{3(a^3 + b^3 + c^3) - (a + b + c)(ab + bc + ca)}{3(a + b + c)} \\ &= \frac{A + B}{3(a + b + c)}, \end{aligned}$$

where

$$A = \sum [a^3 + b^3 - ab(a + b)], \quad B = \sum a^3 - 3abc.$$

Since

$$A = \sum (a+b)(a-b)^2,$$

$$B = \frac{1}{2}(a+b+c) \sum (a-b)^2,$$

we get

$$E = \frac{\sum (3a+3b+c)(a-b)^2}{6(a+b+c)}.$$

Thus, we need to show that

$$\sum (3a+3b+c)(a-b)^2 \geq \frac{10}{3}(a+b+c)(a-c)^2,$$

which is equivalent to

$$3(3a+3b+c)(a-b)^2 + 3(a+3b+3c)(b-c)^2 \geq (a+7b+c)(a-c)^2.$$

Using the substitution

$$a = c + x, \quad b = c + y, \quad x \geq y \geq 0,$$

the inequality becomes

$$3(7c+3x+3y)(x-y)^2 + 3(7c+x+3y)y^2 \geq (9c+x+7y)x^2,$$

which is equivalent to

$$6c(2x^2 - 7xy + 7y^2) + 2(x+y)(2x-3y)^2 \geq 0.$$

This inequality is true since

$$2x^2 - 7xy + 7y^2 = (2x^2 + 7y^2) - 7xy \geq (2\sqrt{14} - 7)xy \geq 0.$$

The equality holds for  $a = b = c = 1$ , and also for  $a = 3/\sqrt{2}$ ,  $b = \sqrt{2}$ ,  $c = 0$ .

□

**P 3.142.** If  $a, b, c$  are nonnegative real numbers such that

$$a \geq b \geq c, \quad ab + bc + ca = 3,$$

then

$$(a) \quad \frac{a^3 + b^3 + c^3}{a + b + c} \geq 1 + \frac{7}{9}(a-b)^2;$$

$$(b) \quad \frac{a^3 + b^3 + c^3}{a + b + c} \geq 1 + \frac{2}{3}(b-c)^2.$$

$$(c) \quad \frac{a^3 + b^3 + c^3}{a + b + c} \geq 1 + \frac{7}{3} \min\{(a-b)^2, (b-c)^2\}.$$

(Vasile Cîrtoaje, 2014)

**Solution.** As we have shown in the proof of the previous problem P 2.69,

$$\frac{a^3 + b^3 + c^3}{a + b + c} - 1 = \frac{\sum(3a + 3b + c)(a - b)^2}{6(a + b + c)}.$$

(a) Write the inequality as

$$\sum(3a + 3b + c)(a - b)^2 \geq \frac{14}{3}(a + b + c)(a - b)^2,$$

$$3(a + 3b + 3c)(b - c)^2 + 3(3a + b + 3c)(a - c)^2 \geq (5a + 5b + 11c)(a - b)^2.$$

It suffices to show that

$$3(3a + b + 3c)(a - c)^2 \geq (5a + 5b + 11c)(a - b)^2.$$

This is true since

$$(a - c)^2 \geq (a - b)^2$$

and

$$3(3a + b + 3c) - (5a + 5b + 11c) = 2(2a - b - c) \geq 0.$$

The equality holds for  $a = b = c = 1$ .

(b) Write the desired inequality as

$$\sum(3a + 3b + c)(a - b)^2 \geq 4(a + b + c)(b - c)^2,$$

$$(3a + 3b + c)(a - b)^2 + (3a + b + 3c)(a - c)^2 \geq (3a + b + c)(b - c)^2.$$

It suffices to show that

$$(3a + b + 3c)(a - c)^2 \geq (3a + b + c)(b - c)^2.$$

This is true since

$$(a - c)^2 \geq (a - b)^2$$

and

$$(3a + b + 3c) - (3a + b + c) = 2c \geq 0.$$

The equality holds for  $a = b = c = 1$ , and also for  $a = b = \sqrt{3}$ ,  $c = 0$ .

(c) Denote

$$m = \min\{(a - b)^2, (b - c)^2\},$$

then write the desired inequality as

$$\frac{\sum(3a + 3b + c)(a - b)^2}{6(a + b + c)} \geq \frac{7}{3}m,$$

$$\sum(3a + 3b + c)(a - b)^2 \geq 14(a + b + c)m,$$

$$(3a + 3b + c)(a - b)^2 + (a + 3b + 3c)(b - c)^2 + (3a + b + 3c)[(a - b) + (b - c)]^2 \geq 14(a + b + c)m,$$

$$(3a + 2b + 2c)(a - b)^2 + (2a + 2b + 3c)(b - c)^2 + (3a + b + 3c)(a - b)(b - c) \geq 7(a + b + c)m.$$

Case 1:  $a - 2b + c \geq 0$ . The inequality is true if

$$(3a + 2b + 2c) + (2a + 2b + 3c) + (3a + b + 3c) \geq 7(a + b + c),$$

which is equivalent to  $a - 2b + c \geq 0$ .

Case 2:  $a - 2b + c \leq 0$ . Since  $a - b \leq b - c$ , we need to show that

$$(3a + 2b + 2c)(a - b)^2 + (2a + 2b + 3c)(b - c)^2 + (3a + b + 3c)(a - b)(b - c) \geq 7(a + b + c)(a - b)^2,$$

which is equivalent to

$$(2a + 2b + 3c)(b - c)^2 + (3a + b + 3c)(a - b)(b - c) \geq (4a + 5b + 5c)(a - b)^2.$$

Since  $b - c \geq a - b \geq 0$ , it suffices to show that

$$(2a + 2b + 3c)(a - b)(b - c) + (3a + b + 3c)(a - b)(b - c) \geq (4a + 5b + 5c)(a - b)^2.$$

This is true if

$$(2a + 2b + 3c)(b - c) + (3a + b + 3c)(b - c) \geq (4a + 5b + 5c)(a - b),$$

which is equivalent to

$$(5a + 3b + 6c)(b - c) \geq (4a + 5b + 5c)(a - b),$$

$$2(2b - a - c)(2b + 2a + 3c) \geq 0.$$

The equality holds for  $2b = a + c$  and  $a^2 + 4ac + c^2 = 6$ .

□

**P 3.143.** If  $a, b, c$  are nonnegative real numbers such that  $ab + bc + ca = 3$ , then

$$a^4 + b^4 + c^4 - a^2 - b^2 - c^2 \geq \frac{11}{4}(a - c)^2.$$

(Vasile Cîrtoaje, 2014)

**Solution.** It suffices to consider the case  $a \geq b \geq c$ . Denote

$$S = a^2 + b^2 + c^2, \quad q = ab + bc + ca.$$

Summing the identities

$$a^4 + b^4 + c^4 - \frac{1}{3}S^2 = \frac{(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2}{3}$$

and

$$\frac{1}{3}S^2 - \frac{1}{3}Sq = S \cdot \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{6},$$

we get

$$a^4 + b^4 + c^4 - a^2 - b^2 - c^2 = \frac{(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2}{3} + S \cdot \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{6}.$$

Therefore, we can write the desired inequality in the homogeneous form

$$\frac{(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2}{3} + S \cdot \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{6} \geq \frac{11}{12}q(a-c)^2.$$

Since

$$(a-b)^2 + (b-c)^2 \geq \frac{1}{2}[(a-b) + (b-c)]^2 = \frac{1}{2}(a-c)^2,$$

it suffices to prove that

$$\frac{(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2}{3} + \frac{S(a-c)^2}{4} \geq \frac{11}{12}q(a-c)^2,$$

which is equivalent to

$$4(a+b)^2(a-b)^2 + 4(b+c)^2(b-c)^2 + E(a-c)^2 \geq 0,$$

where

$$E = 4(a+c)^2 + 3S - 11q.$$

Using the substitution

$$b = c + x, \quad a = c + x + y, \quad x, y \geq 0,$$

the inequality becomes

$$4(2c + 2x + y)^2 y^2 + 4(2c + x)^2 x^2 + E(x + y)^2 \geq 0,$$

where

$$E = -8c^2 - 16xc - x^2 + 7y^2 + 3xy.$$

Write this inequality as

$$Ac^2 + D \geq 2Bc,$$

where

$$A = 8(x-y)^2, \quad B = 8y(x-y)(2x+y), \quad D = 3x^4 + 11y^4 + 28x^2y^2 + 33xy^3 + x^3y.$$

Since  $Ac^2 + D \geq 2c\sqrt{AD}$ , it suffices to show that  $AD \geq B^2$ . Indeed,

$$\begin{aligned} AD - B^2 &= 8(x-y)^2[3x^4 + 11y^4 + 28x^2y^2 + 33xy^3 + x^3y - 8y^2(2x+y)^2] \\ &= 8(x-y)^2[x^4 + y^4 + 2(x^2 - y^2)^2 + xy(x^2 + y^2)] \geq 0. \end{aligned}$$

This completes the proof. The equality holds for  $a = b = c = 1$ .

□

**P 3.144.** If  $a, b, c$  are nonnegative real numbers such that

$$a \geq b \geq c, \quad ab + bc + ca = 3,$$

then

$$(a) \quad a^4 + b^4 + c^4 - a^2 - b^2 - c^2 \geq \frac{11}{3}(a - b)^2;$$

$$(b) \quad a^4 + b^4 + c^4 - a^2 - b^2 - c^2 \geq \frac{10}{3}(b - c)^2.$$

(Vasile Cîrtoaje, 2014)

**Solution.** Denote

$$S = a^2 + b^2 + c^2, \quad q = ab + bc + ca.$$

As we have shown in the proof of the previous problem,

$$\begin{aligned} a^4 + b^4 + c^4 - a^2 - b^2 - c^2 &= \frac{(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2}{3} \\ &\quad + S \cdot \frac{(a - b)^2 + (b - c)^2 + (c - a)^2}{6}. \end{aligned}$$

(a) Write the desired inequality in the homogeneous form

$$(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 + S \cdot \frac{(a - b)^2 + (b - c)^2 + (c - a)^2}{2} \geq \frac{11}{3}q(a - b)^2.$$

Since

$$(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 \geq (a^2 - b^2)^2 + (a^2 - c^2)^2 \geq 2(a^2 - b^2)^2$$

and

$$(a - b)^2 + (b - c)^2 + (c - a)^2 \geq (a - b)^2 + (a - c)^2 \geq 2(a - b)^2,$$

it suffices to prove that

$$2(a + b)^2 + a^2 + b^2 + c^2 \geq \frac{11}{3}(ab + bc + ca);$$

that is,

$$9(a^2 + b^2) + ab + 3c^2 \geq 11c(a + b).$$

Since

$$9(a^2 + b^2) + ab - \frac{19}{4}(a + b)^2 = \frac{17}{4}(a - b)^2 \geq 0,$$

we have

$$\begin{aligned} 9(a^2 + b^2) + ab + 3c^2 - 11c(a + b) &\geq \frac{19}{4}(a + b)^2 + 3c^2 - 11c(a + b) \\ &= \frac{(a + b - 2c)(19a + 19b - 6c)}{4} \geq 0. \end{aligned}$$

The equality holds for  $a = b = c = 1$ .

(b) Write the desired inequality in the homogeneous form

$$(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 + S \cdot \frac{(a - b)^2 + (b - c)^2 + (c - a)^2}{2} \geq \frac{10}{3}q(b - c)^2.$$

Since

$$\begin{aligned} (a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 &\geq (b^2 - c^2)^2 + (a^2 - c^2)^2 \\ &\geq (b + c)^2(b - c)^2 + (a + c)^2(b - c)^2 \end{aligned}$$

and

$$(a - b)^2 + (b - c)^2 + (c - a)^2 \geq (b - c)^2 + (a - c)^2 \geq 2(b - c)^2,$$

it suffices to prove that

$$(b + c)^2 + (a + c)^2 + a^2 + b^2 + c^2 \geq \frac{10}{3}(ab + bc + ca);$$

that is,

$$6(a^2 + b^2) - 10ab + 9c^2 \geq 4c(a + b).$$

Since

$$6(a^2 + b^2) - 10ab - \frac{1}{2}(a + b)^2 = \frac{11}{2}(a - b)^2 \geq 0,$$

we have

$$\begin{aligned} 6(a^2 + b^2) - 10ab + 9c^2 - 4c(a + b) &\geq \frac{1}{2}(a + b)^2 + 9c^2 - 4c(a + b) \\ &\geq 2\sqrt{\frac{9}{2}}c(a + b) - 4c(a + b) \\ &= (3\sqrt{2} - 4)c(a + b) \geq 0. \end{aligned}$$

The equality holds for  $a = b = c = 1$ .

**Remark.** Similarly, we can prove the following refinement of the inequality in (b):

$$a^4 + b^4 + c^4 - a^2 - b^2 - c^2 \geq \frac{1 + \sqrt{33}}{2}(b - c)^2,$$

with equality for  $a = b = c = 1$ , and also for  $a = b = \frac{3 + \sqrt{33}}{4}c$ .

□

**P 3.145.** Let

$$F(a, b, c) = 3(a^2 + b^2 + c^2) - (a + b + c)^2,$$

where  $a, b, c$  are positive real numbers such that  $a = \min\{a, b, c\}$  and  $a^2(b^2 + c^2) \geq 2$ . Then,

$$F(a, b, c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

(Vasile Cîrtoaje, 2020)



**Solution.** Assume that  $a \leq b \leq c$ . From

$$2 \leq a^2(b^2 + c^2) \leq 2a^2c^2,$$

it follows that

$$ac \geq 1, \quad bc \geq 1.$$

We need to show that

$$3(a^2 + b^2 + c^2) - (a + b + c)^2 \geq 3 \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) - \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)^2,$$

which is equivalent to

$$\begin{aligned} (a-b)^2 + (b-c)^2 + (c-a)^2 &\geq \frac{(a-b)^2}{a^2b^2} + \frac{(b-c)^2}{b^2c^2} + \frac{(c-a)^2}{c^2a^2}, \\ (a-b)^2 \left( 1 - \frac{1}{a^2b^2} \right) + (b-c)^2 \left( 1 - \frac{1}{b^2c^2} \right) + (c-a)^2 \left( 1 - \frac{1}{c^2a^2} \right) &\geq 0. \end{aligned}$$

Since  $bc \geq 1$ , it is enough to show that

$$(a-b)^2 \left( 1 - \frac{1}{a^2b^2} \right) + (c-a)^2 \left( 1 - \frac{1}{c^2a^2} \right) \geq 0,$$

that is

$$\left( 1 - \frac{a}{c} \right)^2 \left( c^2 - \frac{1}{a^2} \right) \geq \left( 1 - \frac{a}{b} \right)^2 \left( \frac{1}{a^2} - b^2 \right).$$

Since

$$1 - \frac{a}{c} \geq 1 - \frac{a}{b} \geq 0$$

it suffices to show that

$$c^2 - \frac{1}{a^2} \geq \frac{1}{a^2} - b^2,$$

which is just the hypothesis  $a^2(b^2 + c^2) \geq 2$ .

The proof is completed. The equality occurs for  $a = b = c \geq 1$ , and also for  $\frac{1}{a} = b = c \geq 1$ .

**Remark.** Since  $a(b+c) \geq 2$  implies  $a^2(b^2 + c^2) \geq 2$ , the inequality is true for  $a(b+c) \geq 2$ . In addition, it is true in the particular case  $a, b, c \geq 1$ . □

**P 3.146.** Let

$$F(a, b, c) = a + b + c - 3\sqrt[3]{abc},$$

where  $a, b, c$  are positive real numbers. If  $\min\{a, b, c\} \geq \frac{1}{abc}$ , then

$$F(a, b, c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

(Vasile Cîrtoaje, 2020)

**Solution.** Assume that  $a \leq b \leq c$ . Then, we have

$$a^2bc \geq 1, \quad bc \geq 1.$$

Write the inequality as  $E(a, b, c) \geq 0$ , where

$$E(a, b, c) = a + b + c - 3\sqrt[3]{abc} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} + \frac{3}{\sqrt[3]{abc}},$$

and show that

$$E(a, b, c) \geq E(a, \sqrt{bc}, \sqrt{bc}) \geq 0.$$

The left inequality is equivalent to

$$(\sqrt{b} - \sqrt{c})^2(bc - 1) \geq 0.$$

Substituting

$$a = x^3, \quad bc = y^6, \quad x \leq y, \quad xy \geq 1,$$

the right inequality becomes as follows:

$$E(x^3, y^3, y^3) \geq 0,$$

$$x^3 + 2y^3 - 3xy^2 - \left( \frac{1}{x^3} + \frac{2}{y^3} - \frac{3}{xy^2} \right) \geq 0,$$

$$(x - y)^2(x + 2y) - \left( \frac{1}{x} - \frac{1}{y} \right)^2 \left( \frac{1}{x} + \frac{2}{y} \right) \geq 0.$$

After dividing by  $(x - y)^2$ , we need to show that

$$x + 2y - \frac{2x + y}{x^3y^3} \geq 0.$$

Indeed,

$$x + 2y - \frac{2x + y}{x^3y^3} \geq x + 2y - (2x + y) = y - x \geq 0.$$

The equality holds for  $a = b = c \geq 1$ .

**Remark.** The inequality is true in the particular case  $a, b, c \geq 1$ .

□

**P 3.147.** Let

$$F(a, b, c) = a + b + c - 3\sqrt[3]{abc},$$

where  $a, b, c$  are positive real numbers such that  $a = \min\{a, b, c\}$  and  $a(b + c) \geq 2$ . Then,

$$F(a, b, c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

(Vasile Cîrtoaje, 2020)

**Solution.** Assume that  $a \leq b \leq c$ . Since both side of the inequality are nonnegative, it suffices to prove the homogeneous inequality

$$2 \left( a + b + c - 3\sqrt[3]{abc} \right) \geq a(b+c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{3}{\sqrt[3]{abc}} \right),$$

that is

$$b + c - 6\sqrt[3]{abc} \geq \frac{a(b^2 + c^2)}{bc} - 3(b+c)\sqrt[3]{\frac{a^2}{bc}}.$$

**First Solution.** For fixed  $b$  and  $c$ , write the inequality as  $f(a) \geq 0$ ,  $a \in (0, b]$ . We will show that

$$f(a) \geq f(b) \geq 0.$$

To prove that  $f(a) \geq f(b)$ , we show that  $f'(a) \leq 0$ , which is equivalent to

$$\frac{2(b+c)}{\sqrt[3]{abc}} \leq \frac{b^2 + c^2}{bc} + 2\sqrt[3]{\frac{bc}{a^2}}.$$

Since

$$\frac{b^2 + c^2}{bc} + 2\sqrt[3]{\frac{bc}{a^2}} \geq \frac{2\sqrt{2(b^2 + c^2)}}{\sqrt[3]{abc}},$$

it suffices to show that

$$(b+c) \leq \sqrt{2(b^2 + c^2)},$$

which is true. The inequality  $f(b) \geq 0$  has the form

$$2 \left( 2b + c - 3\sqrt[3]{b^2c} \right) \geq b(b+c) \left( \frac{2}{b} + \frac{1}{c} - \frac{3}{\sqrt[3]{b^2c}} \right).$$

Due to homogeneity, we may set  $b = 1$ , when the inequality has the form

$$2 \left( 2 + c - 3\sqrt[3]{c} \right) \geq (1+c) \left( 2 + \frac{1}{c} - \frac{3}{\sqrt[3]{c}} \right).$$

Denoting

$$t = \sqrt[3]{c}, \quad t \geq 1,$$

the inequality becomes as follows:

$$2c^3(c^3 + 2 - 3c) \geq (c^3 + 1)(2c^3 - 3c^2 + 1),$$

$$2c^3(c-1)^2(c+2) \geq (c^3 + 1)(c-1)^2(2c+1),$$

$$(c-1)^2(3c^3 - 2c - 1) \geq 0,$$

$$(c-1)^3(3c^2 + 3c + 1) \geq 1.$$

The equality holds for  $a = b = c \geq 1$ .

**Second Solution.** Due to homogeneity, we may assume  $a = 1$  and  $1 \leq b \leq c$ , when the inequality becomes

$$b + c - \frac{b^2 + c^2}{bc} \geq 3 \left( 2\sqrt[3]{bc} - \frac{b+c}{\sqrt[3]{bc}} \right).$$

Denote  $s = \frac{b+c}{2}$  and  $p = \sqrt{bc}$ ,  $s \geq p \geq 1$ . From  $(b-1)(c-1) \geq 0$ , we get  $2s \leq p^2 + 1$ . For fixed  $p$ , we need to show that  $F(s) \geq 0$ , where

$$F(s) = s - \frac{2s^2 - p^2}{p^2} - 3 \left( p^{2/3} - \frac{s}{p^{2/3}} \right).$$

Since  $F(s)$  is concave and  $p \leq s \leq \frac{p^2 + 1}{2}$ , it suffices to show that  $F(p) \geq 0$  and  $F\left(\frac{p^2 + 1}{2}\right) \geq 0$ . We have

$$F(p) = p - 1 - 3(p^{2/3} - p^{1/3}) = (p^{1/3} - 1)^3 \geq 0,$$

while the inequality  $F\left(\frac{p^2 + 1}{2}\right) \geq 0$  is equivalent to

$$p^2 - 1 - 3p^{4/3}(2p^{4/3} - p^2 - 1) \geq 0.$$

Denoting  $x = p^{2/3}$ ,  $x \geq 1$ , this inequality becomes

$$\begin{aligned} x^3 - 1 - 3x^2(2x^2 - x^3 - 1) &\geq 0, \\ (x-1)(x^2 + x + 1) + 3x^2(x-1)(x^2 - x - 1) &\geq 0, \\ (x-1)(3x^4 - 3x^3 - 2x^2 + x + 1) &\geq 0, \\ (x-1)^3(3x^2 + 3x + 1) &\geq 0. \end{aligned}$$

**Remark.** Because  $a^2bc \geq 1$  yields  $a(b+c) \geq 2$ , the inequality in P 3.146 follows from the inequality in P 3.150. □

**P 3.148.** Let

$$F(a, b, c, d) = a + b + c + d - 4\sqrt[4]{abcd},$$

where  $a, b, c, d$  are positive real numbers. If

$$\min\{a^2, b^2, c^2, d^2\} \geq \frac{1}{abcd},$$

then

$$F(a, b, c, d) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\right).$$

(Vasile Cîrtoaje, 2020)

**Solution.** By the AM-GM inequality, both sides of the inequality are nonnegative. Assume that  $a \leq b \leq c \leq d$ . Then, we have

$$a^3bcd \geq 1, \quad a \leq x = \sqrt[3]{bcd} \geq 1, \quad a^3x^3 \geq 1.$$

Write the inequality as  $E(a, b, c, d) \geq 0$ , where

$$E(a, b, c, d) = a + b + c + d - 4\sqrt[4]{abcd} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} - \frac{1}{d} + \frac{4}{\sqrt[4]{abcd}},$$

and show that

$$E(a, b, c, d) \geq E(a, x, x, x) \geq 0.$$

The left inequality shows that if  $a$  and  $bcd$  are fixed, then  $E(a, b, c, d)$  has the minimum value for  $b = c = d$ . Using the contradiction method, assume for the sake of contradiction that  $E(a, b, c, d)$  has the minimum value for  $b < d$ . This is false if

$$E(a, b, c, d) > E(a, \sqrt{bd}, c, \sqrt{bd}).$$

Since

$$\begin{aligned} E(a, b, c, d) - E(a, \sqrt{bd}, c, \sqrt{bd}) &= b + d - 2\sqrt{bd} - \frac{1}{b} - \frac{1}{d} + \frac{2}{\sqrt{bd}} \\ &= (\sqrt{b} - \sqrt{d})^2 \left(1 - \frac{1}{bd}\right), \end{aligned}$$

we need to show that  $bd > 1$ . Indeed, we have

$$(bd)^3 - 1 \geq (bd)^3 - a^3bcd \geq a^3(d^3 - bcd) > 0.$$

Write now the right inequality  $E(a, x, x, x) \geq 0$  as

$$a + 3x - 4\sqrt[4]{ax^3} \geq \frac{1}{a} + \frac{3}{x} - \frac{4}{ax^3}.$$

It suffices to prove the homogeneous inequality

$$a + 3x - 4\sqrt[4]{ax^3} \geq ax \left( \frac{1}{a} + \frac{3}{x} - \frac{4}{ax^3} \right),$$

which is equivalent to

$$\begin{aligned} 2(x - a) &\geq 4(\sqrt[4]{ax^3} - \sqrt[4]{a^3x}), \\ (\sqrt{x} - \sqrt{a})(\sqrt[4]{x} - \sqrt[4]{a})^2 &\geq 0. \end{aligned}$$

The equality holds for  $a = b = c = d \geq 1$ .

**Remark.** The inequality is true in the particular case  $a, b, c, d \geq 1$ , which implies

$$\min\{a^2, b^2, c^2, d^2\} \geq \frac{1}{abcd}.$$

□

**P 3.149.** Let

$$F(a, b, c, d) = a + b + c + d - 4\sqrt[4]{abcd},$$

where  $0 < a \leq b \leq c \leq d$  such that  $a(b + c + d) \geq 3$ . Then

$$F(a, b, c, d) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\right).$$

(Vasile Cîrtoaje, 2021)

**Solution.** Since both side of the inequality are nonnegative, it suffices to prove the homogeneous inequality

$$3\left(a + b + c + d - 4\sqrt[4]{abcd}\right) \geq a(b + c + d) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} - \frac{4}{\sqrt[4]{abcd}}\right)$$

for  $0 < a \leq b \leq c \leq d$ . For fixed  $b, c, d$ , write the inequality as  $F(a) \geq 0$ , where

$$F(a) = 3a + 2(b + c + d) - 12\sqrt[4]{abcd} - a(b + c + d) \left(\frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) + 4(b + c + d) \sqrt[4]{\frac{a^3}{bcd}}.$$

We will show that  $F'(a) \leq 0$ , where

$$F'(a) = 3 - 3\sqrt[4]{\frac{bcd}{a^3}} - (b + c + d) \left(\frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) + \frac{3(b + c + d)}{\sqrt[4]{abcd}} = 3 - 3\sqrt[4]{\frac{r}{a^3}} - \frac{pq}{r} + \frac{3p}{\sqrt[4]{ar}},$$

with  $p = b + c + d$ ,  $q = bc + cd + db$ ,  $r = bcd$ . Since  $3 \leq \frac{pq}{3r}$ , we have

$$F'(a) \leq \frac{pq}{3r} - 3\sqrt[4]{\frac{r}{a^3}} - \frac{pq}{r} + \frac{3p}{\sqrt[4]{ar}} = \frac{3p}{\sqrt[4]{ar}} - \frac{2pq}{3r} - 3\sqrt[4]{\frac{r}{a^3}}.$$

So, we need to show that

$$\frac{2pq}{3r} + 3\sqrt[4]{\frac{r}{a^3}} \geq \frac{3p}{\sqrt[4]{ar}}.$$

By the AM-GM inequality and the known inequality  $q^2 \geq 3pr$ , we have

$$\frac{2pq}{3r} + 3\sqrt[4]{\frac{r}{a^3}} = \frac{pq}{3r} + \frac{pq}{3r} + 3\sqrt[4]{\frac{r}{a^3}} \geq 3\sqrt[3]{\frac{p^2q^2}{9r^2}} \cdot 3\sqrt[4]{\frac{r}{a^3}} = \sqrt[3]{\frac{9q^2}{pr}} \cdot \frac{p}{\sqrt[4]{ar}} \geq \frac{3p}{\sqrt[4]{ar}}.$$

Since  $F'(a) \leq 0$ ,  $F(a)$  is decreasing, hence  $F(a) \geq F(b)$ . So, we need to show that  $F(b) \geq 0$ , i.e.

$$4b + c + d - 12\sqrt[4]{b^2cd} - b(b + c + d) \left(\frac{1}{c} + \frac{1}{d}\right) + 4(b + c + d) \sqrt[4]{\frac{b^2}{cd}} \geq 0.$$

Due to homogeneity, we may assume  $b = 1$  and  $1 \leq c \leq d$ . Denote  $s = \frac{c+d}{2}$  and  $p = \sqrt{cd}$ ,  $s \geq p \geq 1$ . From  $(b-1)(c-1) \geq 0$ , we get  $2s \leq p^2 + 1$ . For fixed  $p$ , we need to show that  $G(s) \geq 0$ , where

$$G(s) = 2 + s - 6\sqrt{p} - \frac{(1+2s)s}{p^2} + \frac{2(1+2s)}{\sqrt{p}}.$$

Since  $G(s)$  is concave and  $p \leq s \leq \frac{p^2+1}{2}$ , it suffices to show that  $G(p) \geq 0$  and  $G\left(\frac{p^2+1}{2}\right) \geq 0$ . We have

$$G(p) = 2 + p - 6\sqrt{p} - \frac{(1+2p)p}{p^2} + \frac{2(1+2p)}{\sqrt{p}} = \frac{p^2-1}{p} - \frac{2(p-1)}{\sqrt{p}} = \frac{(p-1)(\sqrt{p}-1)^2}{p} \geq 0,$$

while the inequality  $F\left(\frac{p^2+1}{2}\right) \geq 0$  is equivalent to

$$\frac{p^2-1}{p^2} + \frac{2(p-1)(p-2)}{\sqrt{p}} \geq 0.$$

It is true if

$$p+1+2p\sqrt{p}(p-2) \geq 0,$$

i.e.

$$p^2\sqrt{p} + p^2\sqrt{p} + p + 1 \geq 4p\sqrt{p},$$

which follows immediately from the AM-GM inequality. The equality occurs for  $a = b = c = d = 1$ .

**Remark.** Actually, We claim that the following generalization is true:

• If

$$F(a_1, a_2, \dots, a_n) = a_1 + a_2 + \dots + a_n - n\sqrt[n]{a_1 a_2 \dots a_n},$$

where  $0 < a_1 \leq a_2 \leq \dots \leq a_n$  such that  $a_1(a_2 + a_3 + \dots + a_n) \geq n-1$ , then

$$F(a_1, a_2, \dots, a_n) \geq F\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right)$$

□

**P 3.150.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 1.$$

Prove that

$$(1-a)(1-b)(1-c)(1-d) \geq abcd.$$

(Vasile Cîrtoaje, 2001)

**Solution.** The desired inequality follows by multiplying the inequalities

$$(1-a)(1-b) \geq cd,$$

$$(1-c)(1-d) \geq ab.$$

With regard to the first inequality, we have

$$2cd \leq c^2 + d^2 = 1 - a^2 - b^2,$$

and hence

$$\begin{aligned} 2(1-a)(1-b) - 2cd &\geq 2(1-a)(1-b) - 1 + a^2 + b^2 \\ &= (1-a-b)^2 \geq 0. \end{aligned}$$

The second inequality can be proved similarly. The equality holds for

$$a = b = c = d = 1/2,$$

and also for

$$(a, b, c, d) = (1, 0, 0, 0)$$

or any cyclic permutation.

□

**P 3.151.** Let  $a, b, c, d$  and  $x$  be positive real numbers such that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} = \frac{4}{x^2}.$$

If  $x \geq 2$ , then

$$(a-1)(b-1)(c-1)(d-1) \geq (x-1)^4.$$

(Vasile Cîrtoaje, 2001)

**Solution.** The desired inequality follows by multiplying the inequalities

$$2(a-1)(b-1) \geq (x-1) \left( \frac{ab}{cd}x + x - 2 \right),$$

$$2(c-1)(d-1) \geq (x-1) \left( \frac{cd}{ab}x + x - 2 \right),$$

$$\left( \frac{ab}{cd}x + x - 2 \right) \left( \frac{cd}{ab}x + x - 2 \right) \geq 4(x-1)^2.$$

With regard to the first inequality, we write it as

$$2ab - 2(a+b) + x(3-x) \geq x(x-1) \frac{ab}{cd}.$$

Since

$$\frac{2}{cd} \leq \frac{1}{c^2} + \frac{1}{d^2} = \frac{4}{x^2} - \frac{1}{a^2} - \frac{1}{b^2},$$



it suffices to show that

$$4ab - 4(a + b) + 2x(3 - x) \geq x(x - 1)ab \left( \frac{4}{x^2} - \frac{1}{a^2} - \frac{1}{b^2} \right).$$

This is equivalent to

$$4a^2b^2 - 4ab(a + b)x + 2x^2(3 - x)ab + x^2(x - 1)(a^2 + b^2) \geq 0,$$

which can be written in the obvious form

$$[2ab - (a + b)x]^2 + x^2(x - 2)(a - b)^2 \geq 0.$$

The second inequality can be proved similarly. With regard to the third inequality, we have

$$\begin{aligned} & \left( \frac{ab}{cd}x + x - 2 \right) \left( \frac{cd}{ab}x + x - 2 \right) = \\ &= 2x^2 - 4x + 4 + \left( \frac{ab}{cd} + \frac{cd}{ab} \right) x(x - 2) \\ &\geq 2x^2 - 4x + 4 + 2x(x - 2) = 4(x - 1)^2. \end{aligned}$$

The equality holds for  $a = b = c = d = x$ .

**Remark.** Setting  $x = 2$  and substituting  $a, b, c, d$  by  $1/a, 1/b, 1/c, 1/d$ , respectively, we get the inequality from P 3.150.

□

**P 3.152.** If  $a, b, c, d$  are positive real numbers, then

$$\frac{(1 + a^3)(1 + b^3)(1 + c^3)(1 + d^3)}{(1 + a^2)(1 + b^2)(1 + c^2)(1 + d^2)} \geq \frac{1 + abcd}{2}.$$

(Vasile Cîrtoaje, 1992)

**Solution.** For  $a = b = c = d$ , the inequality can be written as

$$\left( \frac{1 + a^3}{1 + a^2} \right)^4 \geq \frac{1 + a^4}{2}.$$

We will show that

$$\left( \frac{1 + a^3}{1 + a^2} \right)^4 \geq \left( \frac{1 + a^3}{1 + a} \right)^2 \geq \frac{1 + a^4}{2}.$$

The left side inequality is equivalent to

$$(1 + a^3)(1 + a) \geq (1 + a^2)^2,$$

$$a(1-a)^2 \geq 0,$$

while the right side inequality is equivalent to

$$2(1-a+a^2)^2 \geq 1+a^4,$$

$$(1-a)^4 \geq 0.$$

Multiplying the inequalities

$$\begin{aligned} \left(\frac{1+a^3}{1+a^2}\right)^4 &\geq \frac{1+a^4}{2}, & \left(\frac{1+b^3}{1+b^2}\right)^4 &\geq \frac{1+b^4}{2}, \\ \left(\frac{1+c^3}{1+c^2}\right)^4 &\geq \frac{1+c^4}{2}, & \left(\frac{1+d^3}{1+d^2}\right)^4 &\geq \frac{1+d^4}{2}, \end{aligned}$$

yields

$$\frac{(1+a^3)(1+b^3)(1+c^3)(1+d^3)}{(1+a^2)(1+b^2)(1+c^2)(1+d^2)} \geq \frac{1}{2} \sqrt[4]{(1+a^4)(1+b^4)(1+c^4)(1+d^4)}.$$

Applying the Cauchy-Schwarz inequality produces

$$(1+a^4)(1+b^4)(1+c^4)(1+d^4) \geq (1+a^2b^2)^2(1+c^2d^2)^2 \geq (1+abcd)^4,$$

from which the desired inequality follows. The equality holds for

$$a = b = c = d = 1.$$

□

**P 3.153.** Let  $a, b, c, d$  be positive real numbers such that

$$a + b + c + d = 4.$$

Prove that

$$\left(a + \frac{1}{a} - 1\right) \left(b + \frac{1}{b} - 1\right) \left(c + \frac{1}{c} - 1\right) \left(d + \frac{1}{d} - 1\right) + 3 \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

**Solution.** Write the inequality as

$$\prod \left[1 + \left(a + \frac{1}{a} - 2\right)\right] \geq \sum \frac{1}{a} - 3.$$

Since

$$\begin{aligned} a + \frac{1}{a} - 2 &\geq 0, & b + \frac{1}{b} - 2 &\geq 0, \\ c + \frac{1}{c} - 2 &\geq 0, & d + \frac{1}{d} - 2 &\geq 0, \end{aligned}$$

applying Bernoulli's inequality, it suffices to show that

$$1 + \sum \left( a + \frac{1}{a} - 2 \right) \geq \sum \frac{1}{a} - 3.$$

This is an identity, and then the proof is completed. The equality holds for

$$a = b = c = d = 1.$$

□

**P 3.154.** If  $a, b, c, d$  are nonnegative real numbers, then

$$4(a^3 + b^3 + c^3 + d^3) + 15(abc + bcd + cda + dab) \geq (a + b + c + d)^3.$$

**Solution.** Let

$$E(a, b, c, d) = 4(a^3 + b^3 + c^3 + d^3) + 15(abc + bcd + cda + dab) - (a + b + c + d)^3.$$

Without loss of generality, assume that  $a \leq b \leq c \leq d$ . We will show that

$$E(a, b, c, d) \geq E(0, a + b, c, d) \geq 0.$$

We have

$$\begin{aligned} E(a, b, c, d) - E(0, a + b, c, d) &= 4[a^3 + b^3 - (a + b)^3] + 15ab(c + d) \\ &= 3ab[5(c + d) - 4(a + b)] \geq 0. \end{aligned}$$

Now, putting  $x = a + b$ , we need to show that  $E(0, x, c, d) \geq 0$ , where

$$E(0, x, c, d) = 4(x^3 + c^3 + d^3) + 15xcd - (x + c + d)^3.$$

This is equivalent to Schur's inequality

$$x^3 + c^3 + d^3 + 3xcd \geq xc(x + c) + cd(c + d) + dx(d + x).$$

The equality holds for  $a = 0$  and  $b = c = d$  (or any cyclic permutation), and also for  $a = b = 0$  and  $c = d$  (or any permutation thereof).

□

**P 3.155.** Let  $a, b, c, d$  be positive real numbers such that

$$a + b + c + d = 4.$$

Prove that

$$1 + 2(abc + bcd + cda + dab) \geq 9 \min\{a, b, c, d\}.$$

(Vasile Cîrtoaje, 2008)

**Solution.** Assume that

$$a = \min\{a, b, c, d\}$$

and write the inequality in the homogeneous forms

$$(a + b + c + d)^3 + 128bcd + 128a(bc + cd + db) \geq 36a(a + b + c + d)^2.$$

Use now the substitution

$$b = a + x, \quad c = a + y, \quad d = a + z, \quad t = x + y + z,$$

where  $x, y, z, t \geq 0$ . Since

$$\begin{aligned} a + b + c + d &= 4a + t, \\ bcd &= (a + x)(a + y)(a + z) \geq a^3 + a^2t \end{aligned}$$

and

$$\begin{aligned} bc + cd + db &= (a + x)(a + y) + (a + y)(a + z) + (a + z)(a + x) \\ &= 3a^2 + 2at + xy + yz + zx \geq 3a^2 + 2at, \end{aligned}$$

it suffices to prove that

$$(4a + t)^3 + 128(a^3 + a^2t) + 128a(3a^2 + 2at) \geq 36a(4a + t)^2.$$

This inequality is equivalent to

$$t(t - 12a)^2 \geq 0.$$

The equality holds for  $a = b = c = d = 1$ , and also for  $(a, b, c, d) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{13}{4}\right)$  or any cyclic permutation.

□

**P 3.156.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a + b + c + d = 4.$$

Prove that

$$5(a^2 + b^2 + c^2 + d^2) \geq a^3 + b^3 + c^3 + d^3 + 16.$$

(Vasile Cîrtoaje, 2005)

**Solution.** Assume that

$$a \geq b \geq c \geq d.$$

**First Solution.** Use the mixing variables technique. Setting

$$x = \frac{b + c + d}{3},$$

we have

$$a + 3x = 4, \quad x \leq 1.$$

We will show that

$$E(a, b, c, d) \geq E(a, x, x, x) \geq 0,$$

where

$$E(a, b, c, d) = 5(a^2 + b^2 + c^2 + d^2) - a^3 - b^3 - c^3 - d^3 - 16.$$

The left side inequality is equivalent to

$$5(b^2 + c^2 + d^2 - 3x^2) - (b^3 + c^3 + d^3 - 3x^3) \geq 0.$$

Since  $b^2 + c^2 + d^2 - 3x^2 \geq 0$  and  $x \leq 1$ , it suffices to prove the homogeneous inequality

$$5x(b^2 + c^2 + d^2 - 3x^2) - (b^3 + c^3 + d^3 - 3x^3) \geq 0,$$

which is equivalent to

$$2(b^3 + c^3 + d^3) + 3b(c^2 + d^2) + 3c(d^2 + b^2) + 3d(b^2 + c^2) \geq 24bcd.$$

This is true, since  $b^3 + c^3 + d^3 \geq 3bcd$  and

$$b(c^2 + d^2) + c(d^2 + b^2) + d(b^2 + c^2) \geq 2bcd + 2cdb + 2dbc = 6bcd.$$

The right side inequality is also true, since

$$\begin{aligned} E(a, x, x, x) &= 5(a^2 + 3x^2) - a^3 - 3x^3 - 16 \\ &= 5(4 - 3x)^2 + 15x^2 - (4 - 3x)^3 - 3x^3 - 16 \\ &= 24x(x - 1)^2 \geq 0. \end{aligned}$$

This completes the proof. The equality holds for  $a = b = c = d = 1$ , and also for  $(a, b, c, d) = (0, 0, 0, 4)$  or any cyclic permutation.

**Second Solution.** Write the inequality as

$$\sum (5a^2 - a^3 - 7a + 3) \geq 0,$$

$$\sum (1 - a)^2(3 - a) \geq 0.$$

For  $a \leq 3$ , the inequality is clearly true. Otherwise, for  $3 < a \leq 4$ , which involves

$$1 > b \geq c \geq d \geq 0,$$

we get the required inequality by summing the inequalities

$$5a^2 \geq a^3 + 16$$

and

$$5(b^2 + c^2 + d^2) \geq b^3 + c^3 + d^3.$$

We have

$$5a^2 - a^3 - 16 = (4 - a)(a^2 - a - 4) = (4 - a)[a(a - 3) + 2(a - 2)] \geq 0,$$

and

$$5(b^2 + c^2 + d^2) \geq b^2 + c^2 + d^2 \geq b^3 + c^3 + d^3.$$

**Third Solution.** Write the inequality as

$$\sum (a^3 - 5a^2 + 4a) \leq 0,$$

or

$$\sum (a - 1)f(a) \leq 0,$$

where

$$f(a) = a^2 - 4a.$$

Since  $a + b \leq 4$ , we have

$$f(a) - f(b) = (a - b)(a + b - 4) \leq 0,$$

and, similarly,

$$f(b) - f(c) \leq 0, \quad f(c) - f(d) \leq 0.$$

Since

$$a - 1 \geq b - 1 \geq c - 1 \geq d - 1$$

and

$$f(a) \leq f(b) \leq f(c) \leq f(d),$$

by Chebyshev's inequality, we get

$$4 \sum (a - 1)f(a) \leq \left[ \sum (a - 1) \right] \left[ \sum f(a) \right] = 0.$$

**Remark.** Similarly, we can prove the following generalization:

- If  $a_1, a_2, \dots, a_n$  are nonnegative real numbers such that

$$a_1 + a_2 + \dots + a_n = n,$$

then

$$(n + 1)(a_1^2 + a_2^2 + \dots + a_n^2) \geq a_1^3 + a_2^3 + \dots + a_n^3 + n^2.$$

□

**P 3.157.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a + b + c + d = 4.$$

Prove that

$$3(a^2 + b^2 + c^2 + d^2) + 4abcd \geq 16.$$

(Vasile Cîrtoaje, 2004)

**Solution.** We use the mixing variables method. Assume that

$$a = \min\{a, b, c, d\}, \quad a \leq 1.$$

Setting

$$x = \frac{b + c + d}{3},$$

we have

$$a + 3x = 4, \quad 1 \leq x \leq \frac{4}{3}.$$

We will show that

$$E(a, b, c, d) \geq E(a, x, x, x) \geq 0,$$

where

$$E(a, b, c, d) = 3(a^2 + b^2 + c^2 + d^2) + 4abcd - 16.$$

The left side inequality is equivalent to

$$3(b^2 + c^2 + d^2 - 3x^2) \geq 4a(x^3 - bcd),$$

$$3(3x^2 - bc - cd - db) \geq 2a(x^3 - bcd).$$

From Schur's inequality

$$(b + c + d)^3 + 9bcd \geq 4(b + c + d)(bc + cd + db),$$

we get

$$9x^3 + 3bcd \geq 4x(bc + cd + db),$$

$$x^3 - bcd \leq \frac{4x}{3}(3x^2 - bc - cd - db).$$

Therefore, it suffices to prove that

$$3(3x^2 - bc - cd - db) \geq \frac{8ax}{3}(3x^2 - bc - cd - db);$$

that is,

$$(3x^2 - bc - cd - db)(9 - 8ax) \geq 0.$$

This is true since

$$6(3x^2 - bc - cd - db) = (b - c)^2 + (c - d)^2 + (d - b)^2 \geq 0,$$

and

$$3(9 - 8ax) = 27 - 8a(4 - a) = 8(1 - a)^2 + 16(1 - a) + 3 > 0.$$

The right side inequality is also true, since

$$\begin{aligned} E(a, x, x, x) &= 3a^2 + 9x^2 + 4ax^3 - 16 \\ &= 3(4 - 3x)^2 + 9x^2 + 4(4 - 3x)x^3 - 16 \\ &= 4(8 - 18x + 9x^2 + 4x^3 - 3x^4) \\ &= 4(1 - x)^2(2 + x)(4 - 3x) \geq 0. \end{aligned}$$

This completes the proof. The equality holds for  $a = b = c = d = 1$ , and also for  $(a, b, c, d) = \left(0, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$  or any cyclic permutation.

**Remark.** The following generalization holds (Vasile Cîrtoaje, 2005):

- Let  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) be nonnegative real numbers such that

$$a_1 + a_2 + \dots + a_n = n.$$

If  $k$  is a positive integer satisfying

$$2 \leq k \leq n + 2,$$

then

$$\frac{a_1^k + a_2^k + \dots + a_n^k}{n} - 1 \geq m(1 - a_1 a_2 \dots a_n), \quad m = \left(\frac{n}{n-1}\right)^{k-1} - 1.$$

□

**P 3.158.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a + b + c + d = 4.$$

Prove that

$$27(abc + cd + cda + dab) \leq 44abcd + 64.$$

**Solution.** Use the mixing variables method. Without loss of generality, assume that

$$a \geq b \geq c \geq d.$$

Setting  $x = (a + b + c)/3$ , we have

$$3x + d = 4, \quad d \leq x \leq 4/3, \quad x^3 \geq abc.$$

We will show that

$$E(a, b, c, d) \geq E(x, x, x, d) \geq 0.$$



The left inequality is equivalent to

$$(x^3 - abc) + d(3x^2 - ab - bc - ca) \geq \frac{44}{27}d(x^3 - abc).$$

By Schur's inequality

$$(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca),$$

we get

$$9x^3 + 3abc \geq 4x(ab + bc + ca),$$

and hence

$$3x^2 - ab - bc - ca \geq \frac{3(x^3 - abc)}{4x} \geq 0.$$

Therefore, it suffices to prove that

$$1 + \frac{3d}{4x} \geq \frac{44}{27}d.$$

Write this inequality in the homogeneous form

$$27(3x + d)(4x + 3d) \geq 704xd,$$

or, equivalently,

$$81(4x^2 + d^2) \geq 353xd.$$

This inequality is true, since

$$81(4x^2 + d^2) - 353xd \geq 81(4x^2 + d^2 - 5xd) = 81(x - d)(4x - d) \geq 0.$$

The right inequality  $E(x, x, x, t) \geq 0$  is also true, since

$$\begin{aligned} E(x, x, x, d) &= (44x^3 - 81x^2)d - 27x^3 + 64 \\ &= 4(16 - 81x^2 + 98x^3 - 33x^4) \\ &= 4(1 - x)^2(16 + 32x - 33x^2) \\ &= 4(1 - x)^2(4 - 3x)(4 + 11x) \geq 0. \end{aligned}$$

This completes the proof. The equality holds for  $a = b = c = d = 1$ , and also for  $(a, b, c, d) = \left(0, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$  or any cyclic permutation.

□

**P 3.159.** Let  $a, b, c, d$  be positive real numbers such that

$$a + b + c + d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

Prove that

$$(1 - abcd) \left( a^2 + b^2 + c^2 + d^2 - \frac{1}{a^2} - \frac{1}{b^2} - \frac{1}{c^2} - \frac{1}{d^2} \right) \geq 0.$$

(Vasile Cîrtoaje, 2007)

**Solution.** From

$$(a + b + c + d)^2 = \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right)^2,$$

we get

$$\begin{aligned} \sum a^2 - \sum \frac{1}{a^2} &= 2 \sum_{sym} \frac{1}{ab} - 2 \sum_{sym} ab \\ &= 2 \sum_{sym} \left( \frac{1}{ab} - cd \right) \\ &= 2(1 - abcd) \sum_{sym} \frac{1}{ab}. \end{aligned}$$

Thus, the inequality can be restated as

$$2(1 - abcd)^2 \sum_{sym} \frac{1}{ab} \geq 0,$$

which is obviously true. The equality holds for  $ab = cd = 1$ , or  $ac = bd = 1$ , or  $ad = bc = 1$ .  $\square$

**P 3.160.** Let  $a, b, c, d$  be positive real numbers such that

$$a + b + c + d = 1.$$

Prove that

$$(1 - a)(1 - b)(1 - c)(1 - d) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \geq \frac{81}{16}.$$

(Keira, 2007)

**Solution.** Write the inequality as

$$E(a, b, c, d) \geq \frac{81}{16},$$

where

$$E(a, b, c, d) = (1 - a)(1 - b)(1 - c)(1 - d) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right).$$

Without loss of generality, assume that

$$a \leq b \leq c \leq d.$$

First, we show that for

$$a \leq b \leq c \leq d, \quad a + b + c + d = 1,$$

$F(a, b, c, d)$  has the minimum value when  $a = c$ . This is true if

$$E(a, b, c, d) \geq E\left(\frac{a+c}{2}, b, \frac{a+c}{2}, d\right).$$

Since

$$(1-a)(1-c) = 1 - a - c + ac = b + d + ac$$

and

$$(1-b)(1-d) = 1 - b - d + bd = a + c + bd,$$

we have

$$E(a, b, c, d) = (b + d + ac)(a + c + bd) \left( \frac{a+c}{ac} + \frac{b+d}{bd} \right),$$

and the inequality is equivalent to

$$(b + d + ac) \left( \frac{a+c}{ac} + \frac{b+d}{bd} \right) \geq \left[ b + d + \left( \frac{a+c}{2} \right)^2 \right] \left( \frac{4}{a+c} + \frac{b+d}{bd} \right),$$

$$(a-c)^2 \left( \frac{4bd}{ac} - a - c \right) \geq 0.$$

Since

$$\frac{4bd}{ac} - a - c \geq 4 - a - c = 3 + b + d > 0,$$

the last inequality is clearly true. Since  $E(a, b, c, d)$  has the minimum value when  $a = c$ , from  $a \leq b \leq c \leq d$  it follows that  $E(a, b, c, d)$  has the minimum value when  $a = b = c$ . Therefore, it suffices to prove that  $3a + d = 4$  involves

$$E(a, a, a, d) \geq \frac{81}{16}.$$

This is equivalent to

$$21d^4 + 61d^3 - 57d^2 - 153d + 128 \geq 0,$$

$$(d-1)^2(21d^2 + 103d + 128) \geq 0.$$

The equality holds for  $a = b = c = d = 1/4$ .

□

**P 3.161.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a + b + c + d = a^3 + b^3 + c^3 + d^3 = 2.$$

Prove that

$$a^2 + b^2 + c^2 + d^2 \geq \frac{7}{4}.$$

(Vasile Cîrtoaje, 2010)

**Solution.** Let us denote

$$x = a^2 + b^2 + c^2 + d^2.$$

From

$$2 = a^3 + b^3 + c^3 + d^3 \geq a^3 + \frac{1}{9}(b + c + d)^3 = a^3 + \frac{1}{9}(2 - a)^3,$$

it follows that

$$(4a - 5)(a + 1)^2 \leq 0,$$

hence  $a \leq \frac{5}{4}$ . Similarly, we have  $b, c, d \leq \frac{5}{4}$ . On the other hand,

$$5x = 5 \sum a^2 = 4 \sum a^3 + \sum (5a^2 - 4a^3) = 8 + \sum a^2(5 - 4a)$$

and, by the Cauchy-Schwarz inequality,

$$\sum a^2(5 - 4a) \geq \frac{[\sum a(5 - 4a)]^2}{\sum (5 - 4a)} = \frac{(5 \sum a - 4 \sum a^2)^2}{20 - 4 \sum a} = \frac{(5 - 2x)^2}{3}.$$

Therefore, we have

$$5x \geq 8 + \frac{(5 - 2x)^2}{3}.$$

This is equivalent to

$$(4x - 7)(x - 7) \leq 0,$$

which involves  $x \geq \frac{7}{4}$ . The equality holds for  $(a, b, c, d) = \left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{7}{4}\right)$  or any cyclic permutation. □

**P 3.162.** Let  $a, b, c, d \in (0, 4]$  such that

$$abcd = 1.$$

Prove that

$$(1 + 2a)(1 + 2b)(1 + 2c)(1 + 2d) \geq (5 - 2a)(5 - 2b)(5 - 2c)(5 - 2d).$$

(Vasile Cîrtoaje, 2011)

**Solution.** Assume that

$$a \geq b \geq c \geq d.$$

For the nontrivial case where the right side of the inequality is positive, there are two cases to consider.

*Case 1:*  $a < 5/2$ . In virtue of the AM-GM inequality, we have

$$(1 + 2a)(1 + 2b)(1 + 2c)(1 + 2d) \geq (3\sqrt[3]{a^2})(3\sqrt[3]{b^2})(3\sqrt[3]{c^2})(3\sqrt[3]{d^2}) = 81,$$

$$\begin{aligned}
(5-2a)(5-2b)(5-2c)(5-2d) &\leq \left[ \frac{(5-2a) + (5-2b) + (5-2c) + (5-2d)}{4} \right]^4 \\
&= \left[ \frac{10 - (a+b+c+d)}{2} \right]^4 \\
&\leq \left( \frac{10 - 4\sqrt[4]{abcd}}{2} \right)^4 = 81,
\end{aligned}$$

from which the conclusion follows.

*Case 2:*  $a \geq b > 5/2 > c \geq d$ . Write the inequality as

$$\begin{aligned}
\frac{(1+2a)(1+2b)}{(2a-5)(2b-5)} &\geq \frac{(5-2c)(5-2d)}{(1+2c)(1+2d)}, \\
\frac{1+4ab+2(a+b)}{25+4ab-10(a+b)} &\geq \frac{25+4cd-10(c+d)}{1+4cd+2(c+d)}.
\end{aligned}$$

According to the AM-GM inequality, it suffices to prove that

$$\frac{1+4ab+4\sqrt{ab}}{25+4ab-20\sqrt{ab}} \geq \frac{25+4cd-20\sqrt{cd}}{1+4cd+4\sqrt{cd}}.$$

This is equivalent to

$$\begin{aligned}
\frac{2\sqrt{ab}+1}{2\sqrt{ab}-5} &\geq \frac{5-2\sqrt{cd}}{1+2\sqrt{cd}}, \\
\frac{2\sqrt{ab}+1}{2\sqrt{ab}-5} &\geq \frac{5\sqrt{ab}-2}{\sqrt{ab}+2}, \\
(4\sqrt{ab}-1)(4-\sqrt{ab}) &\geq 0.
\end{aligned}$$

The last inequality is true, since  $a, b \in (5/2, 4]$  involves  $4 - \sqrt{ab} \geq 0$ .

The equality holds for  $a = b = c = d = 1$ , and for  $(a, b, c, d) = \left(4, 4, \frac{1}{4}, \frac{1}{4}\right)$  (or any permutation thereof).

□

**P 3.163.** If  $a, b, c, d \in \left[0, 1 + \frac{1}{\sqrt{6}}\right]$  and  $a^2 + b^2 + c^2 + d^2 = 4$ , then

$$a + b + c + d \geq abcd + 3.$$

(Vasile Cîrtoaje, 2018)

**Solution.** Assume that  $a \geq b \geq c \geq d$  and denote

$$k = 1 + \frac{1}{\sqrt{6}}.$$

According to Remark 2 from P 3.57, if  $k \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$  and

$$a_1 + a_2 + \cdots + a_n = \text{constant}, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = \text{constant},$$

then the product  $a_1 a_2 \cdots a_n$  has the maximum value for  $a_1 \geq a_2 = \cdots = a_n$  or  $a_1 = k$ . Thus, it is enough to consider the cases  $a \geq b = c = d$  and  $a = k$ . In addition, for  $a = k$ , it suffices to consider the cases  $b = k$  and  $b \geq c = d$ .

*Case 1:*  $a \geq b = c = d$ . We need to show that

$$a + 3b \geq ab^3 + 3$$

for

$$a^2 + 3b^2 = 4, \quad k \geq a \geq b \geq 0.$$

From

$$4 - 3b^2 = a^2 \leq \left(1 + \frac{1}{\sqrt{6}}\right)^2,$$

we get

$$b \geq \sqrt{\frac{17 - 2\sqrt{6}}{18}} > \frac{1}{\sqrt{3}},$$

hence

$$\frac{1}{\sqrt{3}} < b \leq 1.$$

Write the inequality as

$$a(1 - b^3) - 3(1 - b) \geq 0,$$

$$(1 - b)[a(1 + b + b^2) - 3] \geq 0.$$

Since  $1 - b \geq 0$  and  $1 + b + b^2 \geq 3b$ , it suffices to show that  $ab \geq 1$ . Indeed, we have

$$a^2 b^2 - 1 = (4 - 3b^2)b^2 - 1 = (1 - b^2)(3b^2 - 1) \geq 0.$$

*Case 2:*  $a = b = k$ . We need to show that

$$2k + c + d \geq k^2 cd + 3$$

for

$$c^2 + d^2 = 4 - 2k^2, \quad k \geq c \geq d \geq 0.$$

Denoting

$$x = c + d,$$

from  $c^2 + d^2 = 4 - 2k^2$ , we get

$$2cd = x^2 - 4 + 2k^2.$$

Thus, the required inequality becomes as follows:

$$\begin{aligned} 4k + 2x &\geq k^2(x^2 + 2k^2 - 4) + 6, \\ k^2x^2 - 2x + 2k^4 - 4k^2 - 4k + 6 &\leq 0, \\ 3(7 + 2\sqrt{6})x^2 - 36x + 25 - 4\sqrt{6} &\leq 0, \\ 75x^2 - 36(7 - 2\sqrt{6})x + 271 - 106\sqrt{6} &\leq 0, \\ (3x - 3 + \sqrt{6})(75x - 177 + 47\sqrt{6}) &\leq 0. \end{aligned}$$

This is true if

$$\frac{3 - \sqrt{6}}{3} \leq x \leq \frac{177 - 47\sqrt{6}}{75}.$$

Indeed, from

$$(c + d)^2 \geq c^2 + d^2 = 4 - 2k^2,$$

we get

$$x = c + d \geq \sqrt{4 - 2k^2} = \frac{3 - \sqrt{6}}{3},$$

and from

$$(c + d)^2 \leq 2(c^2 + d^2) = 2(4 - 2k^2),$$

we get

$$x \leq \sqrt{2(4 - 2k^2)} = \frac{3\sqrt{2} - 2\sqrt{3}}{3} < \frac{1}{3} < \frac{177 - 47\sqrt{6}}{75}.$$

*Case 3:*  $a = k$ ,  $b \geq c = d$ . We need to show that

$$k + b + 2d \geq kbd^2 + 3$$

for

$$b^2 + 2d^2 = 4 - k^2, \quad k \geq b \geq d \geq 0.$$

Write the inequality as

$$(1 - kd^2)b \geq 3 - k - 2d.$$

From

$$3d^2 \leq b^2 + 2d^2 = 4 - k^2,$$

we get

$$1 - kd^2 \geq 1 - \frac{k(4 - k^2)}{3} = \frac{3\sqrt{6} - 5}{18\sqrt{6}} > 0.$$

Therefore, consider further the nontrivial case  $3 - k - 2d \geq 0$ , when  $d \leq \frac{3 - k}{2} < \frac{4}{5}$ . On the other hand, from

$$k^2 + 2d^2 \geq b^2 + 2d^2 = 4 - k^2,$$

we get

$$d \geq \sqrt{2 - k^2} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} > \frac{1}{8}.$$

Therefore, it suffices to show that

$$(1 - kd^2)^2(4 - k^2 - 2d^2) - (3 - k - 2d)^2 \geq 0$$

for

$$\frac{1}{8} \leq d \leq \frac{4}{5}.$$

After many calculations, we can show that this inequality is true.

The inequality is an equality for  $a = b = c = d = 1$ , and also for

$$a = b = 1 + \frac{1}{\sqrt{6}}, \quad c = 1 - \sqrt{\frac{2}{3}}, \quad d = 0$$

(or any permutation).

**Open Problem.** If

$$1 + \frac{1}{\sqrt{(n-1)(n-2)}} \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$$

and

$$a_1^2 + a_2^2 + \cdots + a_n^2 = n,$$

then

$$a_1 + a_2 + \cdots + a_n \geq a_1 a_2 \cdots a_n + n - 1.$$

□

**P 3.164.** Let  $a, b, c, d$  be positive real numbers such that

$$(a + b + c + d) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \leq (1 + \sqrt{10})^2.$$

Prove that any three of  $a, b, c, d$  are the lengths of the sides of a triangle (non-degenerate or degenerate).

**Solution.** Without loss of generality, assume that

$$a \geq b \geq c \geq d.$$



Clearly, any three of  $a, b, c, d$  are the lengths of the sides of a triangle if and only if  $a \leq c + d$ . By virtue of the Cauchy-Schwarz inequality, we have

$$\begin{aligned} (1 + \sqrt{10})^2 &\geq (b + a + c + d) \left( \frac{1}{b} + \frac{1}{a} + \frac{1}{c} + \frac{1}{d} \right) \\ &\geq \left[ 1 + \sqrt{(a + c + d) \left( \frac{1}{a} + \frac{1}{c} + \frac{1}{d} \right)} \right]^2 \\ &\geq \left[ 1 + \sqrt{(a + c + d) \left( \frac{1}{a} + \frac{4}{c + d} \right)} \right]^2, \end{aligned}$$

hence

$$(a + c + d) \left( \frac{1}{a} + \frac{4}{c + d} \right) \leq (1 + \sqrt{10}) - 1 = 10.$$

Writing this inequality as

$$(a - c - d)(4a - c - d) \leq 0,$$

we get  $a \leq c + d$ . Thus, the proof is completed.

**Remark 1.** Notice that  $11 + 2\sqrt{10}$  is the largest value of the product

$$(a + b + c + d) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right)$$

such that any three of the positive real numbers  $a, b, c, d$  are the lengths of the sides of a triangle. In order to prove this, assume that

$$(a + b + c + d) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) = k, \quad k > 11 + 2\sqrt{10}.$$

This relation is satisfied for

$$a = p + \sqrt{p^2 - 1}, \quad b = \sqrt{\frac{a(a+2)}{2a+1}}, \quad c = d = 1,$$

where

$$p = \frac{(\sqrt{k} - 1)^2 - 5}{4}.$$

For  $k > 11 + 2\sqrt{10}$ , we get  $p > 5/4$ , then  $a > 2$ . Clearly, the numbers  $a, c$  and  $d$  are not the lengths of the sides of a triangle.

**Remark 2.** In the same manner, we can prove the following generalization:

- If  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) are positive real numbers such that

$$(a_1 + a_2 + \dots + a_n) \left( \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \leq (n + \sqrt{10} - 3)^2,$$

then any three of  $a_1, a_2, \dots, a_n$  are the lengths of the sides of a triangle.

□

**P 3.165.** Let  $a, b, c, d$  be positive real numbers such that

$$(a + b + c + d) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \leq \frac{119}{6}.$$

Prove that there exist three of  $a, b, c, d$  which are the lengths of the sides of a triangle (non-degenerate or degenerate).

(Vasile Cîrtoaje, 2010)

**Solution.** Without loss of generality, assume that

$$a \geq b \geq c \geq d.$$

We need to show that either  $a \leq b + c$  or  $b \leq c + d$ . For the sake of contradiction, consider that

$$a > b + c, \quad b > c + d.$$

It suffices to show that

$$(a + b + c + d) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) > \frac{119}{6}.$$

Notice that for  $a = 3$ ,  $b = 2$  and  $c = d = 1$ , we have  $a = b + c$ ,  $b = c + d$  and

$$(a + b + c + d) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) = \frac{119}{6}.$$

Therefore, we apply the Cauchy-Schwarz inequality in the following manner

$$[a + (b + c + d)] \left( \frac{9}{a} + \frac{16}{b + c + d} \right) \geq (3 + 4)^2 = 49;$$

that is,

$$a + b + c + d \geq \frac{49}{\frac{9}{a} + \frac{16}{b + c + d}}.$$

Thus, it suffices to show that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} > \frac{17}{42} \left( \frac{9}{a} + \frac{16}{b + c + d} \right).$$

From

$$\begin{aligned} (b + c + d) \left( \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) - 10 &\geq (b + c + d) \left( \frac{1}{b} + \frac{4}{c + d} \right) - 10 \\ &= \frac{(b - c - d)(4b - c - d)}{b(c + d)} > 0, \end{aligned}$$

we get

$$\frac{1}{b} + \frac{1}{c} + \frac{1}{d} > \frac{10}{b+c+d}.$$

So, it suffices to show that

$$\frac{1}{a} + \frac{10}{b+c+d} \geq \frac{17}{42} \left( \frac{9}{a} + \frac{16}{b+c+d} \right).$$

This is equivalent to

$$4a \geq 3(b+c+d),$$

which is true, since

$$4a - 3(b+c+d) = 4(a-b-c) + (b-c-d) + 2(c-d) > 0.$$

□

**P 3.166.** Let  $a, b, c, d$  be positive real numbers such that

$$3(a+b+c+d)^2 \geq 11(a^2+b^2+c^2+d^2).$$

Prove that any three of  $a, b, c, d$  are the lengths of the sides of a triangle (non-degenerate or degenerate).

(Vasile Cîrtoaje, 2010)

**Solution.** Without loss of generality, assume that

$$a \geq b \geq c \geq d.$$

Then, any three of  $a, b, c, d$  are the lengths of the sides of a triangle if and only if

$$a \leq c + d.$$

To prove this, let us denote

$$x = \frac{c+d}{2}, \quad x \leq b.$$

Since  $c^2 + d^2 \geq 2x^2$ , we have

$$3(a+b+2x)^2 \geq 11(a^2+b^2+2x^2),$$

which can be written as

$$\begin{aligned} 8b^2 - 6(a+2x)b + 11a^2 + 22x^2 - 3(a+2x)^2 &\leq 0, \\ 8 \left[ b - \frac{3(a+2x)}{8} \right]^2 + 11a^2 + 22x^2 - \frac{33}{8}(a+2x)^2 &\leq 0. \end{aligned}$$

This involves

$$\begin{aligned} a^2 + 2x^2 - \frac{3}{8}(a + 2x)^2 &\leq 0, \\ (a - 2x)(5a - 2x) &\leq 0, \end{aligned}$$

from which we get  $a \leq 2x$ ; that is,

$$a - c - d \leq 0.$$

Thus, the proof is completed.

**Remark 1.** Notice that  $11/3$  is the least value of the ratio

$$\frac{(a + b + c + d)^2}{a^2 + b^2 + c^2 + d^2}$$

such that any three of the positive real numbers  $a, b, c, d$  are the lengths of the sides of a triangle. In order to prove this, assume that

$$(a + b + c + d)^2 = k(a^2 + b^2 + c^2 + d^2), \quad 1 < k < \frac{11}{3}.$$

This relation is satisfied for

$$a = \frac{7 + \sqrt{k(66 - 17k)}}{k - 1}, \quad b = 3, \quad c = d = 2.$$

Since  $a > 4$  for  $1 < k < 11/3$ , the numbers  $a, c$  and  $d$  are not the lengths of the sides of a triangle.

**Remark 2.** In the same manner, we can prove the following generalization:

- If  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) are positive real numbers such that

$$3(a_1 + a_2 + \dots + a_n)^2 \geq (3n - 1)(a_1^2 + a_2^2 + \dots + a_n^2).$$

then any three of  $a_1, a_2, \dots, a_n$  are the lengths of the sides of a triangle.

Notice that  $n - 1/3$  is the least value of the ratio

$$\frac{(a_1 + a_2 + \dots + a_n)^2}{a_1^2 + a_2^2 + \dots + a_n^2}$$

such that any three of the positive real numbers  $a_1, a_2, \dots, a_n$  are the lengths of the sides of a triangle. To prove this, assume that

$$(a_1 + a_2 + \dots + a_n)^2 = k(a_1^2 + a_2^2 + \dots + a_n^2), \quad 1 < k < n - \frac{1}{3}.$$

This relation is satisfied for

$$\begin{aligned} a_1 &= \frac{3n - 5 + \sqrt{k[3(n - 2)(3n - 1) - k(9n - 19)]}}{k - 1}, \\ a_2 &= \dots = a_{n-2} = 3, \quad a_{n-1} = a_n = 2. \end{aligned}$$

Since  $a_1 > 4$  for  $1 < k < n - 1/3$ , the numbers  $a_1, a_{n-1}$  and  $a_n$  are not the lengths of the sides of a triangle.

□

**P 3.167.** Let  $a, b, c, d$  be positive real numbers such that

$$15(a + b + c + d)^2 \geq 49(a^2 + b^2 + c^2 + d^2).$$

Prove that there exist three of  $a, b, c, d$  which are the lengths of the sides of a triangle (non-degenerate or degenerate).

(Vasile Cîrtoaje, 2010)

**Solution.** Without loss of generality, assume that

$$a \geq b \geq c \geq d.$$

We need to show that either  $a \leq b + c$  or  $b \leq c + d$ . For the sake of contradiction, consider that

$$a > b + c, \quad b > c + d.$$

To complete the proof, it suffices to show that

$$15(a + b + c + d)^2 < 49(a^2 + b^2 + c^2 + d^2).$$

Notice that for  $a = 3, b = 2$  and  $c = d = 1$ , we have

$$a = b + c, \quad b = c + d,$$

and

$$15(a + b + c + d)^2 = 49(a^2 + b^2 + c^2 + d^2).$$

Therefore, we apply the Cauchy-Schwarz inequality in the following manner

$$(3 + 4) \left[ \frac{a^2}{3} + \frac{(b + c + d)^2}{4} \right] \geq (a + b + c + d)^2.$$

Based on this result, it suffices to show that

$$\frac{a^2}{3} + \frac{(b + c + d)^2}{4} < \frac{7}{15}(a^2 + b^2 + c^2 + d^2),$$

which is equivalent to

$$2a^2 + 7(b^2 + c^2 + d^2) > \frac{15}{4}(b + c + d)^2.$$

Since

$$\begin{aligned} 8(b^2 + c^2 + d^2) - 3(b + c + d)^2 &= 5b^2 - 6b(c + d) + 5(c^2 + d^2) - 6cd \\ &\geq 5b^2 - 6b(c + d) + (c + d)^2 \\ &= (b - c - d)(5b - c - d) > 0, \end{aligned}$$

it is enough to prove that

$$2a^2 + \frac{21}{8}(b + c + d)^2 \geq \frac{15}{4}(b + c + d)^2,$$

which is equivalent to

$$4a \geq 3(b + c + d).$$

Indeed, we have

$$4a - 3(b + c + d) = 4(a - b - c) + (b - c - d) + 2(c - d) > 0.$$

□

**P 3.168.** Let  $a, b, c, d$  be nonnegative real numbers such that  $a^2 + b^2 + c^2 + d^2 = 4$ . If  $a \geq b \geq c \geq d$ , then

$$a + b + c + d + (2 - \sqrt{2})(a - d)^2 \geq 4.$$

(Vasile Cîrtoaje, 2019)

**Solution.** For  $a = d$ , the inequality is a trivial equality. Next assume  $a > d$ , replace the condition  $a \geq b \geq c \geq d$  with  $b, c \in [d, a]$  and write the inequality in the homogeneous form  $f \geq 0$ , where

$$f = (a + b + c + d)\sqrt{g} + (2 - \sqrt{2})(a - d)^2 - 4g, \quad g = \frac{a^2 + b^2 + c^2 + d^2}{4}.$$

For fixed  $a, c$  and  $d$ ,  $f$  and  $g$  are functions of  $b$ ,  $b \in [d, a]$ . Since

$$g'(b) = \frac{b}{2},$$

we have

$$f'(b) = \sqrt{g} + \frac{(a + b + c + d)b}{4\sqrt{g}} - 2b = bh(b),$$

where

$$\begin{aligned} h(b) &= \frac{\sqrt{g}}{b} + \frac{a + b + c + d}{4\sqrt{g}} - 2, \\ h'(b) &= \frac{-\sqrt{g}}{b^2} + \frac{1}{4\sqrt{g}} + \frac{1}{4\sqrt{g}} - \frac{b(a + b + c + d)}{16g^{3/2}} \\ &\leq \frac{-\sqrt{g}}{b^2} + \frac{1}{2\sqrt{g}} = \frac{-2g + b^2}{2b^2\sqrt{g}} \leq 0. \end{aligned}$$

Since  $h(b)$  is a decreasing function, there are three possible cases: (1)  $h(b) \geq 0$  for  $b \in [d, a]$ , hence  $f(b)$  is increasing on  $[d, a]$ ; (2)  $h(b) \geq 0$  for  $b \in [d, d_1]$  and  $h(b) \leq 0$  for  $b \in [d_1, a]$ , hence  $f(b)$  is increasing on  $[d, d_1]$  and decreasing on  $[d_1, a]$ ; (3)  $h(b) \leq 0$  for  $b \in [d, a]$ , hence  $f(b)$  is decreasing on  $[d, a]$ . In all these cases  $f(b)$  has the minimum value when  $b \in \{d, a\}$ . As a consequence, we only need to prove the required inequality for  $b \in \{d, a\}$ . Similarly, we only need to prove the required inequality for  $c \in \{d, a\}$ . So, we need to show that

$$[ka + (4 - k)d]\sqrt{g} + (2 - \sqrt{2})(a - d)^2 - 4g \geq 0,$$

where

$$g = \frac{ka^2 + (4-k)d^2}{4}, \quad k \in \{1, 2, 3\}.$$

For  $d = 0$ , the inequality reduces to

$$(k\sqrt{k} - 2k + 4 - 2\sqrt{2})a^2 \geq 0,$$

which is true for  $k \in \{1, 2, 3\}$ . Next, due to homogeneity, we may set  $d = 1$  (which involves  $a > 1$ ). The inequality becomes

$$(ka + 4 - k)\sqrt{x} + (2 - \sqrt{2})(a - 1)^2 - 4x \geq 0,$$

with

$$x = \frac{ka^2 + 4 - k}{4}, \quad x \geq 1.$$

Write the inequality as

$$(2 - \sqrt{2})(a - 1)^2 \geq \sqrt{x} (4\sqrt{x} - ka - 4 + k),$$

$$(2 - \sqrt{2})(a - 1)^2 \geq \frac{k(4 - k)(a - 1)^2 \sqrt{x}}{4\sqrt{x} + ka + 4 - k}.$$

This is true if

$$(2 - \sqrt{2})(ka + 4 - k) \geq (4k - k^2 - 8 + 4\sqrt{2})\sqrt{x}.$$

*Case 1:*  $k = 1$ . Since

$$x = \frac{a^2 + 3}{4},$$

the inequality becomes

$$(4 - 2\sqrt{2})(a + 3) \geq (4\sqrt{2} - 5)\sqrt{a^2 + 3}.$$

It follows by multiplying the inequalities

$$4 - 2\sqrt{2} > 4\sqrt{2} - 5$$

and

$$a + 3 > \sqrt{a^2 + 3}.$$

*Case 2:*  $k = 2$ . Since

$$x = \frac{a^2 + 1}{2},$$

the inequality becomes

$$a + 1 \geq \sqrt{a^2 + 1},$$

which is clearly true.

*Case 3:*  $k = 3$ . Since

$$x = \frac{3a^2 + 1}{4},$$

the inequality becomes

$$(4 - 2\sqrt{2})(3a + 1) \geq (4\sqrt{2} - 5)\sqrt{3a^2 + 1}.$$

It follows by multiplying the inequalities

$$4 - 2\sqrt{2} > 4\sqrt{2} - 5$$

and

$$3a + 1 > \sqrt{3a^2 + 1}.$$

The proof is completed. The equality occurs for  $a = b = c = d = 1$ , and also for  $a = b = \sqrt{2}$  and  $c = d = 0$ .

**Remark.** Similarly, we can prove the following statements:

- Let  $a_1, a_2, a_3, a_4, a_5$  be nonnegative real numbers such that

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = 5.$$

If  $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5$ , then

$$1 - \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} \leq \frac{2}{5} \left(1 - \sqrt{\frac{2}{5}}\right) (a_1 - a_5)^2,$$

with equality for  $a_1 = a_2 = a_3 = a_4 = a_5 = 1$ , and also for  $a_1 = a_2 = \sqrt{\frac{5}{2}}$  and  $a_3 = a_4 = a_5 = 0$ .

- Let  $a_1, a_2, a_3, a_4, a_5, a_6$  be nonnegative real numbers such that

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 = 6.$$

If  $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6$ , then

$$1 - \frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6}{6} \leq \frac{1}{2} \left(1 - \frac{1}{\sqrt{2}}\right) (a_1 - a_6)^2,$$

with equality for  $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 1$ , and also for  $a_1 = a_2 = a_3 = \sqrt{2}$  and  $a_4 = a_5 = a_6 = 0$ .

- Let  $a_1, a_2, \dots, a_n$  be nonnegative real numbers such that

$$a_1^2 + a_2^2 + \dots + a_n^2 = n.$$

If  $a_1 \geq a_2 \geq \dots \geq a_n$ , then

$$1 - \frac{a_1 + a_2 + \dots + a_n}{n} \leq \frac{4}{27} (a_1 - a_n)^2,$$

with equality for  $a_1 = a_2 = \dots = a_n = 1$ .

□



**P 3.169.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

If  $a \geq b \geq c \geq d$ , then

$$1 - abcd \leq \frac{\sqrt{3}}{2} (a - d).$$

(Vasile Cîrtoaje, 2019)

**Solution.** For  $a = d$ , the inequality is a trivial equality. It is also true for  $d = 0$ , because we have

$$4 = a^2 + b^2 + c^2 \leq 3a^2,$$

hence

$$\frac{\sqrt{3}}{2} a \geq 1.$$

Next assume  $a > d > 0$ , replace the assumption  $a \geq b \geq c \geq d$  with  $b, c \in [d, a]$  and write the inequality in the homogeneous form  $f \geq 0$ , where

$$f = abcd + \frac{\sqrt{3}}{2}(a - d)g^{3/2} - g^2, \quad g = \frac{a^2 + b^2 + c^2 + d^2}{4}.$$

For fixed  $a, c$  and  $d$ ,  $f$  and  $g$  are functions of  $b$ ,  $b \in [d, a]$ . Since

$$g'(b) = \frac{b}{2},$$

we have

$$f'(b) = acd + \frac{3\sqrt{3}}{8}(a - d)b\sqrt{g(b)} - bg(b) = b\sqrt{g(b)} h(b),$$

where

$$h(b) = \frac{acd}{b\sqrt{g(b)}} + \frac{3\sqrt{3}}{8}(a - d) - \sqrt{g(b)}.$$

Since  $h(b)$  is a decreasing function, there are three possible cases:  $g(b) \geq 0$  for  $b \in [d, a]$ , hence  $f(b)$  is increasing on  $[a, d]$ ;  $g(b) \geq 0$  for  $b \in [d, d_1]$  and  $g(b) \leq 0$  for  $b \in [d_1, a]$ , hence  $f(b)$  is increasing on  $[d, d_1]$  and decreasing on  $[d_1, a]$ ;  $g(b) \leq 0$  for  $b \in [d, a]$ , hence  $f(b)$  is decreasing on  $[a, d]$ . In all these cases  $f(b)$  has the minimum value when  $b \in \{d, a\}$ . As a consequence, we only need to prove the required inequality for  $b \in \{d, a\}$ . Similarly, we only need to prove the required inequality for  $c \in \{d, a\}$ . So, we need to show that

$$a^k d^{4-k} + \frac{\sqrt{3}}{2}(a - d)g^{3/2} - g^2 \geq 0,$$

where

$$g = \frac{ka^2 + (4 - k)d^2}{4}, \quad k \in \{1, 2, 3\}.$$

Due to homogeneity, we may set  $d = 1$  (which involves  $a > 1$ ), when the inequality becomes

$$a^k + \frac{\sqrt{3}}{2}(a-1)x^{3/2} \geq x^2.$$

with

$$x = \frac{ka^2 + 4 - k}{4} > 1.$$

*Case 1:  $k = 1$ .* Since

$$x = \frac{a^2 + 3}{4},$$

the inequality becomes

$$\begin{aligned} 16a + \sqrt{3}(a-1)(a^2+3)^{3/2} &\geq (a^2+3)^2, \\ \sqrt{3}(a-1)(a^2+3)^{3/2} &\geq (a-1)^2(a^2+2a+9). \end{aligned}$$

and, by squaring,

$$2a(a-1)^2(a^5 - a^4 + 6a^3 + 2a^2 + 25a + 63) \geq 0.$$

Clearly, it is true for  $a > 1$ .

*Case 2:  $k = 2$ .* Since

$$x = \frac{a^2 + 1}{2},$$

the inequality becomes

$$\begin{aligned} 8a^2 + \sqrt{6}(a-1)(a^2+1)^{3/2} &\geq 2(a^2+1)^2, \\ \sqrt{6}(a-1)(a^2+1)^{3/2} &\geq 2(a^2-1)^2, \end{aligned}$$

and, by squaring,

$$2(a-1)^2(a^6 - 4a^5 + 11a^4 + 8a^3 + 11a^2 - 4a + 1) \geq 0.$$

This is true because

$$a^6 - 4a^5 + 11a^4 + 8a^3 + 11a^2 - 4a + 1 = a^2(a-1)^4 + 5a^4 + 12a^3 + 6a^2 + (2a-1)^2 > 0.$$

*Case 3:  $k = 3$ .* Since

$$x = \frac{3a^2 + 1}{4},$$

the inequality becomes

$$\begin{aligned} 16a^3 + \sqrt{3}(a-1)(3a^2+1)^{3/2} &\geq (3a^2+1)^2, \\ \sqrt{3}(a-1)(3a^2+1)^{3/2} &\geq (a-1)^2(9a^2+2a+1). \end{aligned}$$

and, by squaring,

$$2(a-1)^2(63a^5 + 25a^4 + 2a^3 + 6a^2 - a + 1) \geq 0.$$

Clearly, it is true for  $a > 1$ .

The proof is completed. The equality occurs for  $a = b = c = d = 1$ , and also for  $a = b = c = \frac{2}{\sqrt{3}}$  and  $d = 0$ .

**Remark.** The following more general statement holds for  $n \leq 6$ :

- If  $a_1, a_2, \dots, a_n$  ( $n \leq 6$ ) are nonnegative real numbers such that

$$a_1^2 + a_2^2 + \dots + a_n^2 = n, \quad a_1 \geq a_2 \geq \dots \geq a_n.$$

then

$$1 - a_1 a_2 \dots a_n \leq \sqrt{1 - \frac{1}{n}} (a_1 - a_n),$$

with equality for  $a_1 = a_2 = \dots = a_n = 1$ , and also for  $a_1 = a_2 = \dots = a_{n-1} = \sqrt{\frac{n}{n-1}}$  and  $a_n = 0$ .

To prove this inequality, we need to show that

$$1 - a^k b^{n-k} \leq \sqrt{1 - \frac{1}{n}} (a - b)$$

for  $a \geq 1 \geq b \geq 0$  and

$$ka^2 + (n-k)b^2 = n, \quad k \in \{1, 2, \dots, n-1\}.$$

□

**P 3.170.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

If  $a \geq b \geq c \geq d$ , then

$$1 - \sqrt{abcd} \leq \frac{3}{4}(a-d)^2.$$

(Vasile Cîrtoaje, 2019)

**Proof.** For  $a = d$ , the inequality is a trivial equality. Next assume  $a > d$ , replace the assumption  $a \geq b \geq c \geq d$  with  $b, c \in [d, a]$  and write the inequality in the homogeneous form  $f \geq 0$ , where

$$f = 4\sqrt{abcd} + 3(a-d)^2 - (a^2 + b^2 + c^2 + d^2)^2.$$

**First Solution.** For fixed  $a$ ,  $c$  and  $d$ ,  $f$  is a function of  $b$ ,  $b \in [d, a]$ . We have

$$f'(b) = 2\sqrt{\frac{acd}{b}} - 2b.$$

Since  $f'(b)$  is a decreasing function, there are three possible cases: (1)  $f'(b) \geq 0$  for  $b \in [d, a]$ , hence  $f(b)$  is increasing on  $[d, a]$ ; (2)  $f'(b) \geq 0$  for  $b \in [d, d_1]$  and  $f'(b) \leq 0$  for  $b \in [d_1, a]$ , hence  $f(b)$  is increasing on  $[d, d_1]$  and decreasing on  $[d_1, a]$ ; (3)  $f'(b) \leq 0$  for  $b \in [d, a]$ , hence  $f(b)$  is decreasing on  $[d, a]$ . In all these cases, if the inequality  $f(b)$  holds for the extreme values of  $b$ , then it holds for all possible values of  $b$ . Thus, we only need to prove the required inequality for  $b \in \{a, d\}$ . Similarly, we only need to prove the required inequality for  $c \in \{a, d\}$ . So, we need to show that

$$4\sqrt{a^k d^{4-k}} + 3(a-d)^2 \geq ka^2 + (4-k)d^2,$$

where

$$k \in \{1, 2, 3\}.$$

For  $d = 0$ , the inequality reduces to

$$(3-k)a^2 \geq 0,$$

which is true for  $k \in \{1, 2, 3\}$ . Next, due to homogeneity, we may set  $d = 1$ . Substituting  $x = \sqrt{a}$ ,  $x \geq 1$ , the required inequality becomes

$$3(x^2 - 1)^2 \geq kx^4 - 4x^k + 4 - k.$$

*Case 1:*  $k = 1$ . We need to show that

$$3(x^2 - 1)^2 \geq x^4 - 4x + 3,$$

that is

$$3(x^2 - 1)^2 \geq (x - 1)^2(x^2 + 2x + 3),$$

$$x(x - 1)^2(3x + 1) \geq 0.$$

*Case 2:*  $k = 2$ . We need to show that

$$3(x^2 - 1)^2 \geq 2x^4 - 4x^2 + 2,$$

that is

$$3(x^2 - 1)^2 \geq 2x^4 - 4x^2 + 2,$$

$$3(x^2 - 1)^2 \geq 2(x^2 - 1)^2.$$

*Case 3:*  $k = 3$ . We need to show that

$$3(x^2 - 1)^2 \geq 3x^4 - 4x^3 + 1,$$

that is

$$\begin{aligned} 3(x^2 - 1)^2 &\geq (x - 1)^2(3x^2 + 2x + 1), \\ (x - 1)^2(2x + 1) &\geq 0. \end{aligned}$$

The proof is completed. The equality occurs for  $a = b = c = d = 1$ , and also for  $a = b = c = \frac{2}{\sqrt{3}}$  and  $d = 0$ .

**Second Solution** (by *Marius Stanean*). Replacing  $a, b, c, d$  with  $a^2, b^2, c^2, d^2$ , respectively, we need to show that  $a \geq b \geq c \geq d \geq 0$  yields the homogeneous inequality  $F(a, b, c, d) \geq 0$ , where

$$F(a, b, c, d) = 4abcd - 3(a^2 - d^2)^2 - a^4 - b^4 - c^4 - d^4.$$

We will show that

$$F(a, b, c, d) \geq F(a, a, c, d) \geq 0.$$

The left inequality is equivalent to

$$a^4 - b^4 \geq 4acd(a - b),$$

which is true if

$$(a + b)(a^2 + b^2) \geq 4acd.$$

Indeed,

$$(a + b)(a^2 + b^2) - 4acd \geq 2ab(a + b) - 4ab^2 = 2ab(a - b) \geq 0.$$

For fixed  $a$  and  $d$ , we have

$$G(c) = F(a, a, c, d) = -c^4 + 4a^2cd + a^4 - 6a^2d^2 + 2d^4, \quad c \in [d, a].$$

Since  $G$  is concave, it is enough to show that  $G(a) \geq 0$  and  $G(d) \geq 0$ . We have

$$G(a) = 2d(a - d)^2(2a + d) \geq 0,$$

$$G(d) = (a^2 - d^2)^2 \geq 0.$$

**Third Solution.** Write the inequality in the homogeneous form

$$a^2 + b^2 + c^2 + d^2 - 4\sqrt{abcd} \leq \frac{3}{4}(a - d)^2.$$

As known (see Remark from the proof of P 2.104), for fixed  $a$  and  $d$ , if  $b, c \in [d, a]$ , then Jensen's difference

$$a^2 + b^2 + c^2 + d^2 - 4\sqrt{abcd}$$

has the maximum value when  $b, c \in \{a, d\}$ , as shown in the first solution.

**Remark.** The following more general statement holds for  $n \leq 6$ :

- If  $a_1, a_2, \dots, a_n$  ( $n \leq 6$ ) are nonnegative real numbers such that

$$a_1^2 + a_2^2 + \dots + a_n^2 = n, \quad a_1 \geq a_2 \geq \dots \geq a_n.$$

then

$$1 - \sqrt{a_1 a_2 \cdots a_n} \leq \left(1 - \frac{1}{n}\right) (a_1 - a_n)^2,$$

with equality for  $a_1 = a_2 = \cdots = a_n = 1$ , and also for  $a_1 = \cdots = a_{n-1} = \sqrt{\frac{n}{n-1}}$  and  $a_n = 0$ .

To prove this inequality, we need to show that

$$1 - \sqrt{a^k b^{n-k}} \leq \left(1 - \frac{1}{n}\right) (a - b)^2$$

for  $a \geq 1 \geq b \geq 0$  and

$$ka^2 + (n-k)b^2 = n, \quad k \in \{1, 2, \dots, n-1\}.$$

□

**P 3.171.** If  $a, b, c, d$  are nonnegative real numbers such that  $a \geq b \geq c \geq d$  and

$$a^2 + b^2 + c^2 + d^2 = 4,$$

then

$$1 - (abcd)^{3/4} \leq \frac{3}{4}(a - d)^2.$$

(Vasile Cîrtoaje, 2020)

**Solution.** For  $a = d$ , the inequality is a trivial equality. Next assume  $a > d$ , replace the assumption  $a \geq b \geq c \geq d$  with  $b, c \in [d, a]$  and write the inequality in the homogeneous form  $f \geq 0$ , where

$$f = \frac{4(abcd)^{3/4}}{g^{1/2}} + 3(a - d)^2 - (a^2 + b^2 + c^2 + d^2),$$

where

$$g = \frac{a^2 + b^2 + c^2 + d^2}{4}.$$

For fixed  $a, c$  and  $d$ ,  $f$  is a function of  $b$ ,  $b \in [d, a]$ . We have

$$g'(b) = \frac{b}{2}$$

and

$$\begin{aligned} f'(b) &= \frac{3(acd)^{3/4}}{b^{1/4}g^{1/2}} - \frac{2(abcd)^{3/4}g'}{g^{3/2}} - 2b \\ &= \frac{3(acd)^{3/4}}{b^{1/4}g^{1/2}} - \frac{b^{7/4}(acd)^{3/4}}{g^{3/2}} - 2b = \frac{(acd)^{3/4}g(b)}{b^{1/4}h^{1/2}}, \end{aligned}$$

where

$$h(b) = 3 - \frac{1}{1 + \frac{a^2 + c^2 + d^2}{b^2}} - 2b^{5/4}g^{1/2}.$$

Since  $h(b)$  is a decreasing function, there are three possible cases: (1)  $h(b) \geq 0$  for  $b \in [d, a]$ , hence  $f(b)$  is increasing on  $[d, a]$ ; (2)  $h(b) \geq 0$  for  $b \in [d, d_1]$  and  $h(b) \leq 0$  for  $b \in [d_1, a]$ , hence  $f(b)$  is increasing on  $[d, d_1]$  and decreasing on  $[d_1, a]$ ; (3)  $h(b) \leq 0$  for  $b \in [d, a]$ , hence  $f(b)$  is decreasing on  $[d, a]$ . In all these cases, if the inequality  $f(b)$  holds for the extreme values of  $b$ , then it holds for all possible values of  $b$ . Thus, we only need to prove the required inequality for  $b \in \{a, d\}$ . Similarly, we only need to prove the required inequality for  $c \in \{a, d\}$ . So, we need to show that  $f \geq 0$  for  $b, c \in \{a, d\}$ , that is

$$\frac{8(a^k d^{4-k})^{3/4}}{\sqrt{ka^2 + (4-k)d^2}} + 3(a-d)^2 - ka^2 - (4-k)d^2 \geq 0,$$

or

$$8(a^k d^{4-k})^{3/4} \geq [(k-3)a^2 + 6ad + (1-k)d^2] \sqrt{ka^2 + (4-k)d^2},$$

where

$$k \in \{1, 2, 3\}.$$

For  $d = 0$ , the inequality reduces to

$$(3-k)a^2 \geq 0,$$

which is true for  $k \in \{1, 2, 3\}$ . Next, due to homogeneity, we may set  $d = 1$ . So, we need to show that  $a \geq 1$  yields

$$8a^{3k/4} \geq [(k-3)a^2 + 6a + 1-k] \sqrt{ka^2 + 4-k}.$$

This is true if

$$64a^{3k/2} \geq [(k-3)a^2 + 6a + 1-k]^2 (ka^2 + 4-k)$$

for  $a \geq 1$  and  $(k-3)a^2 + 6a + 1-k \geq 0$ .

*Case 1:*  $k = 1$ . We need to show that

$$16 \geq \sqrt{a}(3-a)^2(a^2+3)$$

for  $1 \leq a \leq 3$ . Since  $2\sqrt{a} \leq a+1$ , it suffices to show that

$$32 \geq (a+1)(a-3)^2(a^2+3),$$

which is equivalent to

$$\begin{aligned} a^5 - 5a^4 + 6a^3 - 6a^2 + 9a - 5 &\leq 0, \\ (a-1)^2[a^2(a-3) - a - 5] &\leq 0. \end{aligned}$$

*Case 2:*  $k = 2$ . We need to show that

$$32a^3 \geq (-a^2 + 6a - 1)^2(a^2 + 1)$$

for  $1 \leq a \leq 3 + \sqrt{10} < 7$ . Write the required inequality as follows:

$$a^6 - 12a^5 + 39a^4 - 56a^3 + 39a^2 - 12a + 1 \leq 0,$$

$$(a - 1)^4(a^2 - 8a + 1) \leq 0.$$

It is true because

$$a^2 - 8a + 1 = -a(7 - a) - (a - 1) < 0.$$

*Case 3:  $k = 3$ .* We need to show that

$$16a^{9/2} \geq (3a - 1)^2(a^2 + 1)$$

for  $a \geq 1$ . Since

$$a^{9/2} \geq \frac{2a^5}{a + 1},$$

it suffices to show that

$$32a^5 \geq (a + 1)(3a - 1)^2(3a^2 + 1),$$

which is equivalent to

$$5a^5 - 9a^4 + 6a^3 - 6a^2 + 5a - 1 \geq 0,$$

$$(a - 1)^2(5a^3 + a^2 + 3a - 1) \geq 0.$$

The proof is completed. The equality occurs for  $a = b = c = d = 1$ , and also for  $a = b = c = \frac{2}{\sqrt{3}}$  and  $d = 0$ .

**Remark.** Since  $abcd \leq 1$ , the inequality in P 3.171 is stronger than the inequality in the preceding P 3.170. In addition, from P 3.171, we get the nice inequality

$$1 - abcd \leq (a - d)^2.$$

To prove this last inequality, it suffices to show that

$$1 - abcd \leq \frac{4}{3} (1 - (abcd)^{3/4}).$$

Denoting  $x = (abcd)^{1/4}$ , the inequality becomes

$$3(1 - x^4) \leq 4(1 - x^3),$$

i.e.

$$(1 - x)^2(1 + 2x + 3x^2) \geq 0.$$

□



**P 3.172.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

If  $a \geq b \geq c \geq d$ , then

$$(a) \quad 1 - \sqrt{abcd} \geq \frac{1}{2}(b - c)^2;$$

$$(b) \quad 1 - \sqrt{abcd} \geq \frac{1}{4}(a - d)^2.$$

(Vasile Cîrtoaje, 2019)

**Proof.** (a) Since

$$b - c \leq \sqrt{ab} - \sqrt{cd},$$

it suffices to show that

$$1 - \sqrt{abcd} \geq \frac{1}{2}(\sqrt{ab} - \sqrt{cd})^2,$$

which is equivalent to

$$a^2 + b^2 + c^2 + d^2 - 4\sqrt{abcd} \geq 2ab + 2cd - 4\sqrt{abcd},$$

$$(a - b)^2 + (c - d)^2 \geq 0.$$

The equality occurs for

$$a = b = x, \quad c = d = y,$$

where  $x$  and  $y$  are nonnegative numbers such that  $x \geq y$  and  $x^2 + y^2 = 2$ .

(b) Write the inequality as follows:

$$a^2 + b^2 + c^2 + d^2 - 4\sqrt{abcd} \geq (a - d)^2,$$

$$b^2 + c^2 + 2ad \geq 4\sqrt{abcd},$$

$$(b - c)^2 + 2(\sqrt{bc} - \sqrt{ad})^2 \geq 0.$$

The equality occurs for

$$b = c = \sqrt{ad}, \quad a + d = 2.$$

□

**P 3.173.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

If  $a \geq b \geq c \geq d$ , then

$$1 - abcd \geq \frac{3}{4}(c - d)^2.$$

(Vasile Cîrtoaje, 2019)

**Solution.** Write the inequality in the homogeneous form

$$(a^2 + b^2 + c^2 + d^2)^2 - 16abcd \geq 3(c - d)^2(a^2 + b^2 + c^2 + d^2),$$

$$(a^2 + b^2 + c^2 + d^2)(a^2 + b^2 + 6cd - 2c^2 - 2d^2) - 16abcd \geq 0.$$

Denoting

$$x = \sqrt{ab}, \quad x \geq b \geq c,$$

since  $a^2 + b^2 \geq 2x^2$ , it is enough to show that  $F(x) \geq 0$ , where

$$F(x) = (2x^2 + c^2 + d^2)(x^2 + 3cd - c^2 - d^2) - 8x^2cd.$$

We will show that

$$F(x) \geq F(c) \geq 0.$$

We have

$$\begin{aligned} F(x) - F(c) &= 2(x^4 - c^4) + (6cd - c^2 - d^2)(x^2 - c^2) - 8cd(x^2 - c^2) \\ &= (x^2 - c^2)(2x^2 + c^2 - d^2 - 2cd) \geq (x^2 - c^2)(3c^2 - d^2 - 2cd) \\ &= (x^2 - c^2)(c - d)(3c + d) \geq 0 \end{aligned}$$

and

$$F(c) = (2c^2 + d^2)(3cd - d^2) - 8c^3d = d(c - d)^3 \geq 0.$$

The proof is completed. The equality occurs for  $a = b = c = d = 1$ , and also for  $a = b = c = \frac{2}{\sqrt{3}}, d = 0$ . □

**P 3.174.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

If  $a \geq b \geq c \geq d$ , then

$$(a) \quad 1 - abcd \geq (a - b)(c - d);$$

$$(b) \quad 1 - abcd \geq \frac{1 + \sqrt{3}}{2} (a - b)(c - d).$$

(Vasile Cîrtoaje, 2019)

**Solution.** (a) After homogenization, we need to prove that (*kiyoras\_2001*)

$$a^2 + b^2 + c^2 + d^2 - 4\sqrt{abcd} \geq 4(a - b)(c - d),$$

which is equivalent to

$$(a - b)^2 + (c - d)^2 + 2(ab + cd) - 4\sqrt{abcd} \geq 4(a - b)(c - d),$$

$$(a - b - c + d)^2 + 2(ab + cd) - 4\sqrt{abcd} \geq 2(a - b)(c - d).$$

Thus, it suffices to prove that

$$ab + cd - 2\sqrt{abcd} \geq ac + bd - ad - bc,$$

that is

$$\begin{aligned} (ad + bc - 2\sqrt{abcd} + (ab + cd - ac - bd)) &\geq 0, \\ \left(\sqrt{ad} - \sqrt{bc}\right)^2 + (a - d)(b - c) &\geq 0, \end{aligned}$$

which is true. The equality occurs for  $a = b = c = d = 1$ .

(b) Denote

$$k = \frac{1 + \sqrt{3}}{2},$$

and write the inequality in the homogeneous form  $f(a, b, c, d) \geq 0$ , where

$$f(a, b, c, d) = a^2 + b^2 + c^2 + d^2 - 4k(a - b)(c - d) - \frac{16abcd}{a^2 + b^2 + c^2 + d^2}.$$

We will show that

$$f(a, b, c, d) \geq f(a, z, z, d) \geq 0,$$

where

$$z = \frac{b + c}{2}.$$

We have

$$f(a, b, c, d) - f(a, z, z, d) = A + 4kB + 16adC,$$

where

$$\begin{aligned} A &= b^2 + c^2 - 2z^2 = \frac{1}{2}(b - c)^2 \geq 0, \\ B &= (a - z)(z - d) - (a - b)(c - d) = (a + d)z - ac - bd - (z^2 - bc) \\ &= \frac{1}{2}(b - c)(a - d) - \frac{1}{4}(b - c)^2 = \frac{1}{4}(b - c)(2a - b + c - 2d) \\ &\geq \frac{1}{4}(b - c)(b + c - 2d) \geq 0, \\ C &= \frac{z^2}{a^2 + 2z^2 + d^2} - \frac{bc}{a^2 + b^2 + c^2 + d^2} \\ &= \frac{(a^2 + d^2)(z^2 - bc) + z^2(b - c)^2}{(a^2 + 2z^2 + d^2)(a^2 + b^2 + c^2 + d^2)} \geq 0. \end{aligned}$$

Since  $A \geq 0$ ,  $B \geq 0$  and  $C \geq 0$ , the inequality  $f(a, b, c, d) \geq f(a, z, z, d)$  is true. Further, we need to show that  $a \geq z \geq c$  yields  $f(a, z, z, d) \geq 0$ , that is

$$a^2 + 2z^2 + d^2 - \frac{16az^2d}{a^2 + 2z^2 + d^2} \geq 4k(a - z)(z - d).$$

Denote

$$x = ad, \quad y = \frac{a^2 + d^2}{2}.$$

Since

$$(a - z)(z - d) = z(a + d) - ad - z^2 = z\sqrt{2(x + y)} - x - z^2,$$

the inequality is equivalent to

$$y + z^2 - \frac{4xz^2}{y + z^2} \geq 2k \left[ z\sqrt{2(x + y)} - x - z^2 \right].$$

For  $z = 0$ , which involves  $x = 0$ , the inequality is clearly true. For  $z > 0$ , due to homogeneity, we may consider  $z = 1$ . From

$$0 \leq (a - z)(z - d) = \sqrt{2(x + y)} - x - 1,$$

it follows

$$x \leq \sqrt{2y - 1}, \quad y \geq \frac{1}{2}.$$

So, we need to show that for fixed  $y \in \left[\frac{1}{2}, \infty\right)$ , we have  $f(x) \geq 0$ , where

$$f(x) = y + 1 - \frac{4x}{y + 1} - (1 + \sqrt{3}) \left[ \sqrt{2(x + y)} - x - 1 \right], \quad 0 \leq x \leq \sqrt{2y - 1}.$$

There are two cases to consider:  $\frac{1}{2} \leq y \leq 2$  and  $y \geq 2$

*Case 1:*  $\frac{1}{2} \leq y \leq 2$ . Since  $2k < 3$ , it suffices to show that

$$y + 1 - \frac{4x}{y + 1} \geq 3 \left[ \sqrt{2(x + y)} - x - 1 \right].$$

In addition, by Bernoulli's Inequality, we have

$$\begin{aligned} \sqrt{2(x + y)} &= 2\sqrt{\frac{x + y}{2}} = 2\sqrt{1 + \frac{x + y - 2}{2}} \\ &\leq 2 \left( 1 + \frac{x + y - 2}{4} \right) = \frac{x + y + 2}{2}. \end{aligned}$$

Thus, it is enough to show that

$$y + 1 - \frac{4x}{y + 1} \geq 3 \left( \frac{x + y + 2}{2} - x - 1 \right),$$

that is

$$(2 - y)(y + 1) \geq (5 - 3y)x.$$

For the nontrivial case  $\frac{1}{2} \leq y \leq \frac{5}{3}$ , it suffices to show that

$$(2 - y)(y + 1) \geq (5 - 3y)\sqrt{2y - 1}.$$

By squaring, the inequality becomes

$$y^4 - 20y^3 + 66y^2 - 76y + 29 \geq 0,$$

$$(y - 1)^2(y^2 - 18y + 29) \geq 0.$$

It is true because

$$y^2 - 18y + 29 = (y - 3)^2 + 4(5 - 3y) > 0.$$

*Case 2:*  $y \geq 2$ . Since  $x + y \geq y \geq 2$ , we have

$$\begin{aligned} f'(x) &= \frac{-4}{y+1} + (1 + \sqrt{3}) \left( 1 - \frac{1}{\sqrt{2(x+y)}} \right) \\ &\geq \frac{-4}{3} + \frac{1}{2}(1 + \sqrt{3}) = \frac{3\sqrt{3} - 5}{6} > 0. \end{aligned}$$

Therefore,  $f$  is increasing, hence

$$f(x) \geq f(0) = y + 1 - (1 + \sqrt{3}) (\sqrt{2y} - 1) = \frac{1}{2} (\sqrt{2y} - 1 - \sqrt{3})^2 \geq 0.$$

The proof is completed. The equality occurs for  $a = b = c = d = 1$ , and also for

$$a = \sqrt{2 \left( 1 + \frac{1}{\sqrt{3}} \right)}, \quad b = c = \sqrt{1 - \frac{1}{\sqrt{3}}}, \quad d = 0.$$

□

**P 3.175.** Let  $a, b, c, d$  be nonnegative real numbers such that  $a^2 + b^2 + c^2 + d^2 = 4$ . If  $a \geq b \geq c \geq d$ , then

$$1 - abcd \geq 3(a - b)(b - c)(c - d)(a - d).$$

(Vasile Cîrtoaje, 2019)

**Solution.** Write the inequality in the homogeneous form

$$(a^2 + b^2 + c^2 + d^2)^2 - 16abcd - 48(a - b)(b - c)(c - d)(a - d) \geq 0.$$

Using the substitution

$$a = x + d, \quad b = y + d, \quad c = z + d, \quad x \geq y \geq z \geq 0,$$

for fixed  $x, y, z$ , we may write the inequality as  $f(d) \geq 0$ . Since

$$a' = b' = c' = d' = 1,$$

we have

$$\frac{1}{4}f'(d) = (a + b + c + d)(a^2 + b^2 + c^2 + d^2) - 4(bcd + acd + abd + abc).$$

Since

$$\begin{aligned} \left(\sum_{cyc} a\right) \left(\sum_{cyc} a^2\right) &\geq \frac{2}{3} \left(\sum_{cyc} a\right) \left(\sum_{sym} ab\right) = 2 \sum_{cyc} abc + \frac{2}{3} \sum_{cyc} a(b^2 + c^2 + d^2) \\ &\geq 2 \sum_{cyc} abc + \frac{2}{3} \sum_{cyc} a(bc + cd + da) = 4 \sum_{cyc} abc, \end{aligned}$$

we have  $f'(d) \geq 0$ . Therefore,  $f(d)$  is increasing and hence  $f(d) \geq f(0)$ . So, it is enough to show that  $f(0) \geq 0$ , that is

$$(a^2 + b^2 + c^2)^2 \geq 48(a - b)(b - c)ac.$$

Since

$$a^2 + b^2 + c^2 \geq a^2 + b^2$$

and

$$4(b - c)c \leq b^2,$$

it suffices to show that

$$(a^2 + b^2)^2 \geq 12ab^2(a - b),$$

which is equivalent to

$$\begin{aligned} a^4 - 10a^2b^2 + 12ab^3 + b^4 &\geq 0, \\ (a - 2b)^2(a - b)(a + 5b) + 7a^2 - 24ab + 21b^2 &\geq 0. \\ (a - 2b)^2(a - b)(a + 5b) + 7\left(a - \frac{12}{7}b\right)^2 + \frac{3}{7}b^2 &\geq 0. \end{aligned}$$

The proof is completed. The equality occurs for  $a = b = c = d = 1$ .

□

**P 3.176.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

If  $a \geq b \geq c \geq d$ , then

$$\begin{aligned} (a) \quad 1 - \sqrt{abcd} &\geq \frac{1}{3}(b - d)^2; \\ (b) \quad 1 - (abcd)^{3/4} &\geq \frac{1}{2}(b - d)^2. \end{aligned}$$

(Vasile Cîrtoaje, 2019)

**Solution.** (a) Write the inequality in the homogeneous form

$$3(a^2 + b^2 + c^2 + d^2)^2 - 12\sqrt{abcd} \geq 4(b-d)^2,$$

or  $F(a, b, c, d) \geq 0$ , where

$$F(a, b, c, d) = 3a^2 - b^2 + 3c^2 - d^2 + 8bd - 12\sqrt{abcd}.$$

We will show that

$$F(a, b, c, d) \geq F(b, b, c, d) \geq 0.$$

We have

$$\begin{aligned} F(a, b, c, d) - F(b, b, c, d) &= 3(a^2 - b^2) - 12\sqrt{bcd}(\sqrt{a} - \sqrt{b}) \\ &= 3(\sqrt{a} - \sqrt{b}) \left[ (\sqrt{a} + \sqrt{b})(a + b) - 4\sqrt{bcd} \right] \geq 0. \end{aligned}$$

In addition,

$$\begin{aligned} F(b, b, c, d) &= 2b^2 + 4(2d - 3\sqrt{cd})b + 3c^2 - d^2 \\ &= 2(b + 2d - 3\sqrt{cd})^2 + 3(c^2 - 6cd + 8d\sqrt{cd} - 3d^2) \\ &= 2(b + 2d - 3\sqrt{cd})^2 + 3(\sqrt{c} - \sqrt{d})^3(\sqrt{c} + 3\sqrt{d}) \geq 0. \end{aligned}$$

The equality occurs for  $a = b = c = d = 1$ .

(b) Let us denote

$$g = \sqrt{\frac{a^2 + b^2 + c^2 + d^2}{4}}, \quad h = \sqrt[4]{abcd},$$

which satisfy

$$a \geq g \geq h.$$

For fixed  $b, c, d$ , write the inequality in the homogeneous form  $f(a) \geq 0$ , where

$$f(a) = a^2 + b^2 + c^2 + d^2 - \frac{4h^3}{g} - 2(b-d)^2.$$

Since

$$g'(a) = \frac{a}{4g}, \quad h'(a) = \frac{h}{4a},$$

we have

$$\begin{aligned} f'(a) &= 2a - \frac{12h^2h'}{g} + \frac{4h^3g'}{g^2} = 2a - \frac{3h^3}{ag} + \frac{ah^3}{g^3} \\ &= \frac{a^2(2g^3 + h^3) - 3g^2h^3}{ag^3} \geq \frac{g^2(2g^3 + h^3) - 3g^2h^3}{ag^3} = \frac{2(g^3 - h^3)}{ag} \geq 0. \end{aligned}$$

hence  $f$  is increasing, therefore  $f(a) \geq f(b)$ . So, we only need to prove that  $f(b) \geq 0$ , that is

$$2b^2 + c^2 + d^2 - \frac{8(b^2cd)^{3/4}}{\sqrt{2b^2 + c^2 + d^2}} - 2(b-d)^2 \geq 0,$$

$$(4bd + c^2 - d^2)\sqrt{2b^2 + c^2 + d^2} \geq 8(b^2cd)^{3/4}.$$

Due to homogeneity, we may set (for the nontrivial case  $d > 0$ )

$$b = \frac{1}{cd},$$

when the inequality becomes

$$\left(c^2 - d^2 + 4\sqrt{\frac{d}{c}}\right) \sqrt{c^2 + d^2 + \frac{2}{cd}} \geq 8.$$

Denoting

$$d = cx^2, \quad 0 \leq x \leq 1,$$

we need to show that

$$(c^2 - c^2x^4 + 4x)\sqrt{c^2 + c^2x^4 + \frac{2}{c^2x^2}} \geq 8,$$

or, by squaring,

$$(c^2 - c^2x^4 + 4x)^2 \left(c^2 + c^2x^4 + \frac{2}{c^2x^2}\right) \geq 8,$$

that is

$$\begin{aligned} & x^2(1 - x^4)^2(1 + x^4)c^8 + 8x^3(1 - x^4)(1 + x^4)c^6 + 2[(1 - x^4)^2 + 8x^4(1 + x^4)]c^4 + \\ & + 16x(1 - x^4 - 4x)c^2 + 32x^2 \geq 0. \end{aligned}$$

Write the inequality in the form

$$(1 + x^4) [x(1 - x^4)c^4 + 4x^2c^2 - 4x]^2 + Ac^4 + Bc^2 + C,$$

where

$$A = 2(1 - x^4)(4x^6 - x^4 + 4x^2 + 1) \geq 0,$$

$$B = 16x(2x^6 - x^4 + 2x^2 - 4x + 1),$$

$$C = 16x^2(1 - x^4) \geq 0.$$

The inequality is true for  $0 \leq x \leq \frac{1}{4}$  because

$$B \geq 16x[x^2(1 - x^2) + 1 - 4x] > 0.$$

For  $\frac{1}{4} \leq x \leq 1$ , since

$$Ac^4 + Bc^2 + C \geq (2\sqrt{AC} + B)c^2,$$

it suffices to show that  $2\sqrt{AC} + B \geq 0$ . This is true if  $4AC \geq B^2$ , that is

$$4x^{14} - 9x^{12} + 4x^{10} - 15x^8 + 32x^7 - 4x^6 - 16x^5 - 7x^4 + 32x^3 - 36x^2 + 16x - 1 \geq 0,$$



which is equivalent to

$$(x-1)^4 f(x) \geq 0,$$

where

$$f(x) = 4x^{10} + 16x^9 + 31x^8 + 44x^7 + 54x^6 + 60x^5 + 46x^4 + 28x^3 + 18x^2 + 12x - 1 \geq 0.$$

The equality occurs for  $a = b = c = d = 1$ , and also for  $a = b = \sqrt{2}$  and  $c = d = 0$ .

□

**P 3.177.** Let  $a, b, c, d$  be nonnegative real numbers such that

$$a^4 + b^4 + c^4 + d^4 = 4.$$

If  $a \geq b \geq c \geq d$ , then

$$\begin{aligned} (a) \quad & 1 - \sqrt{abcd} \geq \frac{1}{2}(ac - bd)^2; \\ (b) \quad & 1 - abcd \geq \frac{1}{\sqrt{2}}(ac - bd)^2. \end{aligned}$$

(Vasile Cîrtoaje, 2019)

**Solution.** Firstly, we show that

$$a^4 c^4 + b^4 d^4 \leq 2. \quad (*)$$

We have:

$$\begin{aligned} 2 - a^4 c^4 - b^4 d^4 &= 2 + (a^4 - b^4)(b^4 - c^4) - b^4(a^4 + c^4 + d^4 - b^4) \\ &= 2 + (a^4 - b^4)(b^4 - c^4) - b^4(4 - 2b^4) \\ &= (a^4 - b^4)(b^4 - c^4) + 2(b^4 - 1)^2 \geq 0. \end{aligned}$$

(a) Using the notation

$$x = \sqrt{ac}, \quad y = \sqrt{bd},$$

and having in view (\*), it suffices to show that  $x^8 + y^8 \leq 2$  yields

$$2 - 2xy \geq (x^2 - y^2)^2,$$

that is

$$2(1 - xy + x^2 y^2) \geq x^4 + y^4.$$

By squaring, the inequality becomes

$$4(1 - xy + x^2 y^2)^2 \geq x^8 + y^8 + 2x^4 y^4.$$

This is true if

$$2(1 - xy + x^2 y^2)^2 \geq 1 + x^4 y^4,$$

that is equivalent to

$$(1 - xy)^4 \geq 0.$$

The equality occurs for  $a = b = c = d = 1$ .

(b) **First Solution.** Using the notation

$$x = ac, \quad y = bd,$$

and having in view (\*), it suffices to show that  $x^4 + y^4 \leq 2$  yields

$$\sqrt{2} (1 - xy) \geq (x - y)^2,$$

that is

$$\sqrt{2} + \sqrt{2}(\sqrt{2} - 1)xy \geq x^2 + y^2.$$

Since

$$x^2 + y^2 = \sqrt{x^4 + y^4 + 2x^2y^2} \leq \sqrt{2 + 2x^2y^2},$$

we only need to show that

$$1 + (\sqrt{2} - 1)xy \geq \sqrt{1 + x^2y^2}.$$

By squaring, the inequality becomes

$$xy(1 - xy) \geq 0.$$

This is true because

$$2 \geq x^4 + y^4 \geq 2x^2y^2.$$

The equality occurs for  $a = b = c = d = 1$ , and also for  $a = \sqrt[4]{2}$ ,  $b = c = 1$  and  $d = 0$ .

**Second Solution** (by *kiyoras\_2001*). Write the inequality in the homogeneous form  $f(a, b, c, d) \geq 0$ , where

$$f(a, b, c, d) = a^4 + b^4 + c^4 + d^4 - 2\sqrt{2}(a^2c^2 + b^2d^2) + 4(\sqrt{2} - 1)abcd.$$

We will show that

$$f(a, b, c, d) \geq f(a, c, c, d).$$

We have:

$$\begin{aligned} f(a, b, c, d) - f(a, c, c, d) &= b^4 - c^4 - 2\sqrt{2}d^2(b^2 - c^2) + 4(\sqrt{2} - 1)acd(b - c) \\ &= (b - c) \left[ (b + c)(b^2 + c^2) - 2\sqrt{2}d^2(b + c) + 4(\sqrt{2} - 1)acd \right] \\ &\geq (b - c) \left[ 2d^2(b + c) - 2\sqrt{2}d^2(b + c) + 2(\sqrt{2} - 1)(b + c)d^2 \right] = 0. \end{aligned}$$

Further, write the inequality  $f(a, c, c, d) \geq 0$  as

$$2c^4 + \left[ 4(\sqrt{2} - 1)ad - 2\sqrt{2}(a^2 + d^2) \right] c^2 + a^4 + d^4 \geq 0.$$

Since

$$2c^4 + a^4 + d^4 \geq 2c^2 \sqrt{2(a^4 + d^4)} = 2c^2 \sqrt{(2(a^2 + d^2))^2 - 4a^2 d^2},$$

we only need to show that

$$\sqrt{2(a^2 + d^2)^2 - 4a^2 d^2} \geq \sqrt{2}(a^2 + d^2) - 2(\sqrt{2} - 1)ad.$$

This is true because, by squaring, it becomes

$$ad(a - d)^2 \geq 0.$$

**Third Solution.** Write the inequality in the homogeneous form

$$a^4 + b^4 + c^4 + d^4 - 4abcd - 2\sqrt{2} (ac - bd)^2 \geq 0,$$

and use the substitution

$$a = x + d, \quad b = y + d, \quad c = z + d, \quad x \geq y \geq z \geq 0.$$

For fixed  $x, y, z$ , we may write the inequality as  $f(d) \geq 0$ . Since

$$a' = b' = c' = d' = 1,$$

we have

$$\frac{1}{4}f'(d) = a^3 + b^3 + c^3 + d^3 - (abc + bcd + cda + dab) - \sqrt{2} (ac - bd)(a + c - b - d),$$

$$\frac{1}{4}f''(d) = 3(a^2 + b^2 + c^2 + d^2) - 2 \sum_{sym} ab - \sqrt{2} (a + c - b - d)^2.$$

Since

$$\begin{aligned} & 3(a^2 + b^2 + c^2 + d^2) - 2 \sum_{sym} ab - 2(a + c - b - d)^2 = \\ & = (a - b - c + d)^2 + 4(a - d)(b - c) \geq 0, \end{aligned}$$

it follows that  $f'' \geq 0$ ,  $f'$  is increasing, hence

$$\begin{aligned} \frac{1}{4}f'(d) & \geq \frac{1}{4}f'(0) = a^3 + b^3 + c^3 - abc - \sqrt{2} ac(a + c - b) \\ & = a^3 + b^3 + c^3 + (\sqrt{2} - 1)abc - \sqrt{2} ac(a + c) \\ & \geq a^3 + 2c^3 + (\sqrt{2} - 1)ac^2 - \sqrt{2} ac(a + c) \\ & = a(a - c)^2 + (2 - \sqrt{2})a^2c - 2ac^2 + 2c^3 \\ & \geq a(a - c)^2 + \frac{1}{2}a^2c - 2ac^2 + 2c^3 = a(a - c)^2 + \frac{1}{2}c(a - 2c)^2 \geq 0. \end{aligned}$$

Since  $f' \geq 0$ ,  $f$  is increasing, therefore

$$\begin{aligned} f(d) & \geq f(0) = a^4 + b^4 + c^4 - 2\sqrt{2} a^2 c^2 \geq a^4 + 2c^4 - 2\sqrt{2} a^2 c^2 \\ & = (a^2 - \sqrt{2} c^2)^2 \geq 0. \end{aligned}$$

□

**P 3.178.** If  $a, b, c, d$  are nonnegative real numbers such that  $a \geq b \geq c \geq d$  and

$$a^4 + b^4 + c^4 + d^4 = 4,$$

then

$$1 - abcd \geq \frac{3}{4}(ad - bc)^2.$$

(Vasile Cîrtoaje, 2020)

**Solution.** Write the inequality in the homogeneous form  $f(a, b, c, d) \geq 0$ , where

$$f(a, b, c, d) = a^4 + b^4 + c^4 + d^4 + 2abcd - 3(a^2d^2 + b^2c^2).$$

We will show that

$$f(a, b, c, d) \geq f(b, b, c, d) \geq f(c, c, c, d) \geq 0.$$

Since

$$f(a, b, c, d) - f(b, b, c, d) = (a^4 - b^4) + 2bcd(a - b) - 3d^2(a^2 - b^2),$$

we have  $f(a, b, c, d) \geq f(b, b, c, d)$  if

$$a^2 + b^2 + \frac{2bcd}{a + b} - 3d^2 \geq 0,$$

which is true if

$$a^2 + b^2 + \frac{bcd}{a} - 3d^2 \geq 0.$$

We have

$$\begin{aligned} a^2 + b^2 + \frac{bcd}{a} - 3d^2 &\geq a^2 + \frac{bcd}{a} - 2d^2 \\ &\geq 2\sqrt{abcd} - 2d^2 \geq 0. \end{aligned}$$

The inequality  $f(b, b, c, d) \geq f(c, c, c, d)$  is equivalent to

$$2(b^4 - c^4) + 2cd(b^2 - c^2) - 3(c^2 + d^2)(b^2 - c^2) \geq 0,$$

which is true if

$$2(b^2 + c^2) + 2cd - 3(c^2 + d^2) \geq 0.$$

Indeed,

$$\begin{aligned} 2(b^2 + c^2) + 2cd - 3(c^2 + d^2) &\geq 2(c^2 + c^2) + 2cd - 3(c^2 + d^2) \\ &= (c - d)(c + 3d) \geq 0. \end{aligned}$$

Finally, we have

$$\begin{aligned} f(c, c, c, d) &= 3c^4 + d^4 + 2c^3d - 3c^2(c^2 + d^2) = d(2c^3 - 3c^2d + d^3) \\ &= d(c - d)^2(2c + d) \geq 0. \end{aligned}$$

The proof is completed. The equality occurs for  $a = b = c = d = 1$ , and also for  $a = b = c = \sqrt[4]{\frac{4}{3}}$  and  $d = 0$ .

□

**P 3.179.** If  $a, b, c, d$  are nonnegative real numbers such that  $a \geq b \geq c \geq d$  and

$$a + b + c + d = 4,$$

then

$$(a) \quad \frac{a^4 + b^4 + c^4 + d^4}{4} - abcd \geq 2(b - c)^2;$$

$$(b) \quad \frac{a^4 + b^4 + c^4 + d^4}{4} - abcd \geq \frac{3}{2}(a - b)^2;$$

$$(c) \quad \frac{a^4 + b^4 + c^4 + d^4}{4} - abcd \geq \frac{4}{3}(a - c)^2;$$

$$(d) \quad \frac{a^4 + b^4 + c^4 + d^4}{4} - abcd \geq \frac{4}{3}(c - d)^2.$$

(Vasile Cîrtoaje, 2020)

**Solution.** (a) For fixed  $b, c, d$ . write the left inequality in the homogeneous form  $f(a) \geq 0$ , where

$$f(a) = 2(a^4 + b^4 + c^4 + d^4) - 8abcd - (b - c)^2(a + b + c + d)^2.$$

Since

$$\frac{1}{2}f'(a) = 4a^3 - 4bcd - (b - c)^2(a + b + c + d) \geq 4a^3 - 4acd - (a - c)^2(2a + c + d)$$

$$\geq 4a^3 - 4ac^2 - (a - c)^2(2a + 2c) = 2(a - c)(a^2 + 2ac + c^2) \geq 0,$$

$f(a)$  is increasing, hence

$$f(a) \geq f(b) = 2(2b^4 + c^4 + d^4) - 8b^2cd - (b - c)^2(2b + c + d)^2.$$

So, we need to show that  $g(d) \geq 0$ , where

$$g(d) = 2(2b^4 + c^4 + d^4) - 8b^2cd - (b - c)^2(2b + c + d)^2.$$

Since

$$\frac{1}{2}g'(d) = 4(d^3 - b^2c) - (b - c)^2(2b + c + d) \leq 0,$$

$g(d)$  is decreasing, hence

$$g(d) \geq g(c) = 4(b^4 + c^4) - 8b^2c^2 - 4(b - c)^2(b + c)^2 = 0.$$

The equality occurs for  $a = b, c = d, a + c = 2$ .

(b) Write the inequality as

$$f(a, b, c, d) \geq \frac{3}{8}(a - b)^2(a + b + c + d)^2,$$

where  $f(a, b, c, d) = a^4 + b^4 + c^4 + d^4 - 4abcd$ . Since

$$\begin{aligned} f(a, b, c, d) - f(a, b, c, c) &= -(c^4 - d^4) + 4abc(c - d) \\ &= (c - d)(-c^3 - c^2d - cd^2 - d^3 + 4abc) \geq 0, \end{aligned}$$

it suffices to show that

$$f(a, b, c, c) \geq \frac{3}{8}(a - b)^2(a + b + 2c)^2.$$

Since

$$\begin{aligned} f(a, b, c, c) - f(a, b, b, b) &= -2(b^4 - c^4) + 4ab(b^2 - c^2) \\ &= 2(b^2 - c^2)(2ab - b^2 - c^2) \geq 0, \end{aligned}$$

it suffices to show that

$$f(a, b, b, b) \geq \frac{3}{8}(a - b)^2(a + 3b)^2,$$

which is equivalent to

$$\begin{aligned} 8(a - b)^2(a^2 + 2ab + 3b^2) &\geq 3(a - b)^2(a + 3b)^2, \\ (a - b)^3(5a + 3b) &\geq 0. \end{aligned}$$

The equality occurs for  $a = b = c = d = 1$ .

(c) Write the inequality as

$$f(a, b, c, d) \geq \frac{1}{3}(a - c)^2(a + b + c + d)^2,$$

where  $f(a, b, c, d) = a^4 + b^4 + c^4 + d^4 - 4abcd$ . Since

$$\begin{aligned} f(a, b, c, d) - f(a, b, c, c) &= -(c^4 - d^4) + 4abc(c - d) \\ &= (c - d)(-c^3 - c^2d - cd^2 - d^3 + 4abc) \geq 0, \end{aligned}$$

it suffices to show that

$$f(a, b, c, c) \geq \frac{1}{3}(a - c)^2(a + b + 2c)^2,$$

that is

$$3(a^4 + b^4 + 2c^4) - 12abc^2 \geq (a - c)^2(a + b + 2c)^2.$$

Use the substitution

$$a = x + c, \quad b = y + c, \quad x \geq y \geq 0,$$

and, for fixed  $x, y$ , write the inequality in the homogeneous form  $f(c) \geq 0$ , where

$$f(c) = 3(a^4 + b^4 + 2c^4) - 12abc^2 - (a - c)^2(a + b + 2c)^2.$$

Since

$$a' = b' = c' = 1,$$

we have

$$\frac{1}{4}f'(c) = 3(a^3 + b^3 + 2c^3) - 3(2abc + ac^2 + bc^2) - (a - c)^2(a + b + 2c),$$

$$\frac{1}{4}f''(c) = 9(a^2 + b^2 + 2c^2) - 6(ab + 2ac + 2bc + c^2) - 4(a - c)^2,$$

$$\frac{1}{24}f'''(c) = 3(a + b + 2c) - 3(a + b + 2c) = 0,$$

therefore

$$\begin{aligned}\frac{1}{4}f''(c) &= \frac{1}{4}f''(0) = 9(a^2 + b^2) - 6ab - 4a^2 \geq 5a^2 - 6ab + 9b^2 \\ &= 4a^2 + (a - 3b)^2 \geq 0,\end{aligned}$$

$f'(c)$  is increasing, hence

$$\frac{1}{4}f'(c) \geq \frac{1}{4}f'(0) = 3(a^3 + b^3) - a^2(a + b) \geq 0,$$

$f(c)$  is increasing, hence

$$\begin{aligned}f(c) &\geq f(0) = 3(a^4 + b^4) - a^2(a + b)^2 = 2a^4 - 2a^3b - a^2b^2 + 3b^4 \\ &= (a - b)^2(2a^2 + 2ab + b^2) + 2b^2 \geq 0.\end{aligned}$$

The equality occurs for  $a = b = c = d = 1$ .

(d) Write the inequality as  $f(a, b, c, d) \geq 0$ , where

$$F(a, b, c, d) = 3(a^4 + b^4 + c^4 + d^4) - 12abcd - (c - d)^2(a + b + c + d)^2.$$

Denote

$$x = \frac{a + b}{2}, \quad a \geq x \geq b,$$

and show that

$$F(a, b, c, d) \geq F(x, x, c, d) \geq F(c, c, c, d) \geq 0.$$

Since  $a^4 + b^4 \geq 2x^4$  and  $x^2 \geq abc$ , we have

$$F(a, b, c, d) - F(x, x, c, d) = 3(a^4 + b^4 - 2x^4) + 12cd(x^2 - ab) \geq 0.$$

Further, we have

$$\begin{aligned}F(x, x, c, d) &= 6x^4 + 3(c^4 + d^4) - 12x^2cd - (c - d)^2(2x + c + d)^2, \\ F(x, x, c, d) - F(c, c, c, d) &= 6(x^4 - c^4) - 12cd(x^2 - c^2) - 2(c - d)^2(x - c)(x + 2c + d) \\ &= 6(x^2 - c^2)(x^2 + c^2 - 2cd) - 2(c - d)^2(x - c)(x + 2c + d) \\ &\geq 6(x^2 - c^2)(c - d)^2 - 2(c - d)^2(x - c)(x + 2c + d) \\ &= 2(x - c)(c - d)^2(2x + c - d) \geq 0\end{aligned}$$

and

$$\begin{aligned} F(c, c, c, d) &= 9c^4 + 3d^4 - 12c^3d - (c-d)^2(3c+d)^2 \\ &= 3(c-d)^2(3c^3 + 2cd + d^2) - (c-d)^2(3c+d)^2 \\ &= 2(c-d)^2d^2 \geq 0. \end{aligned}$$

The equality occurs for  $a = b = c = d = 1$ , and also for  $a = b = c = 4/3$  and  $d = 0$ .

□

**P 3.180.** If  $a, b, c, d$  are nonnegative real numbers such that  $a \geq b \geq c \geq d$  and  $a + d \geq b + c$ , then

$$a + b + c + d - 4\sqrt[4]{abcd} \leq 2\left(\sqrt{a} - \sqrt{d}\right)^2.$$

(Vasile Cîrtoaje, 2020)

**Solution.** For fixed  $b, c$  and  $d$ , write the inequality as  $f(a) \geq 0$ , where

$$\begin{aligned} f(a) &= 2\left(\sqrt{a} - \sqrt{d}\right)^2 - a - b - c - d + 4\sqrt[4]{abcd} \\ &= a - b - c + d + 4\sqrt[4]{abcd} - 4\sqrt{ad}. \end{aligned}$$

Denote

$$x = \sqrt[4]{\frac{d}{a}}, \quad x \leq 1.$$

Since

$$\begin{aligned} f'(a) &= 1 + \sqrt[4]{\frac{bcd}{a^3}} - 2\sqrt{\frac{d}{a}} \geq 1 + \sqrt[4]{\frac{d^3}{a^3}} - 2\sqrt{\frac{d}{a}} \\ &= 1 + x^3 - 2x^2 = (1-x)(1+x-x^2) \geq 0, \end{aligned}$$

$f$  is increasing, therefore  $f(a) \geq f(b+c-d)$ . So, we only need to show that  $f(b+c-d) \geq 0$ , that is

$$\sqrt[4]{(b+c-d)bcd} \geq \sqrt{(b+c-d)d}.$$

This is true if

$$bc \geq (b+c-d)d,$$

which is equivalent to the obvious inequality

$$(b-d)(c-d) \geq 0.$$

The equality occurs for  $a = b = c = d$ , and also for  $a = b + c$  and  $d = 0$ .

□



**P 3.181.** If  $a, b, c, d$  are nonnegative real numbers such that  $a \geq b \geq c \geq d$  and

$$a + kd \geq b + c, \quad k = (1 + \sqrt{2})^4 \approx 33.970,$$

then

$$a + b + c + d - 4\sqrt[4]{abcd} \leq 2 \left( \sqrt{a} - \sqrt{d} \right)^2.$$

(Vasile Cîrtoaje, 2020)

**Solution.** For fixed  $b, c$  and  $d$ , write the inequality as  $f(a) \geq 0$ , where

$$\begin{aligned} f(a) &= 2 \left( \sqrt{a} - \sqrt{d} \right)^2 - a - b - c - d + 4\sqrt[4]{abcd} \\ &= a - b - c + d + 4\sqrt[4]{abcd} - 4\sqrt{ad}. \end{aligned}$$

Denote

$$x = \sqrt[4]{\frac{d}{a}}, \quad x \leq 1.$$

Since

$$\begin{aligned} f'(a) &= 1 + \sqrt[4]{\frac{bcd}{a^3}} - 2\sqrt{\frac{d}{a}} \geq 1 + \sqrt[4]{\frac{d^3}{a^3}} - 2\sqrt{\frac{d}{a}} \\ &= 1 + x^3 - 2x^2 = (1 - x)(1 + x - x^2) \geq 0, \end{aligned}$$

it follows that  $f(a)$  is increasing, therefore  $f(a) \geq f(b + c - kd)$  (if  $b + c - kd \geq b$ ) and  $f(a) \geq f(b)$ . There are two cases to consider:  $c \geq kd$  and  $d \leq c \leq kd$ .

*Case 1:*  $c \geq kd$ . Since  $f(a) \geq f(b + c - kd)$ , we need to show that

$$f(b + c - kd) \geq 0,$$

that is

$$-(k - 1)d + 4\sqrt[4]{(b + c - kd)bcd} - 4\sqrt{(b + c - kd)d} \geq 0.$$

From

$$(b - kd)(c - kd) \geq 0,$$

we get

$$bc \geq k(b + c - kd)d.$$

Thus, it is enough to show that

$$-(k - 1)d + 4\sqrt[4]{k(b + c - kd)^2d^2} - 4\sqrt{(b + c - kd)d} \geq 0,$$

which is equivalent to

$$-(k - 1)d + 4 \left( \sqrt[4]{k} - 1 \right) \sqrt{(b + c - kd)d} \geq 0.$$

For  $d = 0$ , the inequality is an equality. For  $d > 0$ , due to homogeneity, we may set  $d = 1$  (hence  $c \geq k$ ), when the inequality becomes

$$4 \left( \sqrt[4]{k} - 1 \right) \sqrt{b + c - k} \geq k - 1.$$

Since  $b + c \geq 2c \geq 2k$ , we only need to show that

$$4 \left( \sqrt[4]{k} - 1 \right) \sqrt{k} \geq k - 1.$$

Denoting

$$m = \sqrt[4]{k} = 1 + \sqrt{2},$$

we have

$$\begin{aligned} 4 \left( \sqrt[4]{k} - 1 \right) \sqrt{k} - (k - 1) &= 4(m - 1)m^2 - (m^4 - 1) \\ &= (m - 1)^2(1 + 2m - m^2) = 0. \end{aligned}$$

*Case 2:*  $d \leq c \leq kd$ . Since  $f(a) \geq f(b)$ , we need to show that  $f(b) \geq 0$ , that is  $g(b) \geq 0$ , where

$$g(b) = -c + d + 4\sqrt[4]{b^2cd} - 4\sqrt{bd}.$$

We will show that  $g(b) \geq g(c) \geq 0$ . Since

$$g'(b) = \frac{2 \left( \sqrt[4]{cd} - \sqrt{d} \right)}{\sqrt{b}} \geq 0,$$

$g(b)$  is increasing, therefore  $g(b) \geq g(c)$ . The inequality  $g(c) \geq 0$  has the form

$$-c + d + 4\sqrt[4]{c^3d} - 4\sqrt{c} \geq 0.$$

Due to homogeneity, we may set  $d = 1$ , when  $1 \leq c \leq k$ . We need to show that  $h(c) \geq 0$ , where

$$h(c) = -c + 1 + 4\sqrt[4]{c^3} - 4\sqrt{c} \geq 0.$$

Since  $h(c)$  is concave, it is enough to show that  $h(1) \geq 0$  and  $h(k) \geq 0$ . Indeed,  $h(1) = 0$  and

$$h(k) = -k + 1 + 4\sqrt[4]{k^3} - 4\sqrt{k} = 0.$$

The proof is completed. The equality occurs for  $a = b = c = d$ , for  $a = b + c$  and  $d = 0$ , and for  $a = b = c = kd$ .

**Remark.** Let

$$k_1 \leq k = (1 + \sqrt{2})^4.$$

Since  $a + k_1d \geq b + c$  yields  $a + kd \geq b + c$ , it follows that the inequality is true for all nonnegative real numbers  $a, b, c, d$  such that  $a \geq b \geq c \geq d$  and  $a + k_1d \geq b + c$ . For  $k_1 < k$ , the equality occurs when  $a = b = c = d$ , and when  $a = b + c$  and  $d = 0$ . The particular cases  $k_1 = -1$ ,  $k_1 = 0$  and  $k_1 = 1$  are relevant.

□

**P 3.182.** If  $a, b, c, d$  are nonnegative real numbers such that  $a \geq b \geq c \geq d$  and

$$a + d \geq 2c,$$

then

$$a + b + c + d - 4\sqrt[4]{abcd} \leq \frac{5}{2} \left( \sqrt{a} - \sqrt{d} \right)^2.$$

(Vasile Cîrtoaje, 2020)

**Solution.** For fixed  $b, c$  and  $d$ , write the inequality as  $f(a) \geq 0$ , where

$$\begin{aligned} f(a) &= 5 \left( \sqrt{a} - \sqrt{d} \right)^2 - 2(a + b + c + d) + 8\sqrt[4]{abcd} \\ &= 3a - 2b - 2c + 3d + 8\sqrt[4]{abcd} - 10\sqrt{ad}. \end{aligned}$$

Denote

$$x = \sqrt[4]{\frac{d}{a}}, \quad x \leq 1.$$

Since

$$\begin{aligned} f'(a) &= 3 + 2\sqrt[4]{\frac{bcd}{a^3}} - 5\sqrt{\frac{d}{a}} \geq 3 + 2\sqrt[4]{\frac{d^3}{a^3}} - 5\sqrt{\frac{d}{a}} \\ &= 3 + 2x^3 - 5x^2 = (1 - x)(3 + 3x - 2x^2) \geq 0, \end{aligned}$$

it follows that  $f(a)$  is increasing, therefore  $f(a) \geq f(b)$  and  $f(a) \geq f(2c - d)$  (if  $2c - d \geq b$ ). There are two cases to consider:  $b \geq 2c - d$  and  $c \leq b \leq 2c - d$ .

*Case 1:*  $b \geq 2c - d$ . Since  $f(a) \geq f(b)$ , we need to show that  $f(b) \geq 0$ , that is equivalent to  $g(b) \geq 0$ , where

$$g(b) = b - 2c + 3d + 8\sqrt[4]{b^2cd} - 10\sqrt{bd}.$$

Since

$$\begin{aligned} g'(b) &= 1 + 4\sqrt[4]{\frac{cd}{b^2}} - 5\sqrt{\frac{d}{b}} \geq 1 + 4\sqrt[4]{\frac{d^2}{b^2}} - 5\sqrt{\frac{d}{b}} \\ &= 1 - \sqrt{\frac{d}{b}} \geq 0, \end{aligned}$$

$g(b)$  is increasing, hence

$$\begin{aligned} g(b) &\geq g(2c - d) = 2d + 8\sqrt[4]{(2c - d)^2cd} - 10\sqrt{(2c - d)d} \\ &= 2d + 2\sqrt{2c - d} \left( 4\sqrt[4]{cd} - 5\sqrt{d} \right). \end{aligned}$$

So, we need to show that

$$d \geq \sqrt{2c - d} \left( 5\sqrt{d} - 4\sqrt[4]{cd} \right).$$

For  $d = 0$ , this inequality is an equality. For  $d > 0$ , due to homogeneity, we may set  $d = 1$  (hence  $c \geq 1$ ), when the inequality becomes

$$1 \geq \sqrt{2c - 1} \left( 5 - 4\sqrt[4]{c} \right).$$

Since

$$\sqrt{2c-1} \leq c,$$

it suffices to show that

$$1 \geq c(5 - 4\sqrt[4]{c}).$$

By the AM-GM inequality, we have

$$1 + 4c\sqrt[4]{c} = 1 + c\sqrt[4]{c} + c\sqrt[4]{c} + c\sqrt[4]{c} + c\sqrt[4]{c} \geq 5\sqrt[5]{c^5} = 5c.$$

*Case 2:*  $c \leq b \leq 2c - d$ . Since  $f(a) \geq f(2c - d)$ , we need to show that  $f(2c - d) \geq 0$ , that is equivalent to  $h(b) \geq 0$ , where

$$h(b) = -2b + 4c + 8\sqrt[4]{(2c-d)bcd} - 10\sqrt{(2c-d)d}.$$

We will show that

$$h(b) \geq h(2c - d) \geq 0.$$

Since

$$h'(b) = -2 + 2\sqrt[4]{\frac{(2c-d)cd}{b^3}} \leq -2 + 2\sqrt[4]{\frac{(2c-d)d}{c^2}} \leq 0,$$

it follows that  $h(b)$  is decreasing, hence  $h(b) \geq h(2c - d)$ . Further, the inequality  $h(2c - d) \geq 0$  is equivalent to

$$2d + 8\sqrt[4]{(2c-d)^2cd} - 10\sqrt{(2c-d)d} \geq 0,$$

which was proved in the case 1.

The proof is completed. The equality occurs for  $a = b = c = d$ , and also for  $a = b = 2c$  and  $d = 0$ . □

**P 3.183.** If  $a, b, c, d$  are nonnegative real numbers such that  $a \geq b \geq c \geq d$  and

$$a + b + c + d = 4,$$

then

$$(a - d)^2 \leq \frac{a^4 + b^4 + c^4 + d^4}{4} - abcd \leq 4(a - d)^2.$$

(Vasile Cîrtoaje, 2020)

**Solution.** (a) Write the left inequality in the homogeneous form

$$2(a^4 + b^4 + c^4 + d^4) - 8abcd \geq \frac{1}{2}(a - d)^2(a + b + c + d)^2.$$

Since

$$8 = \frac{1}{2}(a + b + c + d)^2 \leq (b + c)^2 + (a + d)^2,$$

it suffices to prove the homogeneous inequality  $F(a, b, c, d) \geq 0$ , where

$$F(a, b, c, d) = 2(a^4 + b^4 + c^4 + d^4) - 8abcd - (a - d)^2[(b + c)^2 + (a + d)^2].$$

Denote

$$x = \frac{b + c}{2}, \quad a \geq x \geq d,$$

and show that

$$F(a, b, c, d) \geq F(a, x, x, d) \geq 0.$$

Since  $b^4 + c^4 \geq 2x^4$  and  $x^2 \geq bc$ , we have

$$F(a, b, c, d) - F(a, x, x, d) = 4(b^4 + c^4 - 2x^4) + 16ad(x^2 - bc) \geq 0.$$

Further, we have

$$\begin{aligned} F(a, x, x, d) &= 4x^4 + 2(a^4 + d^4) - 8adx^2 - (a - d)^2 [4x^2 + (a + d)^2] \\ &= (2x^2 - a^2 - d^2)^2 \geq 0. \end{aligned}$$

The equality occurs for  $a = b = c = d = 1$ .

(b) Write the right inequality in the homogeneous form

$$a^4 + b^4 + c^4 + d^4 - 4abcd \leq (a - d)^2(a + b + c + d)^2.$$

Use the substitution

$$a = x + d, \quad b = y + d, \quad c = z + d, \quad x \geq y \geq z \geq 0,$$

and, for fixed  $x, y, z$ , write the inequality in the homogeneous form  $f(d) \geq 0$ , where

$$f(d) = a^4 + b^4 + c^4 + d^4 - 4abcd - (a - d)^2(a + b + c + d)^2.$$

Since

$$a' = b' = c' = d' = 1,$$

we have

$$\frac{1}{4}f'(d) = a^3 + b^3 + c^3 + d^3 - (bcd + cda + dab + abc) - 2(a - d)^2(a + b + c + d),$$

$$\frac{1}{4}f''(d) = 3(a^2 + b^2 + c^2 + d^2) - 2 \sum_{sym} ab - 8(a - d)^2,$$

$$\frac{1}{4}f'''(d) = 6(a + b + c + d) - 6(a + b + c + d) = 0,$$

therefore

$$\frac{1}{4}f''(d) = \frac{1}{4}f''(0) = 3(a^2 + b^2 + c^2) - 2(ab + bc + ca) - 8a^2 \leq -5a^2 + 3(b^2 + c^2) - ab$$

$$= 3(b^2 - a^2) + 2(c^2 - a^2) + c^2 - ab \leq 0,$$

$f'(d)$  is decreasing, hence

$$\begin{aligned} \frac{1}{4}f'(d) &\leq \frac{1}{4}f'(0) = a^3 + b^3 + c^3 - abc - 2a^2(a + b + c) \\ &\leq a^3 + b^3 + c^3 - a^2(a + b + c) = b(b^2 - a^2) + c(c^2 - a^2) \leq 0, \end{aligned}$$

$f(d)$  is decreasing, hence

$$\begin{aligned} f(d) &\leq f(0) = a^4 + b^4 + c^4 - a^2(a + b + c)^2 \leq a^4 + b^4 + c^4 - a^2(a^2 + b^2 + c^2) \\ &= b^2(a^2 - b^2) + c^2(c^2 - a^2) \leq 0. \end{aligned}$$

The equality occurs for  $b = c = d = 0$ .

□

**P 3.184.** If  $a \geq b \geq c \geq d \geq 0$ , then

$$\begin{aligned} (a) \quad & a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{3}{2} \left( \sqrt{b} - 2\sqrt{c} + \sqrt{d} \right)^2; \\ (b) \quad & a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{2}{9} \left( 3\sqrt{b} - 2\sqrt{c} - \sqrt{d} \right)^2; \\ (c) \quad & a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{4}{19} \left( 3\sqrt{b} - \sqrt{c} - 2\sqrt{d} \right)^2; \\ (d) \quad & a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{3}{8} \left( \sqrt{b} - 3\sqrt{c} + 2\sqrt{d} \right)^2; \\ (e) \quad & a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{1}{2} \left( 2\sqrt{b} - 3\sqrt{c} + \sqrt{d} \right)^2; \\ (f) \quad & a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{1}{6} \left( 2\sqrt{b} + \sqrt{c} - 3\sqrt{d} \right)^2. \end{aligned}$$

(Vasile Cîrtoaje, 2010)

**Solution.** First, we show that

$$a - 4\sqrt[4]{abcd} \geq b - 4\sqrt[4]{b^2cd}.$$

Write this inequality as

$$a - b \geq 4\sqrt[4]{bcd} \left( \sqrt[4]{a} - \sqrt[4]{b} \right),$$

and prove then the following sharper inequality

$$a - b \geq 4\sqrt[4]{b^3} \left( \sqrt[4]{a} - \sqrt[4]{b} \right).$$

Indeed,

$$a - b - 4\sqrt[4]{b^3} \left( \sqrt[4]{a} - \sqrt[4]{b} \right) = \left( \sqrt[4]{a} - \sqrt[4]{b} \right) \left( \sqrt[4]{a^3} + \sqrt[4]{a^2b} + \sqrt[4]{ab^2} - 3\sqrt[4]{b^3} \right) \geq 0.$$

Thus, we have

$$a + b + c + d - 4\sqrt[4]{abcd} \geq 2b + c + d - 4\sqrt[4]{b^2cd},$$

which is equivalent to

$$a + b + c + d - 4\sqrt[4]{abcd} \geq 2\left(\sqrt{b} - \sqrt[4]{cd}\right)^2 + \left(\sqrt{c} - \sqrt{d}\right)^2.$$

Since

$$\sqrt{b} - \sqrt[4]{cd} \geq \sqrt{b} - \frac{\sqrt{c} + \sqrt{d}}{2} \geq 0,$$

we have

$$a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{1}{2}\left(2\sqrt{b} - \sqrt{c} - \sqrt{d}\right)^2 + \left(\sqrt{c} - \sqrt{d}\right)^2.$$

Using the substitution

$$x = \sqrt{b} - \sqrt{c}, \quad y = \sqrt{c} - \sqrt{d}, \quad x, y \geq 0,$$

we get

$$a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{1}{2}(2x + y)^2 + y^2,$$

that is

$$a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{1}{2}(4x^2 + 4xy + 3y^2). \quad (*)$$

The inequality (\*) is an equality for  $a = b$  and  $c = d$ .

(a) According to (\*), it suffices to show that

$$4x^2 + 4xy + 3y^2 \geq 3(x - y)^2,$$

which is equivalent to

$$x(x + 10y) \geq 0.$$

The equality holds for  $a = b = c = d$ .

(b) According to (\*), it suffices to show that

$$9(4x^2 + 4xy + 3y^2) \geq 4(3x + y)^2,$$

which is equivalent to

$$y(12x + 23y) \geq 0.$$

The equality holds for  $a = b$  and  $c = d$ .

(c) According to (\*), it suffices to show that

$$19(4x^2 + 4xy + 3y^2) \geq 8(3x + 2y)^2,$$

which is equivalent to

$$(2x - 5y)^2 \geq 0.$$

The equality holds for  $a = b = c = d$ .

(d) According to (\*), it suffices to show that

$$4(4x^2 + 4xy + 3y^2) \geq 3(x - 2y)^2,$$

which is equivalent to

$$x(13x + 28y) \geq 0.$$

The equality holds for  $a = b = c = d$ .

(e) According to (\*), it suffices to show that

$$4x^2 + 4xy + 3y^2 \geq (2x - y)^2,$$

which is equivalent to

$$y(4x + y) \geq 0.$$

The equality holds for  $a = b$  and  $c = d$ .

(f) According to (\*), it suffices to show that

$$3(4x^2 + 4xy + 3y^2) \geq (2x + 3y)^2,$$

which is equivalent to

$$x^2 \geq 0.$$

The equality holds for  $a = b = c = d$ .

□

**P 3.185.** If  $a \geq b \geq c \geq d \geq 0$ , then

$$\begin{aligned} (a) \quad & a + b + c + d - 4\sqrt[4]{abcd} \geq \left(\sqrt{a} - \sqrt{d}\right)^2; \\ (b) \quad & a + b + c + d - 4\sqrt[4]{abcd} \geq 2\left(\sqrt{b} - \sqrt{c}\right)^2; \\ (c) \quad & a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{4}{3}\left(\sqrt{b} - \sqrt{d}\right)^2; \\ (d) \quad & a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{3}{2}\left(\sqrt{c} - \sqrt{d}\right)^2. \end{aligned}$$

(Vasile Cîrtoaje, 2010)

**Solution.** (a) Write the inequality as

$$b + c + 2\sqrt{ad} \geq 4\sqrt[4]{abcd},$$

which follows immediately from the AM-GM inequality. The equality holds for

$$b = c = \sqrt{ad}.$$



(b) **First Solution.** Since

$$a + b + c + d - 4\sqrt[4]{abcd} \geq 2\sqrt{ab} + 2\sqrt{cd} - 4\sqrt[4]{abcd} = 2\left(\sqrt[4]{ab} - \sqrt[4]{cd}\right)^2,$$

we only need to show that

$$\sqrt[4]{ab} - \sqrt[4]{cd} \geq \sqrt{b} - \sqrt{c},$$

which is equivalent to the obvious inequality

$$\sqrt[4]{b}\left(\sqrt[4]{a} - \sqrt[4]{b}\right) + \sqrt[4]{c}\left(\sqrt[4]{c} - \sqrt[4]{d}\right) \geq 0.$$

The equality holds for  $a = b$  and  $c = d$ .

**Second Solution.** According to the inequality (\*) from the proof of the previous P 3.184, it suffices to show that

$$4x^2 + 4xy + 3y^2 \geq 4x^2,$$

which is obvious.

(c) According to the inequality (\*) from the proof of the previous P 3.184, it suffices to show that

$$3(4x^2 + 4xy + 3y^2) \geq 8(x + y)^2,$$

which is equivalent to

$$(2x - y)^2 \geq 0.$$

The equality holds for  $a = b = c = d$ .

(d) According to the inequality (\*) from the proof of the previous P 3.184, it suffices to show that

$$4x^2 + 4xy + 3y^2 \geq 3y^2,$$

which is obvious. The equality holds for  $a = b = c = d$ .

□

**P 3.186.** If  $a, b, c, d$  are positive real numbers such that

$$abcd \geq 1, \quad a \geq b \geq c \geq d, \quad ad \geq bc,$$

then

$$a + b + c + d \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

(Vasile Cîrtoaje, 2018)

**Solution.** It suffices to prove the homogeneous inequality

$$a + b + c + d \geq \sqrt{abcd} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right).$$

Replacing  $a, b, c, d$  with  $a^2, b^2, c^2, d^2$ , we need to show that

$$a^2 + b^2 + c^2 + d^2 \geq \frac{bc}{ad}(a^2 + d^2) + \frac{ad}{bc}(b^2 + c^2)$$

for  $a \geq b \geq c \geq d$  and  $ad \geq bc$ . Write the inequality as follows:

$$(a^2 + d^2) \left( 1 - \frac{bc}{ad} \right) + (b^2 + c^2) \left( 1 - \frac{ad}{bc} \right) \geq 0,$$

$$(ad - bc) \left( \frac{a}{d} + \frac{d}{a} - \frac{b}{c} - \frac{c}{b} \right) \geq 0,$$

$$(ad - bc) \left( \frac{ac - bd}{cd} + \frac{bd - ac}{ab} \right) \geq 0.$$

$$\frac{(ad - bc)(ac - bd)(ab - cd)}{abcd} \geq 0.$$

Clearly, the last inequality is true. The equality occurs for  $ad = bc = 1$ .

**Remark.** The following extension is valid.

- If  $a, b, c, d, e$  are positive real numbers such that

$$abcde \geq 1, \quad a \geq b \geq c \geq d \geq e, \quad ae \geq bd \geq c^2,$$

then

$$a + b + c + d + e \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e},$$

with equality for  $af = c^2 = cd = 1$

□

**P 3.187.** If  $a \geq b \geq c \geq d \geq 0$ , then

$$\frac{(a - d)^2}{2(a + d)} \leq a + b + c + d - 4\sqrt[4]{abcd} \leq \frac{3(a - d)^2}{a + 5d}.$$

(Vasile Cîrtoaje, 2009)

**Solution.** (a) To prove the inequality

$$a + b + c + d - 4\sqrt[4]{abcd} \geq \frac{(a-d)^2}{a+3d},$$

we will show that

$$a + b + c + d - 4\sqrt[4]{abcd} \geq a + d - 2\sqrt{ad} \geq \frac{(a-d)^2}{2(a+d)}. \quad (*)$$

The left inequality is equivalent to

$$b + c + 2\sqrt{ad} \geq 4\sqrt[4]{abcd},$$

which is a consequence of the AM-GM inequality. The right inequality in (\*) can be written as follows:

$$\begin{aligned} 2(a+d) \left( \sqrt{a} - \sqrt{d} \right)^2 &\geq (a-d)^2, \\ \left( \sqrt{a} - \sqrt{d} \right)^4 &\geq 0. \end{aligned}$$

The equality holds for  $a = b = c = d$ .

(b) To prove the inequality

$$a + b + c + d - 4\sqrt[4]{abcd} \leq \frac{3(a-d)^2}{a+5d},$$

we will show that

$$a + b + c + d - 4\sqrt[4]{abcd} \leq 2a + c + d - 4\sqrt[4]{a^2cd} \leq \frac{3(a-d)^2}{a+5d}. \quad (**)$$

Write the left inequality as

$$\begin{aligned} a - b - 4\sqrt[4]{acd} \left( \sqrt[4]{a} - \sqrt[4]{b} \right) &\geq 0, \\ \left( \sqrt[4]{a} - \sqrt[4]{b} \right) \left( \sqrt[4]{a^3} + \sqrt[4]{a^2b} + \sqrt[4]{ab^2} + \sqrt[4]{b^3} - 4\sqrt[4]{acd} \right) &\geq 0. \end{aligned}$$

The last inequality follows from the AM-GM inequality:

$$\begin{aligned} \sqrt[4]{a^3} + \sqrt[4]{a^2b} + \sqrt[4]{ab^2} + \sqrt[4]{b^3} - 4\sqrt[4]{acd} &\geq \sqrt[4]{a^3} + \sqrt[4]{a^2b} + \sqrt[4]{b^3} - 3\sqrt[4]{ab^2} \\ &\geq \sqrt[4]{a^3} + \sqrt[4]{b^3} + \sqrt[4]{b^3} - 3\sqrt[4]{ab^2} \geq 0. \end{aligned}$$

Write the right inequality in (\*\*) as

$$F(c) \geq 0,$$

where

$$F(c) = 3(a-d)^2 - (a+5d) \left( 2a + c + d - 4\sqrt[4]{a^2cd} \right).$$

Since  $F$  is a concave function and  $d \leq c \leq a$ , it suffices to show that  $F(d) \geq 0$  and  $F(a) \geq 0$ . We have

$$F(d) = 3(a-d)^2 - 2(a+5d)(\sqrt{a}-\sqrt{d})^2 = (\sqrt{a}-\sqrt{d})^3(\sqrt{a}+7\sqrt{d}) \geq 0$$

and

$$F(a) = 3(a-d)^2 - (a+5d)(3a+d-4\sqrt[4]{a^3d}).$$

Setting  $a = 1$  (due to homogeneity) and substituting  $d = x^4$ ,  $0 \leq x \leq 1$ , the inequality  $F(a) \geq 0$  becomes

$$3(1-x^4)^2 - (1+5x^4)(3+x^4-4x) \geq 0.$$

Since  $3+x^4-4x = (1-x)^2(3+2x+x^2)$ , we need to show that

$$3(1+x+x^2+x^3)^2 - (1+5x^4)(3+2x+x^2) \geq 0,$$

which is equivalent to

$$x(2+4x+6x^2-3x^3-2x^4-x^5) \geq 0.$$

This inequality is true since

$$2+4x+6x^2-3x^3-2x^4-x^5 > 6x^2-3x^3-2x^4-x^5 \geq 0.$$

The equality holds for  $a = b = c = d$ , and also for  $a = b = c$  and  $d = 0$ .

**Remark.** The following generalization holds:

- If  $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$ , then

$$\frac{(a_1 - a_n)^2}{2(a_1 + a_n)} \leq a_1 + a_2 + \dots + a_n - n\sqrt[n]{a_1 a_2 \dots a_n} \leq \frac{(n-1)(a_1 - a_n)^2}{a_1 + k_n a_n},$$

where

$$k_n = 7 - \frac{4}{\left[\frac{n+1}{2}\right]}.$$

□

**P 3.188.** If  $a \geq b \geq c \geq d \geq e \geq 0$ , then

$$a + b + c + d + e - 5\sqrt[5]{abcde} \geq 2(\sqrt{b} - \sqrt{d})^2.$$

(Vasile Cîrtoaje, 2010)

**Solution.** From the AM-GM inequality, we have

$$c + 4\sqrt[4]{abde} \geq 5\sqrt[5]{abcde},$$

which can be rewritten as

$$c - 5\sqrt[5]{abcde} \geq -4\sqrt[4]{abde}.$$

Thus, it suffices to show that

$$a + b + d + e - 4\sqrt[4]{abde} \geq 2 \left( \sqrt{b} - \sqrt{d} \right)^2.$$

Since

$$a + b + d + e - 4\sqrt[4]{abde} \geq 2\sqrt{ab} + 2\sqrt{de} - 4\sqrt[4]{abde} = 2 \left( \sqrt[4]{ab} - \sqrt[4]{de} \right)^2,$$

we only need to prove that

$$\sqrt[4]{ab} - \sqrt[4]{de} \geq \sqrt{b} - \sqrt{d},$$

which is equivalent to the obvious inequality

$$\sqrt[4]{b} \left( \sqrt[4]{a} - \sqrt[4]{b} \right) + \sqrt[4]{d} \left( \sqrt[4]{d} - \sqrt[4]{e} \right) \geq 0.$$

The equality holds for

$$a = b, \quad d = e, \quad c^2 = ad.$$

□

**P 3.189.** Let  $a, b, c, d, e$  be nonnegative real numbers.

(a) If  $a + b + c = 3(d + e)$ , then

$$4(a^4 + b^4 + c^4 + d^4 + e^4) \geq (a^2 + b^2 + c^2 + d^2 + e^2)^2.$$

(b) If  $a + b + c = d + e$ , then

$$12(a^4 + b^4 + c^4 + d^4 + e^4) \leq 7(a^2 + b^2 + c^2 + d^2 + e^2)^2.$$

(Vasile Cîrtoaje, 2010)

**Solution.** (a) Let

$$E(a, b, c, d, e) = 4(a^4 + b^4 + c^4 + d^4 + e^4) - (a^2 + b^2 + c^2 + d^2 + e^2)^2.$$

We will show that

$$E(a, b, c, d, e) \geq E(a, b, c, d + e, 0) \geq 0.$$

The left side inequality is equivalent to

$$de(a^2 + b^2 + c^2 - 3d^2 - 3e^2 - 5de) \geq 0.$$

This is true since

$$\begin{aligned} a^2 + b^2 + c^2 - 3d^2 - 3e^2 - 5de &\geq \frac{1}{3}(a + b + c)^2 - 3d^2 - 3e^2 - 5de \\ &= 3(d + e)^2 - 3d^2 - 3e^2 - 5de = de \geq 0. \end{aligned}$$

Also, in virtue of the Cauchy-Schwarz inequality, we have

$$E(a, b, c, d + e, 0) = 4[a^4 + b^4 + c^4 + (d + e)^4] - [a^2 + b^2 + c^2 + (d + e)^2]^2 \geq 0.$$

The equality holds for  $a = b = c = d$  and  $e = 0$ , or for  $a = b = c = e$  and  $d = 0$ .

(b) Let

$$E(a, b, c, d, e) = 7(a^2 + b^2 + c^2 + d^2 + e^2)^2 - 12(a^4 + b^4 + c^4 + d^4 + e^4).$$

We will show that

$$E(a, b, c, d, e) \geq E(a, b, c, d + e, 0) \geq 0.$$

The left side inequality is equivalent to

$$de[12(d^2 + e^2) + 11de - 7(a^2 + b^2 + c^2)] \geq 0.$$

This is true since

$$\begin{aligned} 12(d^2 + e^2) + 11de - 7(a^2 + b^2 + c^2) &\geq 12(d^2 + e^2) + 11de - 7(a + b + c)^2 \\ &= 12(d^2 + e^2) + 11de - 7(d + e)^2 \\ &= 5(d^2 + e^2) - 3de \geq 0. \end{aligned}$$

Also, we have

$$\begin{aligned} \frac{1}{4}E(a, b, c, d + e, 0) &= \frac{1}{4}E(a, b, c, a + b + c, 0) \\ &= \sum a^4 + 2 \sum ab(a^2 + b^2) + 3 \sum a^2b^2 - 8abc \sum a \\ &\geq \sum a^2b^2 + 4 \sum a^2b^2 + 3 \sum a^2b^2 - 8abc \sum a \\ &= 8 \left( \sum a^2b^2 - abc \sum a \right) \\ &= 4 \sum a^2(b - c)^2 \geq 0. \end{aligned}$$

The equality holds for  $a = b = c = d/3$  and  $e = 0$ , or for  $a = b = c = e/3$  and  $d = 0$ .

□

**P 3.190.** Let  $a, b, c, d, e$  be nonnegative real numbers such that

$$a + b + c + d + e = 5.$$

Prove that

$$a^4 + b^4 + c^4 + d^4 + e^4 + 150 \leq 31(a^2 + b^2 + c^2 + d^2 + e^2).$$

(Vasile Cîrtoaje, 2007)

**Solution.** Write the inequality as

$$\sum (a^4 - 31a^2 + 30a) \leq 0,$$

or

$$\sum (1 - a)f(a) \leq 0,$$

where

$$f(a) = a^3 + a^2 - 30a.$$

Without loss of generality, assume that  $a \geq b \geq c \geq d \geq e$ . Since  $a + b \leq 5$ , we have

$$\begin{aligned} f(a) - f(b) &= (a - b)(a^2 + ab + b^2 + a + b - 30) \\ &\leq (a - b)[(a + b)^2 + a + b - 30] \\ &= (a - b)(a + b - 5)(a + b + 6) \leq 0. \end{aligned}$$

Similarly,

$$f(b) - f(c) \leq 0, \quad f(c) - f(d) \leq 0, \quad f(d) - f(e) \leq 0.$$

Since

$$a - 1 \geq b - 1 \geq c - 1 \geq d - 1 \geq e - 1$$

and

$$f(a) \leq f(b) \leq f(c) \leq f(d) \leq f(e),$$

by Chebyshev's inequality, we get

$$5 \sum (a - 1)f(a) \leq \left[ \sum (a - 1) \right] \left[ \sum f(a) \right] = 0.$$

The equality holds for  $a = b = c = d = e = 1$ , and for  $(a, b, c, d, e) = (5, 0, 0, 0, 0)$  or any cyclic permutation.

**Remark.** Similarly, we can prove the following generalization:

- If  $a_1, a_2, \dots, a_n$  are nonnegative real numbers such that

$$a_1 + a_2 + \dots + a_n = n,$$

then

$$a_1^4 + a_2^4 + \dots + a_n^4 + n^2(n + 1) \leq (n^2 + n + 1)(a_1^2 + a_2^2 + \dots + a_n^2).$$

□

**P 3.191.** Let  $a, b, c, d, e$  be positive real numbers such that

$$a^2 + b^2 + c^2 + d^2 + e^2 = 5.$$

Prove that

$$abcde(a^4 + b^4 + c^4 + d^4 + e^4) \leq 5.$$

(Vasile Cîrtoaje, 2006)

**First Solution.** Without loss of generality, assume that

$$a \leq b \leq c \leq d \leq e.$$

First, we prove that the expression

$$E(a, b, c, d, e) = abcde(a^4 + b^4 + c^4 + d^4 + e^4)$$

has the maximum value for  $a = d$ . We need to show that

$$E(a, b, c, d, e) \leq E\left(\sqrt{\frac{a^2 + d^2}{2}}, b, c, \sqrt{\frac{a^2 + d^2}{2}}, e\right).$$

This inequality is true if

$$4ad(a^4 + b^4 + c^4 + d^4 + e^4) \leq (a^2 + d^2)[(a^2 + d^2)^2 + 2(b^4 + c^4 + e^4)],$$

which is equivalent to

$$(a^2 + d^2)^3 - 4ad(a^4 + d^4) + 2(b^4 + c^4 + e^4)(a - d)^2 \geq 0.$$

Since

$$\begin{aligned} (a^2 + d^2)^3 - 4ad(a^4 + d^4) &= (a^2 + d^2)^3 - 4ad(a^2 + d^2)^2 + 8a^3d^3 \\ &= (a^2 + d^2)[(a^2 + d^2) - 2ad]^2 - 4a^2d^2(a^2 + d^2) + 8a^3d^3 \\ &= (a^2 + d^2)(a - d)^4 - 4a^2d^2(a - d)^2 \\ &\geq -4a^2d^2(a - d)^2, \end{aligned}$$

it suffices to show that

$$b^4 + c^4 + e^4 \geq 2a^2d^2.$$

Indeed, we have

$$b^4 + c^4 + e^4 - 2a^2d^2 \geq b^4 + a^4 + d^4 - 2a^2d^2 = b^4 + (a^2 - d^2)^2 > 0.$$

Since  $E(a, b, c, d, e)$  has the maximum value for  $a = d$  and, on the other hand,  $a \leq b \leq c \leq d$ , it follows that  $E(a, b, c, d, e)$  has the maximum value for  $a = b = c = d$ . Then, it suffices to show that the desired homogeneous inequality

$$\left(\frac{a^2 + b^2 + c^2 + d^2 + e^2}{5}\right)^9 \geq (abcde)^2 \left(\frac{a^4 + b^4 + c^4 + d^4 + e^4}{5}\right)^2$$

holds for  $a = b = c = d = 1$ . Denoting  $e^2$  by  $x$ , we need to show that  $f(x) \geq 0$  for  $x > 0$ , where

$$f(x) = 9 \ln \frac{4+x}{5} - \ln x - 2 \ln \frac{4+x^2}{5}.$$

From

$$f'(x) = \frac{9}{4+x} - \frac{1}{x} - \frac{4x}{4+x^2} = \frac{4(x-1)(x-2)^2}{x(4+x)(4+x^2)},$$



it follows that  $f(x)$  is decreasing for  $0 < x \leq 1$  and increasing for  $x \geq 1$ . Therefore,  $f(x) \geq f(1) = 0$ . This completes the proof. The equality holds if and only if  $a = b = c = d = e = 1$ .

**Second Solution.** Replacing  $a, b, c, d, e$  by  $\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}, \sqrt{e}$ , we need to show the homogeneous inequality

$$\left(\frac{a+b+c+d+e}{5}\right)^9 \geq abcde \left(\frac{a^2+b^2+c^2+d^2+e^2}{5}\right)^2,$$

where  $a, b, c, d, e$  are positive real numbers. According to Remark 1 from P 3.57, it suffices to prove this inequality for  $b = c = d = e = 1$ ; that is, to show that

$$\left(\frac{a+4}{5}\right)^9 \geq a \left(\frac{a^2+4}{5}\right)^2.$$

Taking logarithms of both sides, we need to prove that  $f(a) \geq 0$ , where

$$f(a) = 9 \ln(a+4) - 7 \ln 5 - \ln a - 2 \ln(a^2+4).$$

From

$$f'(a) = \frac{9}{a+4} - \frac{1}{a} - \frac{4a}{a^2+4} = \frac{4(a-1)(a-2)^2}{a(a+4)(a^2+4)},$$

it follows that  $f(a)$  is decreasing for  $0 < a \leq 1$  and increasing for  $a \geq 1$ ; therefore,  $f(a) \geq f(1) = 0$ .

**Remark.** The following more general statement holds (Vasile Cîrtoaje, 2006):

- If  $a_1, a_2, \dots, a_n$  are positive real numbers such that

$$a_1 + a_2 + \dots + a_n = n,$$

then

$$(a_1 a_2 \dots a_n)^{\frac{1}{\sqrt{n-1}}} (a_1^2 + a_2^2 + \dots + a_n^2) \leq n.$$

□

**P 3.192.** Let  $a, b, c, d, e$  be positive real numbers such that

$$a + b + c + d + e = 5.$$

Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{20}{a^2 + b^2 + c^2 + d^2 + e^2} \geq 9.$$

(Vasile Cîrtoaje, 2006)

**Solution.** Without loss of generality, assume that

$$a \leq b \leq c \leq d \leq e.$$

First, we prove that the expression

$$E(a, b, c, d, e) = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{e} + \frac{20}{a^2 + b^2 + c^2 + d^2 + e^2}$$

has the minimum value when  $a = d$ . If this is true, then  $E(a, b, c, d, e)$  has the minimum value when  $a = b = c = d$ , and it suffices to prove the desired inequality for  $a = b = c = d$ , when it is equivalent to the obvious inequality

$$(a - 1)^2(6a - 5)^2 \geq 0.$$

Therefore, it remains to show that

$$E(a, b, c, d, d) \geq E\left(\frac{a+d}{2}, b, c, \frac{a+d}{2}, e\right).$$

This inequality is equivalent to

$$\frac{(a-d)^2}{ad(a+d)} \geq \frac{20(a-d)^2}{(a^2 + b^2 + c^2 + d^2 + e^2)[(a+d)^2 + 2b^2 + 2c^2 + 2e^2]}.$$

Since

$$a^2 + b^2 + c^2 + d^2 + e^2 \geq \frac{1}{5}(a+b+c+d+e)^2 = a+b+c+d+e,$$

it suffices to show that

$$(a+b+c+d+e)[(a+d)^2 + 2b^2 + 2c^2 + 2e^2] \geq 20ad(a+d).$$

Since

$$a+b+c+d+e \geq a+a+a+d+d = 3a+2d$$

and

$$(a+d)^2 + 2b^2 + 2c^2 + 2e^2 \geq (a+d)^2 + 2a^2 + 2a^2 + 2d^2 = 5a^2 + 2ad + 3d^2,$$

it is enough to prove that

$$(3a+2d)(5a^2 + 2ad + 3d^2) \geq 20ad(a+d).$$

This is true, since

$$\begin{aligned} (3a+2d)(5a^2 + 2ad + 3d^2) - 20ad(a+d) &= 15a^3 - 4a^2d - 7ad^2 + 6d^3 \\ &> 5a^3 - 4a^2d - 7ad^2 + 6d^3 \\ &= (a-d)^2(5a+6d) \geq 0. \end{aligned}$$

The proof is completed. The equality holds for  $a = b = c = d = e = 1$ , and also for  $(a, b, c, d, e) = \left(\frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{3}\right)$  or any cyclic permutation.

**Remark.** The following more general statement holds (Vasile Cîrtoaje, 2006):

- If  $a_1, a_2, \dots, a_n$  are positive real numbers such that

$$a_1 + a_2 + \dots + a_n = n,$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} + \frac{2n\sqrt{n-1}}{a_1^2 + a_2^2 + \dots + a_n^2} \geq n + 2\sqrt{n-1}.$$

□

**P 3.193.** If  $a, b, c, d, e \geq 1$ , then

$$\begin{aligned} & \left(a + \frac{1}{a}\right) \left(b + \frac{1}{b}\right) \left(c + \frac{1}{c}\right) \left(d + \frac{1}{d}\right) \left(e + \frac{1}{e}\right) + 68 \geq \\ & \geq 4(a + b + c + d + e) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right). \end{aligned}$$

(Vo Quoc Ba Can and Vasile Cîrtoaje, 2011)

**Solution.** Write the inequality as  $E(a, b, c, d, e) \geq 0$ , and denote

$$A = \left(a + \frac{1}{a}\right) \left(b + \frac{1}{b}\right) \left(c + \frac{1}{c}\right) \left(d + \frac{1}{d}\right), \quad A \geq 16.$$

We claim that

$$E(a, b, c, d, e) \geq E(a, b, c, d, 1).$$

If this is true, then (by symmetry)

$$E(a, b, c, d, e) \geq E(a, b, c, d, 1) \geq \dots \geq E(a, 1, 1, 1, 1) = 0,$$

and the proof is completed. Since

$$E(a, b, c, d, e) - E(a, b, c, d, 1) = (e - 1) \left(B - \frac{C}{e}\right),$$

we need to show that

$$B - \frac{C}{e} \geq 0,$$

where

$$B = A - 4 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right),$$

$$C = A - 4(a + b + c + d).$$

Since  $A \geq 16$  and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq 4,$$

it follows that  $B \geq 0$ . For the non-trivial case  $C \geq 0$ , we have

$$B - \frac{C}{e} \geq B - C = 4 \left( a - \frac{1}{a} \right) + 4 \left( b - \frac{1}{b} \right) + 4 \left( c - \frac{1}{c} \right) + 4 \left( d - \frac{1}{d} \right) \geq 0.$$

The equality holds for  $a = b = c = d = 1$  (or any cyclic permutation). □

**P 3.194.** Let  $a, b, c, d, e, f$  be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 = 6.$$

If  $a \geq b \geq c \geq d \geq e \geq f$ , then

$$1 - abcdef \leq \frac{3}{2}(a - f)^2.$$

(Vasile Cîrtoaje, 2019)

**Solution.** For  $a = f$ , the inequality is a trivial equality. Consider next that  $a > f$ , replace the condition  $a \geq b \geq c \geq d \geq e \geq f$  with  $b, c, d, e \in [f, a]$  and write the inequality in the homogeneous form  $T \geq 0$ , where

$$T = 216abcdef + 9(a - f)^2 G^2 - G^3, \quad G = a^2 + b^2 + c^2 + d^2 + e^2 + f^2.$$

For fixed  $a, c, d, e$  and  $f$ ,  $T$  is a function of  $b$ ,  $b \in [f, a]$ . We have

$$T'(b) = 216acdef + 36(a - f)^2 bG - 6bG^2 = 6bGE(b),$$

where

$$E(b) = \frac{36acdef}{bG} + 6(a - f)^2 - G.$$

Since  $E(b)$  is a decreasing function, there are three possible cases: (1)  $E(b) \geq 0$  for  $b \in [f, a]$ , hence  $T(b)$  is increasing on  $[f, a]$ ; (2)  $E(b) \geq 0$  for  $b \in [f, f_1]$  and  $E(b) \leq 0$  for  $b \in [f_1, a]$ , hence  $T(b)$  is increasing on  $[f, f_1]$  and decreasing on  $[f_1, a]$ ; (3)  $E(b) \leq 0$  for  $b \in [f, a]$ , hence  $T(b)$  is decreasing on  $[f, a]$ . In all these cases  $T(b)$  has the minimum value when  $b \in \{a, f\}$ . As a consequence, we only need to prove the required inequality for  $b \in \{a, f\}$ . Similarly, we only need to prove the required inequality for  $c, d, e \in \{a, f\}$ . So, we need to show that

$$216a^k f^{6-k} + 9(a - f)^2 [ka^2 + (6 - k)f^2]^2 - [ka^2 + (6 - k)f^2]^3 \geq 0,$$

where

$$k \in \{1, 2, 3, 4, 5\}.$$

For  $f = 0$ , the inequality reduces to

$$k(9 - k)a^4 \geq 0,$$

which is true for  $k \in \{1, 2, 3, 4, 5\}$ . Next, due to homogeneity, we may set  $d = 1$  (which involves  $a > 1$ ). The required inequality becomes

$$9(a - 1)^2(ka^2 + 6 - k)^2 \geq (ka^2 + 6 - k)^3 - 216a^k.$$

*Case 1:*  $k = 1$ . We need to show that

$$\begin{aligned} 9(a - 1)^2(a^2 + 5)^2 &\geq (a^2 + 5)^3 - 216a, \\ 9(a - 1)^2(a^2 + 5)^2 &\geq (a - 1)^2(a^4 + 2a^3 + 18a^2 + 34a + 125), \\ 2(a - 1)^2(4a^4 - a^3 + 36a^2 - 17a + 50) &\geq 0. \end{aligned}$$

The last inequality is true because

$$2a^4 - a^3 + 36a^2 - 17a + 50 > 4a^2(a - 1)^2 + 2(2a - 5)^2 > 0.$$

*Case 2:*  $k = 2$ . We need to show that

$$\begin{aligned} 9(a - 1)^2(a^2 + 2)^2 &\geq 2(a^2 + 2)^3 - 54a^2, \\ 9(a - 1)^2(a^2 + 2)^2 &\geq 2(a^2 - 1)^2(a^2 + 8), \\ (a - 1)^2(7a^4 - 4a^3 + 18a^2 - 32a + 20) &\geq 0. \end{aligned}$$

The last inequality is true because

$$7a^4 - 4a^3 + 18a^2 - 32a + 20 > 2a^2(a - 1)^2 + 16(a - 1)^2 > 0.$$

*Case 3:*  $k = 3$ . We need to show that

$$\begin{aligned} 3(a - 1)^2(a^2 + 1)^2 &\geq (a^2 + 1)^3 - 8a^3, \\ 9(a - 1)^2(a^2 + 2)^2 &\geq (a - 1)^2(a^4 + 2a^3 + 6a^2 + 2a + 1), \\ 2(a - 1)^3(a^3 - 1) &\geq 0. \end{aligned}$$

*Case 4:*  $k = 4$ . We need to show that

$$\begin{aligned} 9(a - 1)^2(2a^2 + 1)^2 &\geq 2(2a^2 + 1)^3 - 54a^4, \\ 9(a - 1)^2(2a^2 + 1)^2 &\geq 2(a^2 - 1)^2(8a^2 + 1), \\ (a - 1)^2(20a^4 - 32a^3 + 18a^2 - 4a + 7) &\geq 0. \end{aligned}$$

The last inequality is true because

$$20a^4 - 32a^3 + 18a^2 - 4a + 7 > 16a^2(a - 1)^2 + 2(a - 1)^2 > 0.$$

Case 5:  $k = 5$ . We need to show that

$$\begin{aligned} 9(a-1)^2(5a^2+1)^2 &\geq (5a^2+1)^3 - 216a^5, \\ 9(a-1)^2(5a^2+1)^2 &\geq (a-1)^2(125a^4+34a^3+18a^2+2a+1), \\ 2(a-1)^2(50a^4-17a^3+36a^2-a+4) &\geq 0. \end{aligned}$$

The last inequality is true because

$$50a^4 - 17a^3 + 36a^2 - a + 4 > 2a^2(5a-2)^2 + 4(a-1)^2 > 0.$$

The proof is completed. The equality occurs for  $a = b = c = d = e = f = 1$ .

**Remark.** The following more general statement holds:

- If  $a_1, a_2, \dots, a_n$  are nonnegative real numbers such that

$$a_1^2 + a_2^2 + \dots + a_n^2 = n, \quad a_1 \geq a_2 \geq \dots \geq a_n,$$

then

$$1 - a_1 a_2 \dots a_n \leq \frac{n}{4}(a_1 - a_n)^2,$$

with equality for  $a_1 = a_2 = \dots = a_n = 1$ .

To prove this inequality, we need to show that

$$4 - 4a^k b^{n-k} \leq n(a-b)^2$$

for  $a \geq 1 \geq b \geq 0$  and

$$ka^2 + (n-k)b^2 = n, \quad k \in \{1, 2, \dots, n-1\}.$$

□

**P 3.195.** If  $a \geq b \geq c \geq d \geq e \geq f \geq 0$ , then

$$a + b + c + d + e + f - 6\sqrt[6]{abcdef} \geq 2(\sqrt{b} - \sqrt{e})^2.$$

(Vasile Cîrtoaje, 2010)

**Solution.** Since

$$a + b \geq 2\sqrt{ab}, \quad c + d \geq 2\sqrt{cd}, \quad e + f \geq 2\sqrt{ef},$$

it suffices to show that

$$\sqrt{ab} + \sqrt{cd} + \sqrt{ef} - 3\sqrt[6]{abcdef} \geq (\sqrt{b} - \sqrt{e})^2.$$

By the AM-GM inequality, we have

$$\sqrt{cd} + 2\sqrt[4]{abef} \geq 3\sqrt[6]{abcdef},$$

which can be rewritten as

$$\sqrt{cd} - 3\sqrt[6]{abcdef} \geq -2\sqrt[4]{abef}.$$

Thus, it suffices to show that

$$\sqrt{ab} + \sqrt{ef} - 2\sqrt[4]{abef} \geq (\sqrt{b} - \sqrt{e})^2.$$

Since

$$\sqrt{ab} + \sqrt{ef} - 2\sqrt[4]{abef} = \left(\sqrt[4]{ab} - \sqrt[4]{ef}\right)^2,$$

we only need to prove that

$$\sqrt[4]{ab} - \sqrt[4]{ef} \geq \sqrt{b} - \sqrt{e},$$

which is equivalent to the obvious inequality

$$\sqrt[4]{b} \left(\sqrt[4]{a} - \sqrt[4]{b}\right) + \sqrt[4]{e} \left(\sqrt[4]{e} - \sqrt[4]{f}\right) \geq 0.$$

The equality holds for

$$a = b, \quad c = d, \quad e = f, \quad c^2 = ae.$$

□

**P 3.196.** If  $a, b, c, d, e, f$  are nonnegative real numbers such that

$$a \geq b \geq c \geq d \geq e \geq f,$$

then

$$(a + b + c + d + e + f)^2 \geq 8(ac + bd + ce + df).$$

(Vasile Cîrtoaje, 2005)

**First Solution.** Let us denote

$$x = b + c + d + e + f,$$

and write the inequality as follows:

$$(a + x)^2 - 8(ac + bd + ce + df) \geq 0,$$

$$(a + x - 4c)^2 + 8(a + x)c - 16c^2 - 8(ac + bd + ce + df) \geq 0,$$

$$(a + x - 4c)^2 - 8[c^2 - (b + d + f)c + d(b + f)] \geq 0,$$

$$(a + x - 4c)^2 - 8(c - d)(c - b - f) \geq 0,$$

$$(a + x - 4c)^2 + 8(c - d)(b - c + f) \geq 0.$$

The last inequality is clearly true. The equality holds for  $c = d = (a + b + e + f)/2$ , and for  $c = b + f = (a + d + e)/2$ ; that is, for

$$a = b = c = d, \quad e = f = 0,$$

and for

$$a \geq d + e, \quad b = c = \frac{a + d + e}{2}, \quad f = 0.$$

□

**P 3.197.** If  $a_1 \geq a_2 \geq \cdots \geq a_8 \geq 0$ , then

$$a_1 + a_2 + \cdots + a_8 - 8\sqrt[8]{a_1 a_2 \cdots a_8} \geq 3(\sqrt{a_6} - \sqrt{a_7})^2.$$

**Solution.** Let us denote

$$x = \sqrt[6]{a_1 a_2 \cdots a_6}, \quad y = \sqrt{a_7 a_8}, \quad x \geq a_6 \geq a_7 \geq y.$$

By the AM-GM inequality, we have

$$a_1 + a_2 + \cdots + a_6 \geq 6x, \quad a_7 + a_8 \geq 2y.$$

Also, we have

$$\sqrt{a_6} - \sqrt{a_7} \leq \sqrt{x} - \sqrt{y}.$$

Thus, it suffices to show that

$$6x + 2y - 8\sqrt[8]{x^6 y^2} \geq 3(\sqrt{x} - \sqrt{y})^2.$$

For the nontrivial case  $y \neq 0$ , we can set  $y = 1$  (due to homogeneity) and  $x = t^4$ ,  $t \geq 1$ . The inequality can be restated as

$$6t^4 + 2 - 8t^3 \geq 3(t^2 - 1)^2,$$

which is equivalent to

$$(t - 1)^3(3t + 1) \geq 0.$$

The equality holds for  $a_1 = a_2 = \cdots = a_8$ .

□



**P 3.198.** Let  $a, b, c$  and  $x, y, z$  be real numbers.

(a) If  $ab + bc + ca > 0$ , then

$$[(b+c)x + (c+a)y + (a+b)z]^2 \geq 4(ab + bc + ca)(xy + yz + zx);$$

(b) If  $a, b, c \geq 0$ , then

$$[(b+c)x + (c+a)y + (a+b)z]^2 \geq 4(a+b+c)(ayz + bzx + cxy).$$

(Vasile Cîrtoaje, 1995)

**Solution.** (a) **First Solution.** The condition  $ab + bc + ca > 0$  yields  $b + c \neq 0$ . Indeed, if  $b + c = 0$ , then  $ab + bc + ca = -b^2 \leq 0$ , which is false. The desired inequality is equivalent to  $D \geq 0$ , where  $D$  is the discriminant of the quadratic function

$$f(t) = (at - x)(bt - y) + (bt - y)(ct - z) + (ct - z)(at - x).$$

For the sake of contradiction, assume that  $D < 0$  for some real numbers  $a, b, c$  and  $x, y, z$ . Since the coefficient of  $t^2$  is positive, we have  $f(t) > 0$  for all real  $t$ . This is not true, because for

$$(bt - y) + (ct - z) = 0,$$

we get

$$t = \frac{y+z}{b+c}$$

and

$$f\left(\frac{y+z}{b+c}\right) = -\left(\frac{bz - cy}{b+c}\right)^2 \leq 0.$$

For  $pqr \neq 0$ , the equality holds when

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}.$$

**Second Solution.** If  $xy + yz + zx \leq 0$ , then the inequality is obviously true. Otherwise, due to homogeneity in  $x, y, z$ , we may assume that

$$x + y + z = a + b + c.$$

Then, by the AM-GM inequality, we have

$$\begin{aligned} 2\sqrt{(ab + bc + ca)(xy + yz + zx)} &\leq (ab + bc + ca) + (xy + yz + zx) \\ &= \frac{(a+b+c)^2 - a^2 - b^2 - c^2}{2} + \frac{(x+y+z)^2 - x^2 - y^2 - z^2}{2} \\ &= (a+b+c)(x+y+z) - \frac{a^2 + x^2}{2} - \frac{b^2 + y^2}{2} - \frac{c^2 + z^2}{2} \\ &\leq (a+b+c)(x+y+z) - ax - by - cz = (b+c)x + (c+a)y + (a+b)z. \end{aligned}$$

(b) Assume that  $x$  is between  $y$  and  $z$ , that is,

$$(x - y)(x - z) \leq 0.$$

Consider the non-trivial case

$$a + b + c > 0.$$

The desired inequality is equivalent to  $D \geq 0$ , where  $D$  is the discriminant of the quadratic function

$$f(t) = a(t - y)(t - z) + b(t - z)(t - x) + c(t - x)(t - y).$$

For the sake of contradiction, assume that  $D < 0$  for some  $a, b, c \geq 0$  and real numbers  $x, y, z$ . Since the coefficient of  $t^2$  is positive, we have  $f(t) > 0$  for all real  $t$ . This is false, because

$$f(x) = a(x - y)(x - z) \leq 0.$$

The equality holds for  $x = y = z$ , and also for  $a = 0$  and  $x = \frac{cy + bz}{c + b}$ , or  $b = 0$  and  $y = \frac{az + cx}{a + c}$ , or  $c = 0$  and  $z = \frac{bx + ay}{b + a}$ .

**Remark 1.** For  $x = b$ ,  $y = c$ ,  $z = a$ , from the inequality in (b), we get the following cyclic inequality:

$$(a^2 + b^2 + c^2 + ab + bc + ca)^2 \geq 4(a + b + c)(ab^2 + bc^2 + ca^2),$$

where  $a, b, c \geq 0$ . The equality holds for  $a = b = c$ , and also for  $a = 0$  and  $\frac{b}{c} = \frac{\sqrt{5} - 1}{2}$  (or any cyclic permutation). Notice that this inequality is equivalent to

$$a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 \geq 2(ab^3 + bc^3 + ca^3 - a^3b - b^3c - c^3a).$$

**Remark 2.** For  $x = 1/c$ ,  $y = 1/a$ ,  $z = 1/b$ , from the inequality in (b), we get the following cyclic inequality:

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 3\right)^2 \geq 4(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right),$$

which is the inequality in P 1.49-(c).

**Remark 3.** For  $a = x(x - y + z)$ ,  $b = y(y - z + x)$ ,  $c = z(z - x + y)$ , the inequality in (b) turns into

$$(x^2y + y^2z + z^2x)^2 \geq xyz(x + y + z)(x^2 + y^2 + z^2).$$

where  $x, y, z$  are the lengths of the sides of a triangle (see P 1.166).

□

**P 3.199.** If  $a, b, c$  and  $x, y, z$  are nonnegative real numbers such that

$$x^3 + y^3 + z^3 = a^3 + b^3 + c^3,$$

then

$$(a + b + c)(x + y + z) \geq xy + yz + zx + ab + bc + ca.$$

(Vasile Cîrtoaje, 2020)

**Solution.** Denote

$$t = x + y + z.$$

From the known inequalities

$$(x + y + z)^3 \geq x^3 + y^3 + z^3, \quad (x + y + x)^3 \leq 9(x^3 + y^3 + z^3),$$

we get

$$\begin{aligned} x + y + z &\geq \sqrt[3]{x^3 + y^3 + z^3} = \sqrt[3]{a^3 + b^3 + c^3}, \\ x + y + z &\leq \sqrt[3]{9(x^3 + y^3 + z^3)} = \sqrt[3]{9(a^3 + b^3 + c^3)}, \end{aligned}$$

hence

$$t_1 \leq t \leq t_2,$$

where

$$t_1 = \sqrt[3]{a^3 + b^3 + c^3}, \quad t_2 = \sqrt[3]{9(a^3 + b^3 + c^3)}.$$

On the other hand, it is enough to prove the inequality

$$(a + b + c)(x + y + z) \geq \frac{(a + b + c)^2}{3} + \frac{(x + y + z)^2}{3},$$

or, better, the inequality

$$(a + b + c)(x + y + z) \geq \frac{5(a + b + c)^2}{12} + \frac{(x + y + z)^2}{3},$$

which is equivalent to

$$5t^2 - 12(x + y + z)t + 4(x + y + z)^2 \leq 0,$$

$$[t - 2(x + y + z)][5t - 2(x + y + z)] \leq 0.$$

This is true if  $t_1 \geq \frac{2}{5}(x + y + z)$  and  $t_2 \leq 2(x + y + z)$ . Thus, we need to show that

$$x^3 + y^3 + z^3 \geq \frac{8}{125}(x + y + z)^3$$

and

$$4(x^3 + y^3 + z^3) \leq 8(x + y + z)^3.$$

These follow immediately from the known inequalities

$$x^3 + y^3 + z^3 \geq \frac{1}{9}(x + y + z)^3$$

and

$$x^3 + y^3 + z^3 \leq (x + y + z)^3.$$

The equality holds for  $a = b = x = y = z = 0$ .

**Remark 1.** Using the same method, we can prove the stronger inequalities

$$3k(a + b + c)(x + y + z) \geq (xy + yz + zx)^2 + (ab + bc + ca)^2,$$

$$k(a + b + c)(x + y + z) \geq xy + yz + zx + ab + bc + ca,$$

where

$$k = \frac{1}{\sqrt[3]{3}} + \frac{\sqrt[3]{3}}{9} \approx 0.8536.$$

**Remark 2.** In the same conditions, the following inequality holds:

$$k(a + b + c)(x + y + z) \geq xy + yz + zx + ab + bc + ca, \quad k = \frac{1}{\sqrt[3]{3}} \approx 0.6934,$$

with equality for  $a = b = c = x = y = z$ . If  $a \geq b \geq c$  and  $x \geq y \geq z$ , the equality occurs also for  $a = b = c = \frac{x}{\sqrt[3]{3}}$  and  $y = z = 0$ , and for  $x = y = z = \frac{a}{\sqrt[3]{3}}$  and  $b = c = 0$ . □

**P 3.200.** Let  $a, b, c$  and  $x, y, z$  be positive real numbers such that

$$(a + b + c)(x + y + z) = (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = 4.$$

Prove that

$$abcxyz < \frac{1}{36}.$$

(Vasile Cîrtoaje, 1997)

**Solution.** Using the AM-GM inequality, we have

$$\begin{aligned} & 4(ab + bc + ca)(xy + yz + zx) = \\ & = [(a + b + c)^2 - (a^2 + b^2 + c^2)][(x + y + z)^2 - (x^2 + y^2 + z^2)] \\ & = 20 - (a + b + c)^2(x^2 + y^2 + z^2) - (x + y + z)^2(a^2 + b^2 + c^2) \\ & \leq 20 - 2(a + b + c)(x + y + z)\sqrt{(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)} = 4, \end{aligned}$$

hence

$$(ab + bc + ca)(xy + yz + zx) \leq 1.$$

On the other hand, multiplying the well-known inequalities

$$(ab + bc + ca)^2 \geq 3abc(a + b + c)$$

and

$$(xy + yz + zx)^2 \geq 3xyz(x + y + z),$$

we get

$$(ab + bc + ca)^2(xy + yz + zx)^2 \geq 36abcxyz,$$

hence

$$1 \geq (ab + bc + ca)^2(xy + yz + zx)^2 \geq 36abcxyz.$$

In order to have  $36abcxyz = 1$ , the following relations are necessary:

$$(ab + bc + ca)^2 = 3abc(a + b + c)$$

and

$$(xy + yz + zx)^2 = 3xyz(x + y + z).$$

These relations imply  $a = b = c$  and  $x = y = z$ , which contradict the hypothesis

$$(a + b + c)(x + y + z) = (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = 4.$$

Consequently, we have  $abcxyz < \frac{1}{36}$ .

**Remark.** The following sharper result holds (Vasile Cîrtoaje and Vo Quoc Ba Can, 2008):

- If  $a, b, c$  and  $x, y, z$  are positive real numbers such that

$$(a + b + c)(x + y + z) = (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = 4,$$

then

$$abcxyz \leq \frac{16}{729},$$

with equality for  $(a, b, c) = \left(\frac{1}{3}, \frac{1}{3}, \frac{4}{3}\right)$  (or any cyclic permutation) and  $(x, y, z) = \left(\frac{1}{3}, \frac{1}{3}, \frac{4}{3}\right)$  (or any cyclic permutation). □

**P 3.201.** Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be nonnegative real numbers such that

$$a_1^2 + a_2^2 + \dots + a_n^2 = b_1^2 + b_2^2 + \dots + b_n^2.$$

Then, for  $n = 3$  and  $n = 4$ , the following inequalities holds:

$$(n - 1)(a_1 + a_2 + \dots + a_n)(b_1 + b_2 + \dots + b_n) \geq n \left( \sum_{i < j} a_i a_j + \sum_{i < j} b_i b_j \right).$$

(Vasile Cîrtoaje, 2020)

**Solution.** Denote

$$a = a_1 + a_2 + \cdots + a_n, \quad b = b_1 + b_2 + \cdots + b_n, \quad c = \sqrt{n(b_1^2 + b_2^2 + \cdots + b_n^2)},$$

and write the inequality as

$$2(n-1)ab + 2c^2 \geq na^2 + nb^2.$$

We have

$$a \leq \sqrt{n(a_1^2 + a_2^2 + \cdots + a_n^2)} = \sqrt{n(b_1^2 + b_2^2 + \cdots + b_n^2)} = c$$

and

$$a \geq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} = \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2} = \frac{c}{\sqrt{n}}.$$

For fixed  $b$  and  $c$ , we may write the required inequality as  $f(a) \leq 0$ , where

$$f(a) = na^2 - 2(n-1)ab + nb^2 - 2c^2$$

is a quadratic convex function. Thus, it is enough to show that  $f(c) \leq 0$  and  $f(c/\sqrt{n}) \leq 0$ . We have

$$f(c) = nb^2 - 2(n-1)bc + (n-2)c^2 = (b-c)[nb - (n-2)c] \leq 0.$$

Since  $b \leq c$  and  $nb \geq (n-2)c$  for  $n = 3$  and  $n = 4$ , we have  $f(c) \leq 0$ . The inequality  $f(c/\sqrt{n}) \leq 0$  is equivalent to

$$nb^2 - \frac{2(n-1)}{\sqrt{n}}bc - c^2 \leq 0.$$

Since  $c \geq b$ , it is enough to show that

$$n - \frac{2(n-1)}{\sqrt{n}} - 1 \leq 0,$$

which is true for  $n = 3$  and  $n = 4$ .

The proof is completed. The equality occurs when all  $a_i$  and  $b_i$  are equal.

□

**P 3.202.** Let  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) be positive real numbers such that

$$a_1 + a_2 + \cdots + a_n = a_1^2 + a_2^2 + \cdots + a_n^2 = n - 1.$$

Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq \frac{n^2(2n-3)}{2(n-1)(n-2)}.$$

(Vasile Cîrtoaje, 2010)

**Solution.** By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} n-1 &= a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{(a_1 + a_2 + \cdots + a_{n-1})^2}{n-1} + a_n^2 \\ &= \frac{(n-1-a_n)^2}{n-1} + a_n^2, \end{aligned}$$

which provides

$$a_n \leq \frac{2(n-1)}{n}.$$

Similarly,  $a_i \leq 2(n-1)/n$  for all  $i$ . The hint for proving the given inequality is to apply the Cauchy-Schwarz inequality after we made the numerators nonnegative and as small as possible. So, since  $2n-2-na_i \geq 0$ , we have

$$\begin{aligned} \sum \frac{1}{a_1} &= \sum \left( \frac{1}{a_1} - \frac{n}{2n-2} \right) + \frac{n^2}{2n-2} \\ &= \frac{1}{2(n-1)} \sum \frac{2n-2-na_1}{a_1} + \frac{n^2}{2n-2} \\ &\geq \frac{1}{2(n-1)} \cdot \frac{[\sum (2n-2-na_1)]^2}{\sum a_1(2n-2-na_1)} + \frac{n^2}{2n-2} \\ &= \frac{1}{2(n-1)} \cdot \frac{[n(2n-2)-n \sum a_1]^2}{(2n-2) \sum a_1 - n \sum a_1^2} + \frac{n^2}{2n-2} \\ &= \frac{1}{2(n-1)} \cdot \frac{n^2(n-1)^2}{(n-1)(n-2)} + \frac{n^2}{2n-2} = \frac{n^2(2n-3)}{2(n-1)(n-2)}, \end{aligned}$$

from where the conclusion follows. The equality holds for  $a_1 = a_2 = \cdots = a_{n-1} = 1 - 2/n$  and  $a_n = 2 - 2/n$  (or any cyclic permutation). □

**P 3.203.** Let  $a_1, a_2, \dots, a_n$  be positive real numbers such that

$$a_1 + a_2 + \cdots + a_n = n.$$

Prove that

$$n^2 \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n \right) \geq 4(n-1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

(Vasile Cîrtoaje, 2004)

**Solution.** From

$$\frac{1}{n}(a_1 + a_2 + \cdots + a_n)^2 \leq a_1^2 + a_2^2 + \cdots + a_n^2 < (a_1 + a_2 + \cdots + a_n)^2,$$

it follows that

$$n \leq a_1^2 + a_2^2 + \cdots + a_n^2 < n^2.$$

Thus, we can use the substitution

$$a_1^2 + a_2^2 + \cdots + a_n^2 = n + n(n-1)t^2,$$

where  $0 \leq t < 1$ . On the other hand, from

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq a_1^2 + \frac{(a_2 + \cdots + a_n)^2}{n-1} = a_1^2 + \frac{(n-a_1)^2}{n-1},$$

we get

$$n + n(n-1)t^2 \geq a_1^2 + \frac{(n-a_1)^2}{n-1},$$

which involves  $1 - (n-1)t \leq a_1 \leq 1 + (n-1)t$ ; similarly, we get

$$1 - (n-1)t \leq a_i \leq 1 + (n-1)t$$

for any  $i$ . We will apply now the Cauchy-Schwarz inequality after we made the numerators nonnegative and as small as possible. Since

$$\begin{aligned} \sum \frac{1}{a_1} &= \sum \frac{1}{1 + (n-1)t} + \sum \left[ \frac{1}{a_1} - \frac{1}{1 + (n-1)t} \right] \\ &= \frac{n}{1 + (n-1)t} + \frac{1}{1 + (n-1)t} \sum \frac{1 + (n-1)t - a_1}{a_1} \end{aligned}$$

and

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n = n(n-1)t^2,$$

we can write the desired inequality as

$$\sum \frac{1 + (n-1)t - a_1}{a_1} \geq n(n-1)t + \frac{4(n-1)^2 t^2 [1 + (n-1)t]}{n}.$$

By virtue of the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \sum \frac{1 + (n-1)t - a_1}{a_1} &\geq \frac{[\sum (1 + (n-1)t - a_1)]^2}{\sum a_1 (1 + (n-1)t - a_1)} \\ &= \frac{[n + n(n-1)t - \sum a_1]^2}{(1 + (n-1)t) \sum a_1 - \sum a_1^2} \\ &= \frac{n(n-1)t}{1-t}, \end{aligned}$$

Therefore, it suffices to prove that

$$\frac{n(n-1)t}{1-t} \geq n(n-1)t + \frac{4(n-1)^2 t^2 [(1 + (n-1)t)]}{n}.$$



This inequality is true if

$$4(n-1)(1-t)[1+(n-1)t] \leq n^2.$$

Indeed,

$$4(n-1)(1-t)[1+(n-1)t] \leq [(n-1)(1-t) + 1 + (n-1)t]^2 = n^2.$$

The equality holds for  $a_1 = a_2 = \cdots = a_n = 1$ , and also for  $a_1 = \frac{n}{2}$  and  $a_2 = \cdots = a_n = \frac{n}{2n-2}$  (or any cyclic permutation). □

**P 3.204.** If  $a_1, a_2, \dots, a_n$  are positive real numbers such that

$$a_1 + a_2 + \cdots + a_n = n, \quad a_2, a_3, \dots, a_n \geq 1,$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq a_1^2 + a_2^2 + \cdots + a_n^2.$$

(Vasile Cîrtoaje, 2021)

**Solution.** We will use the induction method. For  $n = 2$ , we need to show that  $a_1 + a_2 = 2$  involves

$$\frac{1}{a_1} + \frac{1}{a_2} \geq a_1^2 + a_2^2.$$

This is equivalent to

$$(a_1 a_2 - 1)^2 \geq 0.$$

Next, assume  $n \geq 3$  and

$$a_1 \leq 1 \leq a_2 \leq \cdots \leq a_n.$$

For fixed  $a_1$  and  $a_3, \dots, a_{n-1}$ , write the required inequality as  $f(a_2) \geq 0$ , where

$$f(a_2) = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - (a_1^2 + a_2^2 + \cdots + a_n^2), \quad a_n = n - a_1 - a_2 - \cdots - a_{n-1}.$$

We will show that

$$f(a_2) \geq f(1) \geq 0.$$

Since

$$2a_2^2 a_n^2 \geq 2a_n^2 \geq 2a_n \geq a_2 + a_n,$$

we have

$$f'(a_2) = \frac{-1}{a_2^2} + \frac{1}{a_n^2} - 2(a_2 - a_n) = (a_n - a_2) \left( 2 - \frac{a_2 + a_n}{a_2^2 a_n^2} \right) \geq 0,$$

$f(a_2)$  is increasing, hence  $f(a_2) \geq f(1)$ . The inequality  $f(1) \geq 0$  has the form

$$\frac{1}{a_1} + \frac{1}{a_3} + \cdots + \frac{1}{a_n} \geq a_1^2 + a_3^2 + \cdots + a_n^2.$$

where  $a_n = n - 1 - a_1 - a_3 - \cdots - a_{n-1}$  and  $a_1 \leq 1 \leq a_3 \leq \cdots \leq a_n$ . Clearly, this is true by the induction hypothesis.

The equality occurs for  $a_1 = a_2 = \cdots = a_n = 1$ .

□

**P 3.205.** Let  $a_1, a_2, \dots, a_n$  be nonnegative real numbers such that

$$a_1 + a_2 + \cdots + a_n = n.$$

Prove that

$$(n+1)(a_1^2 + a_2^2 + \cdots + a_n^2) \geq n^2 + a_1^3 + a_2^3 + \cdots + a_n^3.$$

(Vasile Cîrtoaje, 2002)

**First Solution.** If  $a_1 = a_2 = \cdots = a_n$ , then the equality holds. Otherwise, as in the proof of P 3.203, we will use the substitution

$$a_1^2 + a_2^2 + \cdots + a_n^2 = n + n(n-1)t^2, \quad 0 < t \leq 1;$$

in addition, we have

$$1 - (n-1)t \leq a_i \leq 1 + (n-1)t, \quad i = 1, 2, \dots, n.$$

From the Cauchy-Schwarz inequality

$$\sum [1 + (n-1)t - a_1] a_1^2 \geq \frac{[\sum (1 + (n-1)t - a_1) a_1]^2}{\sum [1 + (n-1)t - a_1]} = n(n-1)t(1-t)^2,$$

we get

$$\begin{aligned} \sum a_1^3 &\leq [1 + (n-1)t] \sum a_1^2 - n(n-1)t(1-t)^2 \\ &= n[(n-1)(n-2)t^3 + 3(n-1)t^2 + 1]. \end{aligned}$$

Therefore, it suffices to show that

$$(n+1)[n + n(n-1)t^2] \geq n^2 + n[(n-1)(n-2)t^3 + 3(n-1)t^2 + 1],$$

which is equivalent to the obvious inequality

$$n(n-1)(n-2)t^2(1-t) \geq 0.$$

For  $n = 2$ , the original inequality is an identity. For  $n \geq 3$ , the equality holds for  $a_1 = a_2 = \cdots = a_n = 1$ , and also for  $a_1 = n$  and  $a_2 = \cdots = a_n = 0$  (or any cyclic permutation).

**Second Solution.** Assume that

$$a_1 \geq a_2 \geq \cdots \geq a_n.$$

Replacing  $n^2$  by  $n(a_1 + a_2 + \cdots + a_n)$ , the desired inequality becomes as follows

$$\sum [(n+1)a_1^2 - na_1 - a_1^3] \geq 0,$$

$$\sum (a_1 - 1)(na_1 - a_1^2) \geq 0.$$

Since

$$a_1 - 1 \geq a_2 - 1 \geq \cdots \geq a_n - 1$$

and

$$na_1 - a_1^2 \geq na_2 - a_2^2 \geq \cdots \geq na_n - a_n^2,$$

we apply Chebyshev's inequality to get

$$n \sum (a_1 - 1)(na_1 - a_1^2) \geq \left[ \sum (a_1 - 1) \right] \left[ \sum (na_1 - a_1^2) \right] = 0.$$

□

**P 3.206.** Let  $a_1, a_2, \dots, a_n$  be nonnegative real numbers such that

$$a_1 + a_2 + \cdots + a_n = n.$$

Prove that

$$(n-1)(a_1^3 + a_2^3 + \cdots + a_n^3) + n^2 \geq (2n-1)(a_1^2 + a_2^2 + \cdots + a_n^2).$$

(Vasile Cîrtoaje, 2002)

**Solution.** For  $a_1 = a_2 = \cdots = a_n$ , the equality holds. Otherwise, as in the proof of P 3.203, we will use the substitution

$$a_1^2 + a_2^2 + \cdots + a_n^2 = n + n(n-1)t^2, \quad 0 < t \leq 1;$$

in addition, for any  $i$ , we have

$$1 - (n-1)t \leq a_i \leq 1 + (n-1)t.$$

By the Cauchy-Schwarz inequality, we have

$$\sum [a_1 - 1 + (n-1)t]a_1^2 \geq \frac{[\sum (a_1 - 1 + (n-1)t)a_1]^2}{\sum [a_1 - 1 + (n-1)t]} = n(n-1)t(t+1)^2,$$

which yields

$$\begin{aligned}\sum a_1^3 &\geq n(n-1)t(t+1)^2 + [1 - (n-1)t] \sum a_1^2 \\ &= n[1 + 3(n-1)t^2 - (n-1)(n-2)t^3].\end{aligned}$$

Therefore, it suffices to show that

$$(n-1)n[1 + 3(n-1)t^2 - (n-1)(n-2)t^3] + n^2 \geq (2n-1)[n + n(n-1)t^2],$$

which is equivalent to the obvious inequality

$$n(n-1)(n-2)t^2[1 - (n-1)t] \geq 0.$$

The equality holds for  $a_1 = a_2 = \cdots = a_n = 1$ , and also for  $a_1 = 0$  and  $a_2 = \cdots = a_n = \frac{n}{n-1}$  (or any cyclic permutation). □

**P 3.207.** Let  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) be nonnegative real numbers such that

$$a_1 + a_2 + \cdots + a_n = n.$$

Prove that

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq \frac{n}{n-1}(1 - a_1 a_2 \cdots a_n).$$

**Solution.** According to Remark 1 from P 3.57, for fixed  $a_1^2 + a_2^2 + \cdots + a_n^2$ , the product  $a_1 a_2 \cdots a_n$  has the minimum value when one of  $a_1, a_2, \dots, a_n$  is zero or  $n-1$  numbers of  $a_1, a_2, \dots, a_n$  are equal. Therefore, it suffices to consider these cases.

*Case 1:*  $a_1 = 0$ . We need to show that  $a_2 + a_3 + \cdots + a_n = n$  involves

$$a_2^2 + a_3^2 + \cdots + a_n^2 \geq \frac{n^2}{n-1}.$$

Indeed, by the Cauchy-Schwarz inequality, we have

$$a_2^2 + a_3^2 + \cdots + a_n^2 \geq \frac{1}{n-1}(a_2 + a_3 + \cdots + a_n)^2 = \frac{n^2}{n-1}.$$

*Case 2:*  $a_2 = a_3 = \cdots = a_n$ . Setting  $a_1 = x$  and  $a_2 = y$ , we need to show that

$$x + (n-1)y = n$$

involves

$$x^2 + (n-1)y^2 - n + \frac{n}{n-1}(xy^{n-1} - 1) \geq 0.$$

By Bernoulli's inequality, we have

$$y^{n-1} = \left(1 + \frac{1-x}{n-1}\right)^{n-1} \geq 1 + (1-x) = 2-x.$$

Therefore, it suffices to prove that

$$x^2 + (n-1)y^2 - n + \frac{n}{n-1} [x(2-x) - 1] \geq 0,$$

which is an identity. The equality holds for  $a_1 = a_2 = \dots = a_n = 1/n$ , and also for  $a_1 = 0$  and  $a_2 = \dots = a_n = \frac{n}{n-1}$  (or any cyclic permutation). □

**P 3.208.** If  $a_1, a_2, \dots, a_n$  are positive numbers such that  $a_1 + a_2 + \dots + a_n = n$  and

$$a_1 a_2 \cdots a_n \leq \frac{1}{(n-1)^{n-2}},$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \geq a_1^2 + a_2^2 + \dots + a_n^2.$$

(Vasile Cîrtoaje, 2018)

**Solution.** According to Remark 3 from P 3.58, the following statement is valid:

- If  $a_1, a_2, \dots, a_n$  are positive real numbers so that

$$a_1 + a_2 + \dots + a_n = n, \quad a_1 a_2 \cdots a_n = \text{constant}, \quad a_1 \geq a_2 \geq \dots \geq a_n,$$

then the sums

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}$$

and

$$-(a_1^2 + a_2^2 + \dots + a_n^2)$$

have the minimum value when  $a_1 \geq a_2 = \dots = a_n$ .

Thus, it is enough to consider the case  $a_1 \geq a_2 = \dots = a_n$ , that means to show that

$$\frac{1}{x} + \frac{n-1}{y} \geq x^2 + (n-1)y^2$$

for

$$x + (n-1)y = n, \quad xy^{n-1} \leq \frac{1}{(n-1)^{n-2}}, \quad n > x \geq 1 \geq y > 0.$$

Write the inequality as follows:

$$\begin{aligned}\frac{(n-1)(1-y^3)}{y} &\geq \frac{x^3-1}{x}, \\ \frac{(n-1)(1-y)(1+y+y^2)}{y} &\geq \frac{(x-1)(x^2+x+1)}{x}, \\ \frac{(n-1)(1-y)(1+y+y^2)}{y} &\geq \frac{(n-1)(1-y)(x^2+x+1)}{x}, \\ (1-y) \left( \frac{1}{y} + y - x - \frac{1}{x} \right) &\geq 0, \\ (1-y)(x-y)(1-xy) &\geq 0, \\ (1-y)^2(1-xy) &\geq 0.\end{aligned}$$

So, we need to show that  $1-xy \geq 0$ . For  $x = 1$ , which implies  $y = 1$ , the inequality is an equality. Consider further  $x > 1$ . Since

$$1-xy = 1 - \frac{(n-x)x}{n-1} = \frac{(x-1)(x-n+1)}{n-1},$$

we need to prove that  $x \geq n-1$ . Write the hypothesis  $xy^{n-1} \leq \frac{1}{(n-1)^{n-2}}$  as  $f(x) \geq 0$ , where

$$f(x) = n-1-x(n-x)^{n-1}.$$

From

$$f'(x) = n(n-x)^{n-2}(x-1) > 0,$$

it follows that  $f$  is strictly increasing. Since  $f(n-1) = 0$ , the hypothesis  $f(x) \geq 0$  involves  $x \geq n-1$ .

The inequality is an equality for  $a_1 = n-1$  and  $a_2 = \dots = a_n = \frac{1}{n-1}$  (or any cyclic permutation). □

**P 3.209.** If  $a_1 \geq a_2 \geq \dots \geq a_n > 0$  such that  $a_1 + a_2 + \dots + a_n = n$ , then

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} - n \geq \frac{4(n-1)^2}{n^3}(a_1 - a_2)^2.$$

(Vasile Cîrtoaje, 2009)

**Solution.** Since

$$\frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n} \geq \frac{(n-1)^2}{a_2 + a_3 + \cdots + a_n} = \frac{(n-1)^2}{n-a_1}$$

and

$$a_1 - a_2 \leq a_1 - \frac{a_2 + a_3 + \cdots + a_n}{n-1} = a_1 - \frac{n-a_1}{n-1} = \frac{n(a_1-1)}{n-1},$$

it suffices to show that

$$\frac{1}{a_1} + \frac{(n-1)^2}{n-a_1} - n \geq \frac{4}{n}(a_1-1)^2.$$

This is equivalent to the obvious inequality

$$(a_1-1)^2(2a_1-n)^2 \geq 0.$$

The equality holds for

$$a_1 = a_2 = \cdots = a_n = 1,$$

and also for

$$a_1 = \frac{n}{2}, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{2(n-1)}.$$

□

**P 3.210.** If  $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ , then

$$a_1 + a_2 + \cdots + a_n - n \sqrt[n]{a_1 a_2 \cdots a_n} \leq (n-1)(\sqrt{a_1} - \sqrt{a_n})^2.$$

(Vasile Cîrtoaje, 2010)

**Solution.** For  $n = 2$ , the inequality is an identity. Next assume  $n \geq 3$ . For  $a_n = 0$ , the inequality becomes  $a_2 + \cdots + a_{n-1} \leq (n-1)a_1$ , which is true. For  $a_n > 0$ , due to homogeneity we may set  $a_n = 1$ . For the nontrivial case  $a_1 > 1$ , consider  $a_1$  fixated and denote  $a_1 = \alpha > 1$ . We need to show that  $f(a_2, \dots, a_{n-1}) \leq 0$  for  $a_2, \dots, a_{n-1} \in [1, \alpha]$ , where

$$f(a_2, \dots, a_{n-1}) = \alpha + a_2 + \cdots + a_{n-1} + 1 - n \sqrt[n]{\alpha a_2 \cdots a_{n-1}} - (n-1)(\sqrt{\alpha} - 1)^2.$$

Since  $f$  is convex with respect to each variable, it has the maximum value when  $a_2, \dots, a_{n-1} \in \{1, \alpha\}$ , that is

$$a_2 = \dots = a_j = \alpha, \quad a_{j+1} = \dots = a_{n-1} = 1,$$

where  $j \in \{1, 2, \dots, n-1\}$ . So, it suffices to show that

$$j\alpha + n - j - n\alpha^{j/n} - (n-1)(\sqrt{\alpha} - 1)^2 \leq 0,$$

that is

$$(n-j-1)\alpha + n\alpha^{j/n} + j - 1 \geq 2(n-1)\sqrt{\alpha}.$$

Clearly, this inequality follows from the AM-GM inequality applied to  $2n-2$  numbers. For  $n \geq 3$ , the equality occurs when  $a_1 = a_2 = \cdots = a_n$ , and also when  $a_n = 0$  and  $a_1 = a_2 = \cdots = a_{n-1}$ .

□

**P 3.211.** If  $0 \leq a_1 \leq a_2 \leq \dots \leq a_n$ , then

$$a_1 + a_2 + \dots + a_n + \frac{k(a_1 - a_2)^2}{a_1 + a_2} \leq \sqrt{n(a_1^2 + a_2^2 + \dots + a_n^2)},$$

where  $k = \frac{1}{1 + \sqrt{\frac{n}{n-1}}}$ .

(Vasile Cîrtoaje, *Math. Reflections*, 4, 2024)

**Solution.** Write the inequality as  $F(a_1, a_2, \dots, a_n) \leq 0$ . We will show that

$$F(a_1, a_2, \dots, a_{n-1}, a_n) \leq F(a_1, a_2, \dots, a_{n-1}, a_{n-1}) \leq \dots \leq F(a_1, a_2, \dots, a_2) \leq 0.$$

Firstly, we show that

$$F(a_1, \dots, a_{n-j-1}, a_{n-j}, a_{n-j+1}, \dots, a_{n-j+1}) \leq F(a_1, \dots, a_{n-j-1}, a_{n-j}, \dots, a_{n-j})$$

for  $j \in \{1, \dots, n-2\}$ , i.e.

$$\begin{aligned} & a_1 + \dots + a_{n-j-1} + a_{n-j} + ja_{n-j+1} + k(\sqrt{a_1} - \sqrt{a_2})^2 - \sqrt{n(a_1^2 + \dots + a_{n-j}^2 + ja_{n-j+1}^2)} \leq \\ & \leq a_1 + \dots + a_{n-j-1} + (j+1)a_{n-j} + k(\sqrt{a_1} - \sqrt{a_2})^2 - \sqrt{n[a_1^2 + \dots + a_{n-j-1}^2 + (j+1)a_{n-j}^2]}, \end{aligned}$$

or

$$\begin{aligned} j(a_{n-j+1} - a_{n-j}) & \leq \sqrt{n(a_1^2 + \dots + a_{n-j}^2 + ja_{n-j+1}^2)} - \sqrt{n[a_1^2 + \dots + a_{n-j-1}^2 + (j+1)a_{n-j}^2]}, \\ j(a_{n-j+1} - a_{n-j}) & \leq \frac{jn(a_{n-j+1}^2 - a_{n-j}^2)}{\sqrt{n(a_1^2 + \dots + a_{n-j}^2 + ja_{n-j+1}^2)} + \sqrt{n[a_1^2 + \dots + a_{n-j-1}^2 + (j+1)a_{n-j}^2]}}. \end{aligned}$$

We need to show that

$$\sqrt{n(a_1^2 + \dots + a_{n-j}^2 + ja_{n-j+1}^2)} + \sqrt{n[a_1^2 + \dots + a_{n-j-1}^2 + (j+1)a_{n-j}^2]} \leq n(a_{n-j+1} + a_{n-j}).$$

It is true because

$$a_1^2 + \dots + a_{n-j}^2 + ja_{n-j+1}^2 \leq na_{n-j+1}^2$$

and

$$a_1^2 + \dots + a_{n-j-1}^2 + (j+1)a_{n-j}^2 \leq na_{n-j}^2.$$

The inequality  $F(a_1, a_2, \dots, a_2) \leq 0$  is equivalent to

$$a_1 + (n-1)a_2 + \frac{k(a_1 - a_2)^2}{a_1 + a_2} \leq \sqrt{na_1^2 + n(n-1)a_2^2}.$$

For  $a_1 = 0$ , this inequality is an identity. For  $a_1 > 0$ , due to homogeneity, we may set  $a_1 = 1$ . Denoting  $a_2 = x$  ( $x \geq 1$ ), the inequality becomes

$$1 + (n-1)x + \frac{k(1-x)^2}{1+x} \leq \sqrt{n + n(n-1)x^2},$$



$$\frac{k(1-x)^2}{1+x} \leq \frac{(n-1)(x-1)^2}{\sqrt{n+n(n-1)x^2+(n-1)x+1}}.$$

So, we need to show that

$$k \leq \frac{(n-1)(x+1)}{\sqrt{n+n(n-1)x^2+(n-1)x+1}},$$

i.e.

$$k\sqrt{n+n(n-1)x^2} \leq (1-k)(n-1)x + n - k - 1.$$

It is true if

$$k^2[n+n(n-1)x^2] \leq (1-k)^2(n-1)^2x^2 + (n-k-1)^2,$$

which is equivalent to

$$k^2n \leq (n-k-1)^2,$$

i.e.

$$k\sqrt{n} \leq n - k - 1,$$

$$k \leq \frac{n-1}{\sqrt{n}+1} = \sqrt{n} - 1.$$

The inequality  $k \leq \sqrt{n} - 1$  is true, with equality for  $n = 2$ . Thus, the proof is completed. The equality occurs for  $a_1 = a_2 = \cdots = a_n$ , and also for  $a_1 = 0$  and  $a_2 = \cdots = a_n$ .

**Remark.** Since

$$\frac{(a_1 - a_2)^2}{a_1 + a_2} \geq (\sqrt{a_1} - \sqrt{a_2})^2,$$

we get the following statement:

- If  $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ , then

$$a_1 + a_2 + \cdots + a_n + k(\sqrt{a_1} - \sqrt{a_2})^2 \leq \sqrt{n(a_1^2 + a_2^2 + \cdots + a_n^2)},$$

where  $k = \frac{1}{1 + \sqrt{\frac{n}{n-1}}}$ .

□

**P 3.212.** For given  $n \geq 2$ , find the largest integer  $k$  such that

$$\left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right)^2 \geq \frac{a_1^2 + a_2^2 + \cdots + a_k^2}{k}$$

for all nonnegative numbers  $a_i$  satisfying  $a_1 \leq a_2 \leq \cdots \leq a_n$ .

(Vasile Cîrtoaje, *Cruz Mathematicorum*, 7, 2024)

**Solution.** Setting  $a_1 = 0$  and  $a_2 = \cdots = a_n = 1$ , the inequality leads to

$$\left(\frac{n-1}{n}\right)^2 \geq \frac{k-1}{k}, \quad k \leq \frac{n^2}{2n-1} = \frac{n}{2} + \frac{n}{2(2n-1)}.$$

Since  $\frac{n}{2(2n-1)} < \frac{1}{2}$ , we have

$$k \leq k_0 = \left\lfloor \frac{n}{2} \right\rfloor.$$

To show that  $k_0$  is the largest value of  $k$ , we need to prove the inequality for

$$k = k_0 \leq \frac{n}{2}.$$

**First Solution.** For  $n = 2$  and  $n = 3$  (when  $k = 1$ ), the inequality reduces to  $a_2 \geq a_1$  and  $a_2 + a_3 \geq 2a_1$ , respectively. Next assume  $n \geq 4$ , when  $k \geq 2$ . Since  $a_k \leq a_{k+1} \leq \cdots \leq a_n$ , we have

$$a_1 + a_2 + \cdots + a_n \geq a_1 + \cdots + a_{k-1} + (n - k + 1)a_k.$$

So, it suffices to show that  $F \geq 0$ , where

$$F = [a_1 + \cdots + a_{k-1} + (n - k + 1)a_k]^2 - \frac{n^2}{k}(a_1^2 + \cdots + a_{k-1}^2 + a_k^2).$$

We will prove this inequality for the hypothesis  $a_k = \max\{a_1, a_2, \dots, a_k\}$ , which is weaker than the original hypothesis  $a_1 \leq a_2 \leq \cdots \leq a_k$ . For fixed  $a_2, \dots, a_k$ , the inequality  $F \geq 0$  can be written as  $f(a_1) \geq 0$ , where  $a_1 \in [0, a_k]$ . We have

$$f(a_1) - f(a_k) = (a_1 - a_k)[a_1 + 2(a_2 + \cdots + a_{k-1}) + (2n - 2k + 3)a_k] - \frac{n^2}{k}(a_1^2 - a_k^2) = (a_1 - a_k)A,$$

where

$$A = a_1 + 2(a_2 + \cdots + a_{k-1}) + (2n - 2k + 3)a_k - \frac{n^2}{k}(a_1 + a_k).$$

Since  $k \leq \frac{n}{2}$ , we have

$$\begin{aligned} A &\leq a_1 + 2(a_2 + \cdots + a_{k-1}) + (2n - 2k + 3)a_k - 2n(a_1 + a_k) = -(2n - 1)a_1 + 2(a_2 + \cdots + a_{k-1}) - (2k - 3)a_k \\ &\leq 2(a_2 + \cdots + a_{k-1}) - (2k - 3)a_k \leq 2(k - 2)a_k - (2k - 3)a_k = -a_k \leq 0, \end{aligned}$$

therefore

$$f(a_1) - f(a_k) = (a_1 - a_k)A \geq 0.$$

Thus,  $F$  has the minimum value for  $a_1 = a_k$ . Similarly,  $F$  has the minimum value for  $a_2 = a_k, \dots, a_{k-1} = a_k$ . Consequently, it suffices to consider the case  $a_1 = \cdots = a_{k-1} = a_k$ , when the desired inequality  $F \geq 0$  is an identity.

The proof is completed. For  $k = k_0 = \left\lfloor \frac{n}{2} \right\rfloor$ , the equality occurs when  $a_1 = a_2 = \cdots = a_n$ .

**Second Solution.** First consider the case  $n$  is even. So, for  $n = 2m$ , we need to show that

$$(a_1 + a_2 + \cdots + a_{2m})^2 \geq 4m(a_1^2 + a_2^2 + \cdots + a_m^2) \\ \iff a_1^2 + a_2^2 + \cdots + a_{2m}^2 + \sum_{1 \leq i < j \leq 2m} 2a_i a_j \geq 4m(a_1^2 + a_2^2 + \cdots + a_m^2).$$

Since

$$a_1^2 + a_2^2 + \cdots + a_{2m}^2 \geq 2(a_1^2 + a_2^2 + \cdots + a_m^2),$$

it suffices to show that

$$\sum_{1 \leq i < j \leq 2m} a_i a_j \geq (2m - 1)(a_1^2 + a_2^2 + \cdots + a_m^2).$$

We get this inequality by adding the following inequalities:

$$\begin{aligned} a_1(a_2 + a_3 + \cdots + a_{2m}) &\geq (2m - 1)a_1^2, \\ a_2(a_3 + a_4 + \cdots + a_{2m}) + a_{2m-1}a_{2m} &\geq (2m - 1)a_2^2, \\ a_3(a_4 + a_5 + \cdots + a_{2m}) + a_{2m-2}(a_{2m-1} + a_{2m}) &\geq (2m - 1)a_3^2, \\ &\dots\dots\dots \\ a_m(a_{m+1} + a_{m+2} + \cdots + a_{2m}) + a_{m+1}(a_{m+2} + \cdots + a_{2m}) &\geq (2m - 1)a_m^2. \end{aligned}$$

Consider now the case  $n$  is odd. For  $n = 2m + 1$ , we have

$$\left( \frac{a_1 + a_2 + \cdots + a_{2m+1}}{2m + 1} \right)^2 \geq \left( \frac{a_1 + a_2 + \cdots + a_{2m}}{2m} \right)^2 \geq \frac{a_1^2 + a_2^2 + \cdots + a_m^2}{m}.$$

□

**P 3.213.** For given  $n \geq 2$ , find the largest integer  $k$  such that

$$\left( \frac{a_1 + a_2 + \cdots + a_k}{k} \right)^2 \geq \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}$$

for all nonnegative numbers  $a_i$  satisfying  $a_1 \geq a_2 \geq \cdots \geq a_n$ .

(Vasile Cîrtoaje, *Mathematical Reflections*, 2, 2024)

**Solution** (by Nermin Hodzic). Setting  $a_1 = 1$  and  $a_2 = \cdots = a_n = 0$ , the inequality leads to  $k \leq \sqrt{n}$ , hence

$$k \leq k_0 = \lfloor \sqrt{n} \rfloor.$$

To show that  $k_0$  is the largest value of  $k$ , we need to prove the inequality for  $k = k_0 \leq \sqrt{n}$ . For  $n = 2$  and  $n = 3$  (when  $k = 1$ ), the inequality reduces to  $a_1^2 \geq a_2^2$  and  $2a_1^2 \geq a_2^2 + a_3^2$ , respectively. Next assume  $n \geq 4$ , hence  $k \geq 2$ . Since  $a_k \geq a_{k+1} \geq \cdots \geq a_n$ , we have

$$\left( \frac{a_1 + a_2 + \cdots + a_k}{k} \right)^2 = \frac{1}{k^2} \left( a_1^2 + a_2^2 + \cdots + a_k^2 + 2 \sum_{1 \leq i < j \leq k} a_i a_j \right)$$

$$\begin{aligned}
&\geq \frac{1}{k^2} [a_1^2 + a_2^2 + \cdots + a_k^2 + k(k-1)a_{k+1}^2] \geq \frac{1}{k^2} \left( a_1^2 + a_2^2 + \cdots + a_k^2 + \sum_{i=k+1}^{k^2} a_i^2 \right) \\
&= \frac{1}{k^2} \sum_{i=1}^{k^2} a_i^2 \geq \frac{1}{n} \sum_{i=1}^n a_i^2.
\end{aligned}$$

The last inequality follows from  $k^2 \leq n$ .

The proof is completed. For  $k = k_0 = \lfloor \sqrt{n} \rfloor$ , the equality occurs when  $a_1 = a_2 = \cdots = a_n$ . If  $n$  is a perfect square, then the equality also occurs when  $a_2 = \cdots = a_n = 0$ . □

**P 3.214.** If  $a_1 \geq a_2 \geq \cdots \geq a_9 \geq 0$ , then

$$\left( \frac{5a_1 + a_2 + \cdots + a_8}{12} \right)^2 \geq \frac{a_1^2 + a_2^2 + \cdots + a_9^2}{9}.$$

(Vasile Cîrtoaje, SSMJ, 5, 2024)

**Solution.** Since

$$a_1^2 + a_2^2 + \cdots + a_9^2 \leq a_1^2 + a_2^2 + \cdots + a_7^2 + 2a_8^2,$$

it suffices to prove that  $F(a_1, a_2, \dots, a_8) \geq 0$ , where

$$F(a_1, a_2, \dots, a_8) = (5a_1 + a_2 + \cdots + a_7 + a_8)^2 - 16(a_1^2 + a_2^2 + \cdots + a_7^2 + 2a_8^2).$$

We will show that

$$F(a_1, a_2, a_3, \dots, a_8) \geq F(a_2, a_2, a_3, \dots, a_8) \geq \cdots \geq F(a_7, a_7, \dots, a_7, a_8) \geq 0.$$

Since

$$F(a_7, a_7, \dots, a_7, a_8) = (11a_7 + a_8)^2 - 16(7a_7^2 + 2a_8^2) = (a_7 - a_8)(9a_7 + 31a_8) \geq 0,$$

we only need to show that

$$F(a_i, \dots, a_i, a_{i+1}, a_{i+2}, \dots, a_8) \geq F(a_{i+1}, \dots, a_{i+1}, a_{i+1}, a_{i+2}, \dots, a_8)$$

for  $i = 1, \dots, 6$ . Write the inequality as follows:

$$\begin{aligned}
&[(i+4)a_i + a_{i+1} + a_{i+2} + \cdots + a_8]^2 - 16(ia_i^2 + a_{i+1}^2 + a_{i+2}^2 + \cdots + a_7^2 + 2a_8^2) \geq \\
&\geq [(i+5)a_{i+1} + a_{i+2} + \cdots + a_8]^2 - 16[(i+1)a_{i+1}^2 + a_{i+2}^2 + \cdots + a_7^2 + 2a_8^2],
\end{aligned}$$

$$(i+4)(a_i - a_{i+1})[(i+4)a_i + (i+6)a_{i+1} + 2a_{i+2} + \cdots + 2a_8]^2 \geq 16i(a_i^2 - a_{i+1}^2),$$

$$(a_i - a_{i+1})[(i-4)^2 a_i + (i^2 - 6i + 24)a_{i+1} + 2(i+4)a_{i+2} + \cdots + 2(i+4)a_8] \geq 0.$$

Since the last inequality is true, the proof is completed. For  $k = 5$ , the equality occurs when  $a_1 = a_2 = \cdots = a_9$ , and also when  $a_1 = a_2 = a_3 = a_4$  and  $a_5 = a_6 = a_7 = a_8 = a_9 = 0$ .

**Remark.** Similarly, we can prove the following generalization:

• Let  $n \geq 4$  be a perfect square, and let  $k = (\sqrt{n} - 1)^2 + 1$ . If  $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ , then

$$\left( \frac{ka_1 + a_2 + \cdots + a_{n-1}}{k + n - 2} \right)^2 \geq \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n},$$

with equality for  $a_1 = a_2 = \cdots = a_n$ , and also for  $a_1 = a_2 = \cdots = a_{k-1}$  and  $a_k = \cdots = a_n = 0$ .

□

**P 3.215.** If  $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ , then

$$(a) \quad a_1 + a_2 + \cdots + a_n - n \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{1}{3} (\sqrt{a_1} + \sqrt{a_2} - 2\sqrt{a_n})^2;$$

$$(b) \quad a_1 + a_2 + \cdots + a_n - n \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{1}{4} (2\sqrt{a_1} - \sqrt{a_{n-1}} - \sqrt{a_n})^2.$$

(Vasile Cîrtoaje, 2010)

**Solution.** (a) For  $n = 2$ , the inequality is equivalent to  $(\sqrt{a_1} - \sqrt{a_2})^2 \geq 0$ . Consider further  $n \geq 3$ . By the AM-GM inequality, we have

$$a_3 + \cdots + a_{n-1} + 3\sqrt[3]{a_1 a_2 a_n} \geq n \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Therefore, it suffices to prove that

$$a_1 + a_2 + a_n - 3\sqrt[3]{a_1 a_2 a_n} \geq \frac{1}{3} (\sqrt{a_1} + \sqrt{a_2} - 2\sqrt{a_n})^2.$$

Setting

$$x = \left( \frac{\sqrt{a_1} + \sqrt{a_2}}{2} \right)^2, \quad x \geq a_n,$$

since  $a_1 + a_2 \geq 2x$  and  $a_1 a_2 \leq x^2$ , it suffices to show that

$$2x + a_n - 3\sqrt[3]{x^2 a_n} \geq \frac{4}{3} (\sqrt{x} - \sqrt{a_n})^2.$$

For the nontrivial case  $a_n \neq 0$ , we may consider  $a_n = 1$  (due to homogeneity). In addition, substituting  $x = y^6$ ,  $y \geq 1$ , the inequality can be restated as

$$2y^6 + 1 - 3y^4 \geq \frac{4}{3}(y^3 - 1)^2,$$

$$(y-1)^2[3(y+1)^2(2y^2+1)-4(y^2+y+1)^2] \geq 0.$$

This inequality is true if

$$(y+1)\sqrt{3(2y^2+1)} \geq 2(y^2+y+1).$$

Since

$$\sqrt{3(2y^2+1)} \geq 2y+1,$$

we have

$$(y+1)\sqrt{3(2y^2+1)} - 2(y^2+y+1) \geq (y+1)(2y+1) - 2(y^2+y+1) = y-1 \geq 0.$$

This completes the proof. The equality holds for  $a_1 = a_2 = \dots = a_n$ .

(b) For  $n = 2$ , the inequality is equivalent to  $(\sqrt{a_1} - \sqrt{a_2})^2 \geq 0$ . Consider further  $n \geq 3$ . By the AM-GM inequality, we have

$$a_2 + a_3 + \dots + a_{n-2} + 3\sqrt[3]{a_1 a_{n-1} a_n} \geq n\sqrt[n]{a_1 a_2 \dots a_n}.$$

Therefore, it suffices to prove that

$$a_1 + a_{n-1} + a_n - 3\sqrt[3]{a_1 a_{n-1} a_n} \geq \frac{1}{4} (2\sqrt{a_1} - \sqrt{a_{n-1}} - \sqrt{a_n})^2.$$

Setting

$$x = \sqrt{a_{n-1} a_n}, \quad x \leq a_1,$$

since  $a_{n-1} + a_n \geq 2x$  and  $\sqrt{a_{n-1}} + \sqrt{a_n} \geq 2\sqrt{x}$ , it suffices to show that

$$a_1 + 2x - 3\sqrt[3]{a_1 x^2} \geq (\sqrt{a_1} - \sqrt{x})^2.$$

Due to homogeneity, we may consider  $a_1 = 1$ . In addition, substituting  $x = y^6$ ,  $y \leq 1$ , the inequality becomes

$$1 + 2y^6 - 3y^4 \geq (1 - y^3)^2,$$

which is equivalent to the obvious inequality

$$y^3(y-1)^2(y+2) \geq 0.$$

The equality holds for  $a_1 = a_2 = \dots = a_n$ . If  $n \geq 3$ , then the equality also holds for  $a_2 = \dots = a_n = 0$ .

□

**P 3.216.** If  $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$ ,  $n \geq 3$ , then

$$a_1 + a_2 + \dots + a_n - n\sqrt[n]{a_1 a_2 \dots a_n} \geq \frac{n-1}{2n} (\sqrt{a_{n-2}} + \sqrt{a_{n-1}} - 2\sqrt{a_n})^2.$$

(Vasile Cîrtoaje, 2010)

**Solution.** Let us denote

$$x = \frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}, \quad x \geq a_n.$$

By the AM-GM inequality, we have

$$a_1 a_2 \cdots a_{n-1} \leq x^{n-1}.$$

Also,

$$\frac{\sqrt{a_{n-2}} + \sqrt{a_{n-1}}}{2} \leq \sqrt{\frac{a_{n-2} + a_{n-1}}{2}} \leq \sqrt{\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}} = \sqrt{x}.$$

Then, it suffices to show that

$$(n-1)x + a_n - n\sqrt[n]{x^{n-1}a_n} \geq \frac{2(n-1)}{n} (\sqrt{x} - \sqrt{a_n})^2.$$

For the nontrivial case  $a_n \neq 0$ , we may consider  $a_n = 1$  (due to homogeneity). In addition, substituting  $x = t^{2n}$ ,  $t \geq 1$ , the inequality becomes  $g(t) \geq 0$ , where

$$g(t) = (n-1)t^{2n} + 1 - nt^{2n-2} - \frac{2(n-1)}{n}(t^n - 1)^2.$$

We have

$$g'(t) = 2(n-1)t^{n-1}h(t),$$

where

$$h(t) = n(t^n - t^{n-2}) - 2(t^n - 1).$$

Since

$$h'(t) = n(n-2)t^{n-3}(t^2 - 1) \geq 0,$$

$h(t)$  is increasing,  $h(t) \geq h(1) = 0$ ,  $g'(t) \geq 0$ ,  $g(t)$  is increasing, hence  $g(t) \geq g(1) = 0$ . This completes the proof. The equality holds for  $a_1 = a_2 = \cdots = a_n$ .

□

**P 3.217.** Let  $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ . If

$$1 \leq k < j \leq n, \quad k + j \geq n + 1,$$

then

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{2k(n-j+1)}{n+k-j+1} (\sqrt{a_k} - \sqrt{a_j})^2.$$

(Vasile Cîrtoaje, 2010)

**Solution.** Let us denote

$$P = \frac{k(n-j+1)}{n+k-j+1}$$

and

$$x = \sqrt[k]{a_1 a_2 \cdots a_k}, \quad y = \sqrt[n-j+1]{a_j a_{j+1} \cdots a_n}$$

where

$$x \geq a_k \geq a_j \geq y.$$

By the AM-GM inequality, we have

$$a_1 + a_2 + \cdots + a_k \geq kx, \quad a_j + a_{j+1} + \cdots + a_n \geq (n-j+1)y$$

and

$$a_{k+1} + \cdots + a_{j-1} \geq (j-k-1) \cdot \sqrt[j-k-1]{a_{k+1} \cdots a_{j-1}}.$$

Also, we have

$$\sqrt{a_k} - \sqrt{a_j} \leq \sqrt{x} - \sqrt{y}.$$

Thus, it suffices to show that

$$kx + (n-j+1)y + (j-k-1) \cdot \sqrt[j-k-1]{a_{k+1} \cdots a_{j-1}} - n \sqrt[n]{a_1 a_2 \cdots a_n} \geq 2P(\sqrt{x} - \sqrt{y})^2.$$

By the AM-GM inequality, we have

$$(j-k-1) \cdot \sqrt[j-k-1]{a_{k+1} \cdots a_{j-1}} + (n-j+k+1) \cdot \sqrt[n-j+k+1]{(a_1 \cdots a_k)(a_j \cdots a_n)} \geq n \sqrt[n]{a_1 a_2 \cdots a_n},$$

which is equivalent to

$$\begin{aligned} & (j-k-1) \cdot \sqrt[j-k-1]{a_{k+1} \cdots a_{j-1}} - n \sqrt[n]{a_1 a_2 \cdots a_n} \geq \\ & \geq -(n-j+k+1) \cdot \sqrt[n-j+k+1]{(a_1 \cdots a_k)(a_j \cdots a_n)} \\ & = -(n-j+k+1)x^{\frac{k}{n-j+k+1}}y^{\frac{n-j+1}{n-j+k+1}}. \end{aligned}$$

Therefore, we only need to show that

$$kx + (n-j+1)y - (n-j+k+1)x^{\frac{k}{n-j+k+1}}y^{\frac{n-j+1}{n-j+k+1}} \geq 2P(\sqrt{x} - \sqrt{y})^2.$$

For the nontrivial case  $y \neq 0$ , we can set  $y = 1$  (due to homogeneity). Thus, we need to prove that  $f(x) \geq 0$  for  $x \geq 1$ , where

$$f(x) = kx + n-j+1 - (n-j+k+1)x^{\frac{k}{n-j+k+1}} - 2P(\sqrt{x} - 1)^2.$$

We have the derivatives

$$f'(x) = k - kx^{\frac{k}{n-j+k+1}-1} + 2P\left(\frac{1}{\sqrt{x}} - 1\right),$$



$$f''(x) = P \left( x^{\frac{k}{n-j+k+1}-2} - x^{\frac{-3}{2}} \right).$$

Since  $k + j \geq n + 1$  and  $x \geq 1$ , we have  $f''(x) \geq 0$ ,  $f'$  is increasing,  $f'(x) \geq f'(1) = 0$ ,  $f(x)$  is increasing,  $f(x) \geq f(1) = 0$ . This completes the proof. The equality holds for  $a_1 = a_2 = \cdots = a_n$ . If  $k + j = n + 1$ , then the equality holds for  $a_1 = a_2 = \cdots = a_k = \alpha$ ,  $a_j = a_{j+1} = \cdots = a_n = \beta$  and  $a_{k+1} = \cdots = a_{j-1} = \sqrt{\alpha\beta}$ , where  $\alpha \geq \beta \geq 0$ .

**Remark 1.** For  $k + j = n + 1$ , we get the statement:

- If  $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$  and  $1 \leq k \leq \frac{n}{2}$ , then

$$a_1 + a_2 + \cdots + a_n - n \sqrt[n]{a_1 a_2 \cdots a_n} \geq k (\sqrt{a_1} - \sqrt{a_{n-k+1}})^2,$$

with equality for  $a_1 = a_2 = \cdots = a_k = \alpha$ ,  $a_{n-k+1} = a_{n-k+2} = \cdots = a_n = \beta$  and  $a_{k+1} = \cdots = a_{n-k} = \sqrt{\alpha\beta}$ , where  $\alpha \geq \beta \geq 0$ .

In addition, for  $k = 1$ , we get:

- If  $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ , then

$$a_1 + a_2 + \cdots + a_n - n \sqrt[n]{a_1 a_2 \cdots a_n} \geq (\sqrt{a_1} - \sqrt{a_n})^2,$$

with equality for  $a_1 = \alpha$ ,  $a_n = \beta$  and  $a_2 = \cdots = a_{n-1} = \sqrt{\alpha\beta}$ , where  $\alpha \geq \beta \geq 0$ .

**Remark 2.** For  $j = k + 1$ , we get the the statement:

- Let  $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ . If  $\frac{n}{2} \leq k \leq n - 1$ , then

$$a_1 + a_2 + \cdots + a_n - n \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{2k(n-k)}{n} (\sqrt{a_k} - \sqrt{a_{k+1}})^2,$$

with equality for  $a_1 = a_2 = \cdots = a_n$ . If  $n$  is even and  $2k = n$ , then the equality holds for  $a_1 = a_2 = \cdots = a_k$  and  $a_{k+1} = a_{k+2} = \cdots = a_n$ .

□

**P 3.218.** If  $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ ,  $n \geq 4$ , then

$$(a) \quad a_1 + a_2 + \cdots + a_n - n \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{1}{2} \left( 1 - \frac{1}{n} \right) (\sqrt{a_{n-2}} - 3\sqrt{a_{n-1}} + 2\sqrt{a_n})^2;$$

$$(b) \quad a_1 + a_2 + \cdots + a_n - n \sqrt[n]{a_1 a_2 \cdots a_n} \geq \left( 1 - \frac{2}{n} \right) (2\sqrt{a_{n-2}} - 3\sqrt{a_{n-1}} + \sqrt{a_n})^2.$$

(Vasile Cîrtoaje, 2010)

**Solution.** Let

$$x = \sqrt{a_{n-2}} - \sqrt{a_{n-1}} \geq 0, \quad y = \sqrt{a_{n-1}} - \sqrt{a_n} \geq 0.$$

For  $k = n - 2$  and  $j = n - 1$ , and for  $k = n - 1$  and  $j = n$ , the inequality in P 3.217 becomes respectively

$$a_1 + a_2 + \cdots + a_n - n \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{4(n-2)x^2}{n}$$

and

$$a_1 + a_2 + \cdots + a_n - n \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{2(n-1)y^2}{n}.$$

Therefore,

$$a_1 + a_2 + \cdots + a_n - n \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{2}{n} \max\{2(n-2)x^2, (n-1)y^2\}.$$

(a) It suffices to show that

$$\max\{8(n-2)x^2, 4(n-1)y^2\} \geq (n-1)(x-2y)^2.$$

This is true since

$$8(n-2)x^2 \geq (n-1)x^2 \geq (n-1)(x-2y)^2$$

for  $x - 2y \geq 0$ , and

$$4(n-1)y^2 \geq (n-1)(2y-x)^2$$

for  $2y - x \geq 0$ . The equality holds for  $a_1 = a_2 = \cdots = a_n$ .

(b) **First Solution.** It suffices to show that

$$\max\{4(n-2)x^2, 2(n-1)y^2\} \geq (n-2)(2x-y)^2.$$

This is true since

$$4(n-2)x^2 \geq (n-2)(2x-y)^2$$

for  $2x - y \geq 0$ , and

$$2(n-1)y^2 \geq (n-2)y^2 \geq (n-2)(y-2x)^2$$

for  $y - 2x \geq 0$ . The equality holds for  $a_1 = a_2 = \cdots = a_n$ . If  $n = 4$ , then the equality holds for  $a_1 = a_2$  and  $a_3 = a_4$ .

**Second Solution.** Let us denote

$$A = \sqrt[n-2]{a_1 a_2 \cdots a_{n-2}}, \quad B = \sqrt{a_{n-1} a_n}, \quad A \geq a_{n-2} \geq B.$$

By the AM-GM inequality, we have

$$a_1 + a_2 + \cdots + a_{n-2} \geq (n-2)A,$$

$$a_{n-1} + a_n \geq 2B,$$

and

$$\sqrt{a_{n-1}} + \sqrt{a_n} \geq 2\sqrt{B}.$$

Then, it suffices to show that

$$(n-2)A + 2B - n\sqrt[n]{A^{n-2}B^2} \geq \frac{4(n-2)}{n} \left(\sqrt{A} - \sqrt{B}\right)^2.$$

For the nontrivial case  $B \neq 0$ , we may consider  $B = 1$  (due to homogeneity). In addition, substituting  $A = t^{2n}$ ,  $t \geq 1$ , the inequality becomes  $g(t) \geq 0$ , where

$$g(t) = (n-2)t^{2n} + 2 - nt^{2n-4} - \frac{4(n-2)}{n}(t^n - 1)^2.$$

We have

$$g'(t) = 2(n-2)t^{n-1}h(t),$$

where

$$h(t) = (n-4)t^n - nt^{n-4} + 4.$$

Since

$$h'(t) = n(n-4)t^{n-5}(t^4 - 1) \geq 0,$$

$h(t)$  is increasing,  $h(t) \geq h(1) = 0$ ,  $g'(t) \geq 0$ ,  $g(t)$  is increasing, hence  $g(t) \geq g(1) = 0$ . This completes the proof. □

**P 3.219.** Let  $a_1, a_2, \dots, a_n$  ( $n \geq 4$ ) be nonnegative real numbers such that

$$a_1 \geq a_2 \geq \dots \geq a_n$$

and

$$(a_1 + a_2 + \dots + a_n)^2 = 4(a_1^2 + a_2^2 + \dots + a_n^2).$$

Prove that

$$1 \leq \frac{a_1 + a_2}{a_3 + a_4 + \dots + a_n} \leq 1 + \sqrt{\frac{2n-8}{n-2}}.$$

(Vasile Cîrtoaje, 2007)

**Solution.** Denote

$$A = a_1 + a_2, \quad B = a_3 + a_4 + \dots + a_n.$$

Since

$$2(a_1^2 + a_2^2) \geq A^2, \quad (n-2)(a_3^2 + a_4^2 + \dots + a_n^2) \geq B^2,$$

from the hypothesis

$$(a_1 + a_2 + \dots + a_n)^2 = 4(a_1^2 + a_2^2) + 4(a_3^2 + \dots + a_n^2),$$

we get

$$(A + B)^2 \geq 2A^2 + \frac{4}{n-2}B^2,$$

$$A \leq \left(1 + \sqrt{\frac{2n-8}{n-2}}\right) B.$$

The right inequality is an equality for

$$a_1 = a_2 = ka_3 = \cdots = ka_n, \quad k = \frac{n-2 + \sqrt{2(n-2)(n-4)}}{2}.$$

To prove the left inequality, let

$$a_1 \geq a_2 \geq x \geq a_3 \geq \cdots \geq a_n.$$

From

$$\frac{A}{a_1 a_2} = \frac{1}{a_1} + \frac{1}{a_2} \leq \frac{1}{x} + \frac{1}{x} = \frac{2}{x},$$

we get

$$2a_1 a_2 \geq Ax,$$

hence

$$a_1^2 + a_2^2 = A^2 - 2a_1 a_2 \leq A^2 - Ax.$$

In addition,

$$a_3^2 + \cdots + a_n^2 \leq a_3 x + \cdots + a_n x = Bx.$$

Therefore, from the hypothesis

$$(a_1 + a_2 + \cdots + a_n)^2 = 4(a_1^2 + a_2^2) + 4(a_3^2 + \cdots + a_n^2),$$

we get

$$(A + B)^2 \leq 4(A^2 - Ax) + 4Bx,$$

$$4(A - B)x - 3A^2 + 2AB + B^2 \leq 0,$$

$$(A - B)(3A + B - 4x) \geq 0.$$

Since

$$3A + B - 4x \geq 3A - 4x \geq 6x - 4x \geq 0,$$

it follows that  $A - B \geq 0$ . The left inequality is an equality only for  $n = 4$  and  $a_1 = a_2 = a_3 = a_4$ .

□

**P 3.220.** Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be real numbers. Prove that

$$a_1 b_1 + \cdots + a_n b_n + \sqrt{(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)} \geq \frac{2}{n}(a_1 + \cdots + a_n)(b_1 + \cdots + b_n).$$

(Vasile Cîrtoaje, 1989)

**First Solution.** Write the inequality as

$$\sqrt{(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)} \geq a_1(2b - b_1) + \cdots + a_n(2b - b_n),$$

where

$$b = \frac{1}{n}(b_1 + \cdots + b_n).$$

Using the substitution

$$x_i = 2b - b_i, \quad i = 1, 2, \dots, n,$$

we have

$$\begin{aligned} \sum_{i=1}^n x_i &= 2nb - \sum_{i=1}^n b_i = 2nb - nb = nb, \\ \sum_{i=1}^n b_i^2 &= \sum_{i=1}^n (2b - x_i)^2 = 4nb^2 - 4b \sum_{i=1}^n x_i + \sum_{i=1}^n x_i^2 = \sum_{i=1}^n x_i^2. \end{aligned}$$

Therefore, the desired inequality can be restated as

$$\sqrt{(a_1^2 + \cdots + a_n^2)(x_1^2 + \cdots + x_n^2)} \geq a_1x_1 + \cdots + a_nx_n,$$

which is just the Cauchy-Schwarz inequality. If  $a_1a_2 \cdots a_n \neq 0$ , then the equality holds for

$$\frac{2b - b_1}{a_1} = \frac{2b - b_2}{a_2} = \cdots = \frac{2b - b_n}{a_n} \geq 0.$$

**Second Solution.** Consider the nontrivial case where  $a_1^2 + \cdots + a_n^2 \neq 0$  and  $b_1^2 + \cdots + b_n^2 \neq 0$ , denote

$$p = \sqrt{\frac{b_1^2 + \cdots + b_n^2}{a_1^2 + \cdots + a_n^2}},$$

and use the substitution

$$b_i = px_i, \quad i = 1, 2, \dots, n$$

to have

$$a_1^2 + \cdots + a_n^2 = x_1^2 + \cdots + x_n^2.$$

The desired inequality becomes

$$(a_1x_1 + \cdots + a_nx_n) + (a_1^2 + \cdots + a_n^2) \geq \frac{2}{n}(a_1 + \cdots + a_n)(x_1 + \cdots + x_n),$$

$$(a_1 + x_1)^2 + \cdots + (a_n + x_n)^2 \geq \frac{4}{n}(a_1 + \cdots + a_n)(x_1 + \cdots + x_n).$$

Since

$$4(a_1 + \cdots + a_n)(x_1 + \cdots + x_n) \leq [(a_1 + \cdots + a_n) + (x_1 + \cdots + x_n)]^2,$$

it suffices to show that

$$(a_1 + x_1)^2 + \cdots + (a_n + x_n)^2 \geq \frac{1}{n}[(a_1 + x_1) + \cdots + (a_n + x_n)]^2.$$

This follows immediately from the Cauchy-Schwarz inequality.

**Remark.** Substituting  $b_i = 1/a_i$  for all  $i$ , we get the following inequality

$$n^2 + n\sqrt{(a_1^2 + \cdots + a_n^2)\left(\frac{1}{a_1^2} + \cdots + \frac{1}{a_n^2}\right)} \geq 2(a_1 + \cdots + a_n)\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right).$$

If  $a_1 \leq a_2 \leq \cdots \leq a_n$  and  $n$  is even,  $n = 2k$ , then the equality holds for

$$a_1 = a_2 = \cdots = a_k, \quad a_{k+1} = a_{k+2} = \cdots = a_{2k}.$$

If  $n$  is odd, then the equality holds only if  $a_1 = a_2 = \cdots = a_n$ .

**Conjecture.** If  $a_1, a_2, \dots, a_n$  are positive real numbers and  $n$  is odd, then

$$\begin{aligned} n^2 + 1 + \sqrt{(n^2 - 1)(a_1^2 + \cdots + a_n^2)\left(\frac{1}{a_1^2} + \cdots + \frac{1}{a_n^2}\right) - n^2 + 1} &\geq \\ &\geq 2(a_1 + \cdots + a_n)\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right). \end{aligned}$$

If  $a_1 \leq a_2 \leq \cdots \leq a_n$  and  $n$  is odd,  $n = 2k + 1$ , then the equality holds for

$$a_1 = a_2 = \cdots = a_k, \quad a_{k+1} = a_{k+2} = \cdots = a_{2k+1},$$

and for

$$a_1 = a_2 = \cdots = a_{k+1}, \quad a_{k+2} = a_{k+3} = \cdots = a_{2k+1}.$$

□

**P 3.221.** Let  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) be positive real numbers such that

$$a_1 \leq a_2 \leq \cdots \leq a_n,$$

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) = k.$$

(a) If  $n^2 < k \leq n^2 + \frac{i(n-i)}{2}$ ,  $i \in \{2, 3, \dots, n-1\}$ , then  $a_{i-1}$ ,  $a_i$  and  $a_{i+1}$  are the lengths of the sides of a non-degenerate or degenerate triangle.

(b) If  $n^2 < k \leq \alpha_n$ , where  $\alpha_n = \frac{9n^2}{8}$  for even  $n$ , and  $\alpha_n = \frac{9n^2 - 1}{8}$  for odd  $n$ , then there exist three numbers  $a_i$  which are the lengths of the sides of a non-degenerate or degenerate triangle.

(Vasile Cîrtoaje, 2010)

**Solution.** From the AM-HM inequality, we have

$$(a_1 + a_2 + \cdots + a_n) \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) \geq n^2,$$

with equality if and only if  $a_1 = a_2 = \cdots = a_n$ . Therefore, the hypothesis  $k > n^2$  in (a) and (b) involves  $a_1 < a_n$ .

(a) For the sake of contradiction, assume that  $a_{i-1}$ ,  $a_i$  and  $a_{i+1}$  are not the lengths of the sides of a triangle; that is,

$$a_{i+1} > a_{i-1} + a_i.$$

Let us denote

$$\begin{aligned} x &= \frac{a_1 + \cdots + a_{i-1}}{i-1}, \\ y &= \frac{a_{i+1} + \cdots + a_n}{n-i}, \\ A(x, y) &= (i-1)x + a_i + (n-i)y, \\ B(x, y) &= \frac{i-1}{x} + \frac{1}{a_i} + \frac{n-i}{y}, \\ f(x, y) &= A(x, y)B(x, y). \end{aligned}$$

We have

$$\begin{aligned} x &\leq a_{i-1} \leq a_i < a_{i+1} \leq y, \quad x < y, \\ A(x, y) &= a_1 + a_2 + \cdots + a_n, \\ B(x, y) &\leq \left( \frac{1}{a_1} + \cdots + \frac{1}{a_{i-1}} \right) + \frac{1}{a_i} + \left( \frac{1}{a_{i+1}} + \cdots + \frac{1}{a_n} \right) = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}, \\ f(x, y) &\leq (a_1 + a_2 + \cdots + a_n) \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) = k. \end{aligned}$$

On the other hand, we claim that

$$f(x, y) > f(a_{i-1}, a_{i-1} + a_i).$$

This inequality is equivalent to

$$[a_{i-1}y - (a_{i-1} + a_i)x][(a_{i-1} + a_i)y - a_{i-1}x] > 0,$$

and is true since  $y \geq a_{i+1} > a_{i-1} + a_i$  and  $x \leq a_{i-1}$  imply

$$a_{i-1}y - (a_{i-1} + a_i)x > a_{i-1}(a_{i-1} + a_i) - (a_{i-1} + a_i)a_{i-1} = 0.$$

Then, we have

$$\begin{aligned} k &\geq f(x, y) > f(a_{i-1}, a_{i-1} + a_i) \\ &= [(n-1)a_{i-1} + (n-i+1)a_i] \left( \frac{i-1}{a_{i-1}} + \frac{1}{a_i} + \frac{n-i}{a_{i-1} + a_i} \right) \\ &\geq n^2 + \frac{i(n-i)}{2}, \end{aligned}$$

which contradicts the hypothesis  $k \leq n^2 + \frac{i(n-i)}{2}$ . Setting  $a_{i-1} = 1$  and denoting  $a_i = t$ ,  $t \geq 1$ , the last inequality becomes

$$\frac{[n-1+(n-i+1)t][1+nt+(i-1)t^2]}{t(1+t)} \geq n^2 + \frac{i(n-i)}{2},$$

or

$$(t-1)(Ct^2 + Dt + E) \geq 0,$$

where

$$C = 2(i-1)(n-i+1), \quad D = (i-2)(n-i), \quad E = -2(n-1).$$

This is true, since

$$Ct^2 + Dt + E \geq C + D + E = 3(i-2)(n-i) \geq 0.$$

(b) We apply the result of (a). If  $n$  is even, then  $n^2 + \frac{i(n-i)}{2}$  attains its maximum  $\frac{9n^2}{8}$  for  $i = \frac{n}{2}$ . If  $n$  is odd, then  $n^2 + \frac{i(n-i)}{2}$  attains its maximum  $\frac{9n^2-1}{8}$  for  $i = \frac{n \pm 1}{2}$ .  $\square$

**P 3.222.** Let  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) be positive real numbers such that

$$a_1 \leq a_2 \leq \dots \leq a_n,$$

$$(a_1 + a_2 + \dots + a_n)^2 = k(a_1^2 + a_2^2 + \dots + a_n^2).$$

(a) If  $\frac{(2n-i)^2}{4n-3i} \leq k < n$ ,  $i \in \{2, 3, \dots, n-1\}$ , then  $a_{i-1}$ ,  $a_i$  and  $a_{i+1}$  are the lengths of the sides of a non-degenerate or degenerate triangle.

(b) If  $\frac{8n+1}{9} \leq k < n$ , then there exist three numbers  $a_i$  which are the lengths of the sides of a non-degenerate or degenerate triangle.

(Vasile Cîrtoaje, 2010)

**First Solution.** From the the Cauchy-Schwarz inequality, we have

$$(a_1 + a_2 + \dots + a_n)^2 \leq n(a_1^2 + a_2^2 + \dots + a_n^2),$$

with equality if and only if  $a_1 = a_2 = \dots = a_n$ . Therefore, the hypothesis  $k < n$  in (a) and (b) involves  $a_1 < a_n$ .

(a) For the sake of contradiction, assume that  $a_{i-1}$ ,  $a_i$  and  $a_{i+1}$  are not the lengths of the sides of a triangle; that is,

$$a_{i+1} > a_{i-1} + a_i.$$

Let us denote

$$x = \frac{a_1 + \dots + a_{i-1}}{i-1},$$



$$\begin{aligned}
y &= \frac{a_{i+1} + \cdots + a_n}{n-i}, \\
A(x, y) &= (i-1)x + a_i + (n-i)y, \\
B(x, y) &= (i-1)x^2 + a_i^2 + (n-i)y^2, \\
f(x, y) &= \frac{A^2(x, y)}{B(x, y)}.
\end{aligned}$$

We have

$$\begin{aligned}
x &\leq a_{i-1} \leq a_i < a_{i+1} \leq y, \quad x < y, \\
A(x, y) &= a_1 + a_2 + \cdots + a_n, \\
B(x, y) &\leq (a_1^2 + \cdots + a_{i-1}^2) + a_i^2 + (a_{i+1}^2 + \cdots + a_n^2) = a_1^2 + a_2^2 + \cdots + a_n^2, \\
f(x, y) &\geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{a_1^2 + a_2^2 + \cdots + a_n^2} = k.
\end{aligned}$$

On the other hand, from

$$\frac{\partial f(x, y)}{\partial x} = \frac{2(i-1)AC}{B^2} > 0$$

and

$$\frac{\partial f(x, y)}{\partial y} = \frac{2(n-i)AD}{B^2} < 0,$$

where

$$\begin{aligned}
C &= a_i(a_i - x) + (n-i)y(y - x) > 0, \\
D &= a_i(a_i - y) + (i-1)x(x - y) < 0,
\end{aligned}$$

it follows that  $f(x, y)$  is strictly increasing with respect to  $x$  and strictly decreasing with respect to  $y$ . Then, since  $x \leq a_{i-1}$  and  $y \geq a_{i+1} > a_{i-1} + a_i$ , we have

$$f(x, y) < f(a_{i-1}, a_{i-1} + a_i).$$

This involves

$$k < f(a_{i-1}, a_{i-1} + a_i),$$

hence

$$\begin{aligned}
k &< \frac{[(i-1)a_{i-1} + a_i + (n-i)(a_{i-1} + a_i)]^2}{(i-1)a_{i-1}^2 + a_i^2 + (n-i)(a_{i-1} + a_i)^2} \\
&= \frac{[(n-1)a_{i-1} + (n-i+1)a_i]^2}{(n-1)a_{i-1}^2 + 2(n-i)a_{i-1}a_i + (n-i+1)a_i^2} \\
&\leq \frac{(2n-i)^2}{4n-3i},
\end{aligned}$$

which contradicts the hypothesis  $k \geq \frac{(2n-i)^2}{4n-3i}$ . Setting  $a_{i-1} = 1$  and denoting  $a_i = t$ ,  $t \geq 1$ , the last inequality becomes

$$\frac{[n-1 + (n-i+1)t]^2}{n-1 + 2(n-i)t + (n-i+1)t^2} \leq \frac{(2n-i)^2}{4n-3i},$$

or

$$(t-1)(Et-F) \geq 0,$$

where

$$E = (n-i+1)[(3i-4)n-2i^2+3i],$$

$$F = (n-1)[(4-i)n+i^2-3i].$$

Since

$$E = (n-i+1)[(3i-4)(n-i-1)+i^2+2i-4] > 0,$$

we get

$$Et-F \geq E-F = 2(i-2)(n-i)(2n-i) \geq 0.$$

(b) According to (a), it suffices to show that there exists  $i \in \{2, 3, \dots, n-1\}$  such that

$$\frac{(2n-i)^2}{4n-3i} \leq \frac{8n+1}{9};$$

that is,

$$4n-3i \geq (2n-3i)^2.$$

Since one of the numbers  $\frac{2n-1}{3}$ ,  $\frac{2n}{3}$  and  $\frac{2n+1}{3}$  is integer, it suffices to prove this inequality for all  $i \in \left\{ \frac{2n-1}{3}, \frac{2n}{3}, \frac{2n+1}{3} \right\}$ . Indeed, for these cases, the inequality reduces to  $2n \geq 0$ ,  $2n \geq 0$  and  $2n-2 \geq 0$ , respectively.

□

**P 3.223.** Let  $q$  be a fixed real number such that there exist three nonnegative real numbers  $a, b, c$  satisfying

$$a \geq 1 \geq b \geq c \geq 0, \quad a+b+c=3, \quad ab+bc+ca=q.$$

Prove that the product  $r = abc$  has the maximum value for  $b = c$ , and the minimum value for  $b = 1$  or  $c = 0$ .

(Vasile Cîrtoaje, 2015)

**First Solution.** According to the proof of P 3.57-(a),  $r(b)$  has the maximum value when  $b$  is minimum, i.e. when  $b = c$ , and has the minimum value when  $b$  is maximum or  $c$  is minimum (because  $c'(b) \leq 0$ ), hence when  $b = 1$  or  $c = 0$ .

**Second Solution.** By P 3.57-(a), under the weaker condition  $a \geq b \geq c \geq 0$  instead of  $a \geq 1 \geq b \geq c \geq 0$ , the product  $r = abc$  has the maximum value for  $b = c$ . Since  $a+b+c=3$ , this maximum case remains valid even for the constraint  $b \geq 1$ . Indeed, if  $b = c$ , then  $b = (b+c)/2 \leq (a+b+c)/3 = 1$ .

For  $c = 0$ , when  $a + b = 3$  and  $ab = q$ , from  $(a - 1)(b - 1) \leq 0$ , i.e.  $ab + 1 \leq a + b$ , we get  $q + 1 \leq 3$ , hence  $q \leq 2$ . So, to prove that  $r$  has the minimum value for  $b = 1$  or  $c = 0$ , we only need to show that if  $q \geq 2$ , then  $r$  has the minimum value for  $b = 1$ . Indeed, from

$$(a - 1)(b - 1)(c - 1) \geq 0,$$

we get

$$r \geq q - 2,$$

with equality for  $b = 1$ . □

**P 3.224.** Let  $p$  and  $q$  be fixed real numbers such that there exist three nonnegative real numbers  $a, b, c$  satisfying

$$a \geq 1 \geq b \geq c \geq 0, \quad a + b + c = p, \quad ab + bc + ca = q.$$

Prove that the product  $r = abc$  has the maximum value for  $b = c$ , and the minimum value for  $a = 1$  or  $b = 1$  or  $c = 0$ .

(Vasile Cîrtoaje, 2015)

**Solution.** According to the proof of P 3.57-(a),  $r(b)$  has the maximum value when  $b$  is minimum, i.e. when  $b = c$ , and has the minimum value when  $b$  is maximum ( $a$  and  $c$  are minimum), i.e. when  $b = 1$  or  $a = 1$  or  $c = 0$ .

**Remark.** If  $ab + bc + ca = 3$  and  $a = 1$ , then  $a = b = c = 1$ . Therefore, the product  $r = abc$  has the minimum value only for  $b = 1$  or  $c = 0$ . □

**P 3.225.** (a) If  $p$  and  $q$  are fixed real numbers such that there exist three positive real numbers  $a, b, c$  satisfying

$$a \geq b \geq c \geq 1, \quad a + b + c = p, \quad ab + bc + ca = q,$$

then the product  $r = abc$  has the maximum value for  $b = c$ , and the minimum value for  $a = b$  or  $c = 1$ .

(b) If  $p$  and  $q$  are fixed real numbers such that there exist three nonnegative real numbers  $a, b, c$  satisfying

$$1 \geq a \geq b \geq c \geq 0, \quad a + b + c = p, \quad ab + bc + ca = q,$$

then the product  $r = abc$  has the maximum value for  $a = 1$  or  $b = c$ , and the minimum value for  $a = b$  or  $c = 0$ .

(Vasile Cîrtoaje, 2015)

**Solution.** (a) According to the proof of P 3.57-(a),  $r(b)$  has the maximum value when  $b$  is minimum ( $c$  is maximum), i.e. when  $b = c$ , and has the minimum value when  $b$  is maximum or  $c$  is minimum (because  $c'(b) \leq 0$ ), i.e. when  $b = a$  or  $c = 1$ .

(b) According to the proof of P 3.57-(a),  $r(b)$  has the maximum value when  $b$  is minimum or  $a$  is maximum (because  $a'(b) \leq 0$ ), i.e. when  $a = 1$  or  $b = c$ , and has the minimum value when  $b$  is maximum or  $c$  is minimum (because  $c'(b) \leq 0$ ), i.e. when  $b = a$  or  $c = 0$ . □

**P 3.226.** Let  $q$  be a fixed real number such that there exist three nonnegative real numbers  $a, b, c$  satisfying

$$a \geq b \geq 1 \geq c \geq 0, \quad a + b + c = 3, \quad ab + bc + ca = q.$$

Prove that the product  $r = abc$  has the maximum value for  $b = 1$ , and the minimum value for  $a = b$  or  $c = 0$ .

(Vasile Cîrtoaje, 2015)

**First Solution.** According to the proof of P 3.57-(a),  $r(b)$  has the maximum value when  $b$  is minimum, i.e. when  $b = 1$ , and has the minimum value when  $b$  is maximum or  $c$  is minimum (because  $c'(b) \leq 0$ ), hence when  $b = a$  or  $c = 0$ .

**Second Solution.** By P 3.57-(a), under the weaker condition  $a \geq b \geq c \geq 0$  instead of  $a \geq b \geq 1 \geq c \geq 0$ , the product  $r = abc$  has the minimum value for  $a = b$  or  $c = 0$ . Since  $a + b + c = 3$ , these minimum cases remain valid even for the constraint  $b \geq 1$ . Indeed, if  $a = b$ , then  $b = (a + b)/2 \geq (a + b + c)/3 = 1$ , and if  $c = 0$ , then the minimum value of  $r$  is 0 for any value of  $b$ .

To prove that  $r = abc$  has the maximum value for  $b = 1$ , we use the inequality

$$(a - 1)(b - 1)(c - 1) \leq 0,$$

which is equivalent to

$$r \leq q - 2,$$

with equality for  $b = 1$ . □

**P 3.227.** Let  $p$  and  $q$  be fixed real numbers such that there exist three nonnegative real numbers  $a, b, c$  satisfying

$$a \geq b \geq 1 \geq c \geq 0, \quad a + b + c = p, \quad ab + bc + ca = q.$$

Prove that the product  $r = abc$  has the maximum value for  $b = 1$  or  $c = 1$ , and the minimum value for  $a = b$  or  $c = 0$ .

(Vasile Cîrtoaje, 2015)

**Solution.** According to the proof of P 3.57-(a),  $r(b)$  has the maximum value when  $b$  is minimum or  $c$  is maximum (because  $c'(b) \leq 0$ ), i.e. when  $b = 1$  or  $c = 1$ , and has the minimum value when  $b$  is maximum or  $c$  is minimum, i.e. when  $b = a$  or  $c = 0$ .

**Remark.** If  $a + b + c = 3$  and  $c = 1$ , then  $a = b = c = 1$ . Therefore, the product  $r = abc$  has the maximum value only for  $b = 1$ .

□

**P 3.228.** (a) If  $p$  is a real number such that there exist three positive real numbers  $a, b, c$  satisfying

$$a \geq 1 \geq b \geq c, \quad a + b + c = p, \quad abc = 1,$$

then the sum  $q = ab + bc + ca$  has the maximum value for  $b = 1$ , and the minimum value for  $b = c$ .

(b) If  $p$  is a real number such that there exist three positive real numbers  $a, b, c$  satisfying

$$a \geq b \geq 1 \geq c, \quad a + b + c = p, \quad abc = 1,$$

then the sum  $q = ab + bc + ca$  has the maximum value for  $a = b$ , and the minimum value for  $b = 1$ .

(Vasile Cîrtoaje, 2015)

**Solution.** (a) According to the proof of P 3.58,  $q(b)$  has the maximum value when  $b$  is maximum, i.e. when  $b = 1$ , and has the minimum value when  $b$  is minimum, i.e. when  $b = c$ . The maximum value of  $q$  can be also find from

$$(a - 1)(b - 1)(c - 1) \geq 0,$$

which is equivalent tot  $q \leq p$ , with equality for  $b = 1$ .

(b) According to the proof of P 3.58,  $q(b)$  has the maximum value when  $b$  is maximum ( $a$  is minimum), i.e. when  $a = b$ , and has the minimum value when  $b$  is minimum, i.e. when  $b = 1$ . The minimum value of  $q$  can be also find from

$$(a - 1)(b - 1)(c - 1) \leq 0,$$

which is equivalent tot  $q \geq p$ , with equality for  $b = 1$ .

□

**P 3.229.** If  $a \geq 1 \geq b \geq c \geq 0$  such that  $a + b + c = 3$ , then

$$abc + \frac{9}{ab + bc + ca} \geq 4.$$

(Vasile Cîrtoaje, 2015)

**Solution.** Let

$$q = ab + bc + ca.$$

**First Solution.** According to P 3.223, for fixed  $q$ , the product  $abc$  has the minimum value when  $b = 1$  or  $c = 0$ . Therefore, it suffices to consider these cases. If  $b = 1$ , then  $a + c = 2$ , and the inequality becomes

$$ac + \frac{9}{2 + ac} \geq 4,$$

$$(ac - 1)^2 \geq 0.$$

For  $c = 0$ , we need to show that  $a + b = 3$  involves  $4ab \leq 9$ . Indeed,

$$4ab < (a + b)^2 = 9.$$

The equality holds for  $a = b = c = 1$ .

**Second Solution.** From  $(a - 1)(b - 1)(c - 1) \geq 0$ , we get

$$abc \geq q - 2.$$

Therefore,

$$abc + \frac{9}{ab + bc + ca} - 4 \geq q - 2 + \frac{9}{q} - 4 = \frac{(q - 3)^2}{q} \geq 0.$$

□

**P 3.230.** If  $a \geq 1 \geq b \geq c \geq 0$  such that  $a + b + c = 3$ , then

$$abc + \frac{2}{ab + bc + ca} \geq \frac{5}{a^2 + b^2 + c^2}.$$

(Vasile Cîrtoaje, 2015)

**Solution.** Write the inequality as

$$abc + \frac{2}{q} \geq \frac{5}{9 - 2q},$$

where  $q = ab + bc + ca$ ,  $q \leq (a + b + c)^2/3 = 3$ .

**First Solution.** According to P 3.223, for fixed  $q$ , the product  $abc$  has the minimum value when  $b = 1$  or  $c = 0$ . Therefore, it suffices to consider these cases. For  $b = 1$ , when  $a + c = 2$  and  $q = a + c + ac = 2 + ac$ , the inequality becomes

$$ac + \frac{2}{2 + ac} \geq \frac{5}{5 - 2ac},$$

$$ac(1 - ac)(1 + 2ac) \geq 0.$$

The last inequality is true since

$$4 = (a + c)^2 \geq 4ac.$$

For  $c = 0$ , we need to show that  $a + b = 3$  involves

$$\frac{2}{ab} \geq \frac{5}{9 - 2ab},$$

that is  $ab \leq 2$ . Indeed,

$$ab - 2 = ab - a - b + 1 = (a - 1)(b - 1) \leq 0.$$

The equality holds for  $a = b = c = 1$ , and also for  $a = 2$ ,  $b = 1$  and  $c = 0$ .

**Second Solution.** If  $q \leq 2$ , then

$$abc + \frac{2}{q} - \frac{5}{9 - 2q} \geq \frac{2}{q} - \frac{5}{9 - 2q} = \frac{9(q - 2)}{q(9 - 2q)} \geq 0.$$

If  $2 \leq q \leq 3$ , from  $(a - 1)(b - 1)(c - 1) \geq 0$ , we get

$$abc \geq q - 2,$$

hence

$$abc + \frac{2}{q} - \frac{5}{9 - 2q} \geq q - 2 + \frac{2}{q} - \frac{5}{9 - 2q} = \frac{(3 - q)(q - 2)(2q - 3)}{q(9 - 2q)} \geq 0.$$

□

**P 3.231.** If  $a \geq b \geq 1 \geq c > 0$  such that  $a + b + c = 3$ , then

$$\frac{1}{abc} + 2 \geq \frac{9}{ab + bc + ca}.$$

(Vasile Cîrtoaje, 2015)

**Solution.** Let

$$q = ab + bc + ca.$$

**First Solution.** According to P 3.226, for fixed  $q$ , the product  $abc$  has the maximum value for  $b = 1$ . Therefore, it suffices to consider the case  $b = 1$ , when  $a + c = 2$ , and the inequality becomes

$$\begin{aligned} \frac{1}{ac} + 2 &\geq \frac{9}{2 + ac}, \\ (ac - 1)^2 &\geq 0. \end{aligned}$$

The equality holds for  $a = b = c = 1$ .

**Second Solution.** From  $(a-1)(b-1)(c-1) \leq 0$ , we get

$$abc \leq q - 2, \quad q > 2.$$

Thus, it suffices to show that

$$\frac{1}{q-2} + 2 \geq \frac{9}{q},$$

which is equivalent to

$$(q-3)^2 \geq 0.$$

□

**P 3.232.** If  $a \geq b \geq 1 \geq c > 0$  such that  $a + b + c = 3$ , then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 11 \geq 4(a^2 + b^2 + c^2).$$

(Vasile Cîrtoaje, 2015)

**Solution.** Let

$$q = ab + bc + ca.$$

**First Solution.** Write the inequality as

$$\frac{q}{abc} + 8q \geq 25.$$

According to P 3.226, for fixed  $q$ , the product  $abc$  has the maximum value when  $b = 1$ . Therefore, it suffices to consider the case  $b = 1$ , when  $a + c = 2$ , and the inequality becomes

$$\frac{1}{ac} + 4ac \geq 4,$$

$$(2ac - 1)^2 \geq 0.$$

The equality holds for  $a = 1 + \frac{1}{\sqrt{2}}$ ,  $b = 1$  and  $c = 1 - \frac{1}{\sqrt{2}}$ .

**Second Solution.** From  $(a-1)(b-1)(c-1) \leq 0$ , we get

$$abc \leq q - 2, \quad q > 2.$$

Thus, it suffices to show that

$$\frac{q}{q-2} + 11 \geq 4(9-2q),$$

which is equivalent to

$$(2q-5)^2 \geq 0.$$

□



**P 3.233.** If  $a \geq b \geq 1 \geq c > 0$  such that  $a + b + c = 3$ , then

$$\frac{1}{abc} + \frac{2}{a^2 + b^2 + c^2} \geq \frac{5}{ab + bc + ca}.$$

(Vasile Cîrtoaje, 2015)

**Solution.** Let

$$q = ab + bc + ca.$$

**First Solution.** According to P 3.226, for fixed  $q$ , the product  $abc$  has the maximum value when  $b = 1$ . Therefore, it suffices to consider the case  $b = 1$ , when the inequality becomes

$$\frac{1}{ac} + \frac{2}{5 - 2ac} \geq \frac{5}{2 + ac},$$

$$(ac - 1)^2 \geq 0.$$

The equality holds for  $a = b = c = 1$ .

**Second Solution.** From  $(a - 1)(b - 1)(c - 1) \leq 0$ , we get

$$abc \leq q - 2, \quad q > 2.$$

Thus, it suffices to show that

$$\frac{1}{q - 2} + \frac{2}{9 - 2q} \geq \frac{5}{q},$$

which is equivalent to

$$(q - 3)^2 \geq 0.$$

□

**P 3.234.** If  $a \geq b \geq 1 \geq c \geq 0$  such that  $a + b + c = 3$ , then

$$\frac{9}{a^3 + b^3 + c^3} + 2 \leq \frac{15}{a^2 + b^2 + c^2}.$$

(Vasile Cîrtoaje, 2015)

**Solution.** Write the inequality as

$$\frac{3}{abc + 9 - 3q} + 2 \leq \frac{15}{9 - 2q},$$

where  $q = ab + bc + ca$ . From

$$3q \leq (a + b + c)^2 = 9$$

and

$$0 \geq (1 - a)(1 - b)(1 - c) = abc - q + (a + b + c) - 1 \geq -q + (a + b + c) - 1 = 2 - q,$$

it follows that

$$2 \leq q \leq 3.$$

**First Solution.** Consider the following two cases.

*Case 1:*  $2 \leq q \leq 9/4$ . Since  $abc \geq 0$ , it suffices to prove that

$$\frac{1}{3-q} + 2 \leq \frac{15}{9-2q},$$

which is equivalent to the obvious inequality

$$(4q-9)(q-2) \leq 0.$$

*Case 2:*  $9/4 \leq q \leq 3$ . By Schur's inequality of third degree

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

we get

$$3abc \geq 4q-9.$$

Therefore, it suffices to show that

$$\frac{9}{4q-9+3(9-3q)} + 2 \leq \frac{15}{9-2q},$$

which is equivalent to

$$\frac{9}{18-5q} + 2 \leq \frac{15}{9-2q},$$

$$4q^2 - 21q + 27 \leq 0, \quad (q-3)(4q-9) \leq 0.$$

The equality holds for  $a = b = c = 1$ , for  $a = b = 3/2$  and  $c = 0$ , and also for  $a = 2$ ,  $b = 1$  and  $c = 0$ .

**Second Solution.** According to P 3.226, for fixed  $q$ , the product  $abc$  has the minimum value when  $a = b$  or  $c = 0$ . Therefore, it suffices to consider these cases.

*Case 1:*  $a = b \in [1, 3/2]$ . The desired inequality is equivalent to

$$\frac{9}{2a^3 + (3-2a)^3} + 2 \leq \frac{15}{2a^2 + (3-2a)^2},$$

$$(a-1)^2(3-2a)(9a-2a^2-3) \geq 0,$$

which is true since

$$9a - 2a^2 - 3 > 3(3a - a^2 - 2) = 3(a-1)(2-a) \geq 0.$$

*Case 2:*  $c = 0$ . Since

$$q = ab \leq \frac{1}{4}(a+b)^2 = \frac{9}{4},$$

we have  $2 \leq q \leq 9/4$ . The desired inequality is equivalent to

$$\frac{1}{3-q} + 2 \leq \frac{15}{9-2q},$$

$$(4q-9)(q-2) \leq 0.$$

Clearly, the last inequality is true. □

**P 3.235.** If  $a \geq b \geq 1 \geq c \geq 0$  such that  $a + b + c = 3$ , then

$$\frac{36}{a^3 + b^3 + c^3} + 9 \leq \frac{65}{a^2 + b^2 + c^2}.$$

(Vasile Cîrtoaje, 2015)

**Solution.** Write the inequality as

$$\frac{12}{abc + 9 - 3q} + 9 \leq \frac{65}{9 - 2q},$$

where

$$q = ab + bc + ca.$$

From

$$3q \leq (a + b + c)^2 = 9$$

and

$$q = (1-a)(1-b)(1-c) + abc - 1 + a + b + c \geq -1 + a + b + c = 2,$$

it follows that

$$2 \leq q \leq 3.$$

**First Solution.** Consider the following two cases.

*Case 1:*  $2 \leq q \leq 7/3$ . Since  $\frac{12}{abc + 9 - 3q} \leq \frac{4}{3-q}$ , it suffices to prove that

$$\frac{4}{3-q} + 9 \leq \frac{65}{9-2q},$$

which is equivalent to the obvious inequality

$$(3q-7)(q-2) \leq 0.$$

*Case 2:*  $7/3 \leq q \leq 3$ . By Schur's inequality of third degree

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

we get

$$3abc \geq 4q - 9.$$

Therefore, it suffices to show that

$$\frac{36}{4q - 9 + 3(9 - 3q)} + 9 \leq \frac{65}{9 - 2q},$$

which is equivalent to

$$\frac{198 - 45q}{18 - 5q} \leq \frac{65}{9 - 2q}.$$

We will prove the sharper inequality

$$\frac{200 - 45q}{18 - 5q} \leq \frac{65}{9 - 2q},$$

which is equivalent to

$$\begin{aligned} \frac{40 - 9q}{18 - 5q} &\leq \frac{13}{9 - 2q}, \\ (q - 3)(3q - 7) &\leq 0. \end{aligned}$$

The last inequality is clearly true. The equality holds for  $a = 2$ ,  $b = 1$  and  $c = 0$ .

**Second Solution.** According to the preceding P 3.234, it suffices to show that

$$4 \left( \frac{15}{a^2 + b^2 + c^2} - 2 \right) + 9 \leq \frac{65}{a^2 + b^2 + c^2},$$

which is equivalent to

$$\begin{aligned} a^2 + b^2 + c^2 &\geq 5, \\ ab + bc + ca &\geq 2. \end{aligned}$$

□

**P 3.236.** If  $a, b, c$  are real numbers such that

$$a \geq b \geq 1 \geq c, \quad a^2 + b^2 + c^2 = 3,$$

then

$$\begin{aligned} (a) \quad & 1 - abc \leq 2(b - c)^2; \\ (b) \quad & 1 - abc \geq 2(a - b)^2; \\ (c) \quad & 1 - abc \geq \frac{1}{2}(a - c)^2; \\ (d) \quad & 1 - abc \leq \frac{3}{4}(a - c)^2. \end{aligned}$$

(Vasile Cîrtoaje, 2020)

**Solution.** (a) Write the inequality as follows:

$$1 - abc \leq 2(3 - a^2 - 2bc),$$

$$5 - 2a^2 \geq (4 - a)bc.$$

From  $(b^2 - 1)(c^2 - 1) \leq 0$ , we get

$$b^2c^2 \leq b^2 + c^2 - 1 = 2 - a^2, \quad bc \leq \sqrt{2 - a^2}, \quad a \leq \sqrt{2}.$$

Thus, it is enough to show that

$$5 - 2a^2 \geq (4 - a)\sqrt{2 - a^2},$$

which, by squaring, becomes

$$5a^4 - 8a^3 - 6a^2 + 16a - 7 \geq 0,$$

$$(a - 1)^3(5a + 7) \geq 0.$$

The equality occurs for  $a = b = c = 1$ .

(b) From

$$3 = a^2 + b^2 + c^2 \geq 1 + 1 + c^2,$$

it follows that  $c \in [-1, 1]$ . Write the required inequality as follows:

$$1 - abc \geq 2(3 - c^2 - 2ab),$$

$$(4 - c)ab \geq 5 - 2c^2.$$

From  $(a^2 - 1)(b^2 - 1) \geq 0$ , we get

$$ab \geq \sqrt{a^2 + b^2 - 1} = \sqrt{2 - c^2}.$$

Thus, it is enough to show that

$$(4 - c)\sqrt{2 - c^2} \geq 5 - 2c^2,$$

which, by squaring, becomes

$$5c^4 - 8c^3 - 6c^2 + 16c - 7 \leq 0,$$

$$(c - 1)^3(5c + 7) \leq 0.$$

(c) Write the inequality as follows:

$$2 - 2abc \geq 3 - b^2 - 2ac,$$

$$b^2 - 1 \geq 2ac(b - 1),$$

which is true if

$$b + 1 \geq 2ac.$$

It is enough to show that

$$b + 1 \geq a^2 + c^2,$$

which is equivalent to

$$\begin{aligned} b + 1 &\geq 3 - b^2, \\ (b - 1)(b + 2) &\geq 0. \end{aligned}$$

The equality occurs for  $a = b = c = 1$ .

(d) Write the inequality as follows:

$$\begin{aligned} 4 - 4abc &\leq 3(3 - b^2) - 6ac, \\ 2(3 - 2b)ac &\leq 5 - 3b^2. \end{aligned}$$

From

$$(a^2 - b^2)(b^2 - c^2) \geq 0,$$

it follows that

$$ac \leq \sqrt{b^2(a^2 + c^2) - b^4} = b\sqrt{3 - 2b^2}, \quad 1 \leq b \leq \sqrt{\frac{3}{2}}.$$

Thus, it suffices to show that

$$2b(3 - 2b)\sqrt{3 - 2b^2} \leq 5 - 3b^2.$$

By squaring, the inequality becomes

$$\begin{aligned} 32b^6 - 96b^5 + 33b^4 + 144b^3 - 138b^2 + 25 &\geq 0, \\ (b - 1)^2(32b^4 - 32b^3 - 63b^2 + 50b + 25) &\geq 0. \end{aligned}$$

It is true because

$$\begin{aligned} 32b^4 - 32b^3 - 63b^2 + 50b + 25 &> 32b^4 - 32b^3 - 64b^2 + 48b + 24 \\ &= 8(3 - 2b^2)(1 + 2b - 2b^2) \geq 0. \end{aligned}$$

The equality occurs for  $a = b = c = 1$ .

□

**P 3.237.** If  $a, b, c$  are real numbers such that

$$a \geq 1 \geq b \geq c, \quad a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \leq \frac{2}{3}(a - c)^2.$$

**First Solution.** There are two cases to consider:  $b \leq 0$  and  $b \geq 0$ .

*Case 1:*  $b \leq 0$ . Since  $0 \geq b \geq c$ , hence  $c^2 \geq b^2$ , we have

$$\begin{aligned} 3abc + 2(a-c)^2 - 3 &\geq 2(a-c)^2 - 3 \\ &> a^2 + 2c^2 - 3 \geq a^2 + b^2 + c^2 - 3 = 0. \end{aligned}$$

*Case 2:*  $b \geq 0$ . Write the inequality as follows:

$$\begin{aligned} 3abc + 2(3 - b^2 - 2ac) &\geq 3, \\ 3 - 2b^2 &\geq (4 - 3b)ac. \end{aligned}$$

From  $(b^2 - c^2)(b^2 - a^2) \leq 0$ , we get

$$a^2c^2 \leq b^2(a^2 + c^2) - b^4 = b^2(3 - b^2) - b^4, \quad ac \leq b\sqrt{3 - 2b^2}.$$

Thus, it is enough to show that

$$3 - 2b^2 \geq b(4 - 3b)\sqrt{3 - 2b^2},$$

which, by squaring, becomes

$$\begin{aligned} 6b^6 - 16b^5 + 3b^4 + 24b^3 - 20b^2 + 3 &\geq 0, \\ (1 - b)^3(3 + 9b - 2b^2 - 6b^3) &\geq 0. \\ (1 - b)^3[3 + b + 2b(1 - b) + 6b(1 - b^2)] &\geq 0. \end{aligned}$$

The equality occurs for  $a = b = c = 1$ .

**Second Solution** ( by Mudok). We will prove the stronger inequality

$$1 - abc \leq \frac{2}{3} [(a-c)^2 - (a-b)(b-c)],$$

which is equivalent to

$$\begin{aligned} 1 - abc &\leq \frac{2}{3}(3 - ab - bc - ca), \\ 3 - 3abc &\leq 9 - p^2, \end{aligned}$$

where

$$p = a + b + c.$$

From

$$(a-1)(b-1)(c-1) \geq 0,$$

we get

$$abc \geq \frac{p^2 - 2p - 1}{2}.$$

Thus it suffices to show that

$$3 - \frac{3(p^2 - 2p - 1)}{2} \leq 9 - p^2,$$

which is equivalent to

$$(p - 3)^2 \geq 0.$$

□

**P 3.238.** If  $a \geq 1 \geq b \geq c \geq 0$  and  $a^2 + b^2 + c^2 = 3$ , then

$$1 - abc \leq \frac{1}{\sqrt{2}}(a - c).$$

(Vasile Cîrtoaje, 2020)

**Solution.** Denoting  $x = ac$ , we need to show that  $f(x) \geq 0$ , where

$$f(x) = bx + \sqrt{\frac{3 - b^2 - 2x}{2}} - 1.$$

For fixed  $b$ , we have  $x \in [0, M]$ , where

$$M = b\sqrt{3 - 2b^2}.$$

Indeed,  $(b^2 - a^2)(b^2 - c^2) \leq 0$  yields

$$a^2c^2 \leq b^2(a^2 + c^2) - b^4 = 3b^2 - 2b^4, \quad x \leq b\sqrt{3 - 2b^2} = M.$$

We have  $x = 0$  for  $c = 0$ , and  $x = M$  for  $a = b = 1$  or  $b = c$ . Since

$$f''(x) = -\frac{1}{\sqrt{2}} \cdot \frac{1}{(3 - b^2 - 2x)^{3/2}} \leq 0,$$

$f$  is a concave function, therefore it suffices to show that  $f(0) \geq 0$  and  $f(M) \geq 0$ . We have

$$f(0) = \sqrt{\frac{3 - b^2}{2}} - 1 \geq 0.$$

Since

$$\sqrt{3 - b^2 - 2M} = \sqrt{3 - b^2 - 2b\sqrt{3 - 2b^2}} = \sqrt{3 - 2b^2} - b,$$

we have

$$f(M) \geq bM + \frac{2}{3} \cdot \sqrt{3 - b^2 - 2M} - 1 = \frac{2}{3} \cdot \sqrt{3 - b^2 - 2M} - (1 - bM)$$



$$\begin{aligned}
&= \frac{2}{3} \cdot \left( \sqrt{3-2b^2} - b \right) - (1 - b^2 \sqrt{3-2b^2}) \\
&= \left( b^2 + \frac{2}{3} \right) \sqrt{3-2b^2} - \frac{2b}{3} - 1.
\end{aligned}$$

So, we need to show that

$$(3b^2 + 2)\sqrt{3-2b^2} \geq 2b + 3.$$

By squaring, the inequality becomes

$$\begin{aligned}
6b^6 - b^4 - 8b^2 + 4b - 1 &\leq 0, \\
(b-1)(6b^5 + 6b^4 + 5b^3 + 5b^2 - 3b + 1) &\leq 0.
\end{aligned}$$

It is true because

$$5b^2 - 3b + 1 = b^2 + b + (2b-1)^2 > 0.$$

The equality occurs for  $a = b = c = 1$ , and also for  $a = \sqrt{2}$ ,  $b = 1$ ,  $c = 0$ .

□

**P 3.239.** If  $a \geq 1 \geq b \geq c \geq 0$  and  $a^2 + b^2 + c^2 = 3$ , then

$$1 - abc \leq (1 + \sqrt{2})(a - b).$$

(Vasile Cîrtoaje, 2020)

**Solution.** Denoting  $x = ab$ , we need to show that  $f(x) \geq 0$ , where

$$f(x) = cx + k\sqrt{3-c^2-2x} - 1, \quad k = 1 + \sqrt{2}.$$

For fixed  $c$ , we have  $x \in [m, M]$ , where

$$m = c\sqrt{3-2c^2}, \quad M = \sqrt{2-c^2}.$$

Indeed,  $(a^2 - c^2)(b^2 - c^2) \geq 0$  yields

$$a^2b^2 \geq c^2(a^2 + b^2) - c^4 = 3c^2 - 2c^4, \quad x \geq c\sqrt{3-2c^2} = m,$$

and  $(a^2 - 1)(b^2 - 1) \leq 0$  yields

$$ab \leq \sqrt{a^2 + b^2 - 1} = \sqrt{2 - c^2} = M.$$

We have  $x = m$  for  $b = c$ , and  $x = M$  for  $b = 1$ . Since

$$f''(x) = \frac{-k}{(3-c^2-2x)^{3/2}} \leq 0,$$

$f$  is a concave function, therefore it suffices to show that  $f(m) \geq 0$  and  $f(M) \geq 0$ . We have

$$\begin{aligned}
 f(m) &= c^2\sqrt{3-2c^2} + k\sqrt{3-c^2-2c\sqrt{3-2c^2}} - 1 \\
 &= c^2\sqrt{3-2c^2} + k\left(\sqrt{3-2c^2} - c\right) - 1 \\
 &\geq c^2\sqrt{3-2c^2} + 2\left(\sqrt{3-2c^2} - c\right) - 1 \\
 &= (c^2+2)\sqrt{3-2c^2} - 2c - 1 \geq (2c+1)\left(\sqrt{3-2c^2} - 1\right) \geq 0.
 \end{aligned}$$

Also,

$$\begin{aligned}
 f(M) &= c\sqrt{2-c^2} + k\sqrt{3-c^2-2\sqrt{2-c^2}} - 1 \\
 &= c\sqrt{2-c^2} - 1 + k\left(\sqrt{2-c^2} - 1\right) \\
 &= \frac{-(1-c^2)^2}{c\sqrt{2-c^2}+1} + \frac{k(1-c^2)}{\sqrt{2-c^2}+1}.
 \end{aligned}$$

So, we need to show that

$$\frac{k}{\sqrt{2-c^2}+1} \geq \frac{1-c^2}{c\sqrt{2-c^2}+1}.$$

It is true because

$$\frac{k}{\sqrt{2-c^2}+1} \geq 1 \geq \frac{1-c^2}{1+c\sqrt{2-c^2}}.$$

The equality occurs for  $a = b = c = 1$ , and also for  $a = \sqrt{2}$ ,  $b = 1$ ,  $c = 0$ .

□

**P 3.240.** If  $a \geq 1 \geq b \geq c \geq 0$  and  $a^2 + b^2 + c^2 = 3$ , then

$$1 - abc \leq (3 + 2\sqrt{2})(a - b)^2.$$

(Vasile Cîrtoaje, 2020)

**Solution.** Write the inequality as follows:

$$abc + k(3 - c^2 - 2ab) \geq 1,$$

$$3k - 1 - kc^2 \geq (2k - c)ab, \quad k = 3 + 2\sqrt{2}.$$

From  $(a^2 - 1)(b^2 - 1) \leq 0$ , we get

$$ab \leq \sqrt{a^2 + b^2 - 1} = \sqrt{2 - c^2}.$$

Thus, it is enough to show that

$$3k - 1 - kc^2 \geq (2k - c)\sqrt{2 - c^2}.$$

Write this inequality as follows:

$$k(3 - c^2 - 2\sqrt{2 - c^2}) \geq 1 - c\sqrt{2 - c^2},$$

$$\frac{k(1 - c^2)^2}{3 - c^2 + 2\sqrt{2 - c^2}} \geq \frac{(1 - c^2)^2}{1 + c\sqrt{2 - c^2}},$$

which is true if

$$k(1 + c\sqrt{2 - c^2}) \geq 3 - c^2 + 2\sqrt{2 - c^2},$$

$$k - 3 + c^2 \geq (2 - kc)\sqrt{2 - c^2}.$$

For the nontrivial case  $2 - kc \geq 0$ , we have

$$k - 3 + c^2 \geq k - 3 = 2\sqrt{2} \geq (2 - kc)\sqrt{2 - c^2}.$$

The equality occurs for  $a = b = c = 1$ , and also for  $a = \sqrt{2}$ ,  $b = 1$ ,  $c = 0$ .

□

**P 3.241.** If  $a, b, c, d$  are nonnegative real numbers such that

$$a \geq b \geq 1 \geq c \geq d, \quad a + b + c + d = 3,$$

then

$$a^2 + b^2 + c^2 + d^2 + 10abcd \leq 5.$$

(Vasile Cîrtoaje, 2015)

**First Solution.** Let

$$E(a, b, c, d) = a^2 + b^2 + c^2 + d^2 + 10abcd.$$

We will show that

$$E(a, b, c, d) \leq E(a, b, x, x) \leq 5,$$

where

$$x = (c + d)/2, \quad a + b + 2x = 3.$$

The left inequality is true since

$$E(a, b, c, d) - E(a, b, x, x) = \frac{1}{2}(c - d)^2(1 - 5ab) \leq 0.$$

The right inequality can be written as follows:

$$a^2 + b^2 + 2x^2 + 10abx^2 \leq 5,$$

$$(a + b)^2 + 2x^2 + 2ab(5x^2 - 1) \leq 5,$$

$$2s^2 + (3 - s)^2 + ab[5(3 - s)^2 - 4] \leq 10,$$

where

$$s = a + b, \quad s \in [2, 3].$$

*Case 1:*  $5(3 - s)^2 - 4 \geq 0$ . Since  $ab \leq s^2/4$ , it suffices to show that

$$2s^2 + (3 - s)^2 + \frac{1}{4}s^2[5(3 - s)^2 - 4] \leq 10,$$

which is equivalent to the obvious inequality

$$(s - 1)(s - 2)[5s(s - 3) - 2] \leq 0.$$

*Case 2:*  $5(3 - s)^2 - 4 \leq 0$ . From  $(a - 1)(b - 1) \geq 0$ , we get  $ab \geq s - 1$ . Therefore, it suffices to show that

$$2s^2 + (3 - s)^2 + (s - 1)[5(3 - s)^2 - 4] \leq 10,$$

which is equivalent to the obvious inequality

$$(s - 2)(s - 3)(5s - 7) \leq 0.$$

The equality holds for  $a = b = 1$ ,  $c = d = 1/2$ , and for  $a = 2$ ,  $b = 1$ ,  $c = d = 0$ .

**Second Solution.** From

$$(a - 1)(b - 1)(c - 1)(d - 1) \geq 0,$$

we have

$$-2 + \sum_{sym} ab - \sum_{sym} abc + abcd \geq 0.$$

Since

$$2 \sum_{sym} ab = 9 - a^2 - b^2 - c^2 - d^2,$$

we get

$$\begin{aligned} -2 + \frac{9 - a^2 - b^2 - c^2 - d^2}{2} - \sum_{sym} abc + abcd &\geq 0, \\ a^2 + b^2 + c^2 + d^2 &\leq 5 - 2 \sum_{sym} abc + 2abcd. \end{aligned}$$

Therefore, it suffices to show that

$$(5 - 2 \sum_{sym} abc + 2abcd) + 10abcd \leq 5,$$

which is equivalent to

$$\sum_{sym} abc \geq 6abcd.$$

For the non-trivial case  $d \neq 0$ , this inequality is equivalent to

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \geq 6.$$

Since

$$\frac{1}{a} + \frac{1}{b} \geq \frac{4}{a+b}$$

and

$$\frac{1}{c} + \frac{1}{d} \geq \frac{4}{c+d},$$

it suffices to show that

$$\frac{2}{a+b} + \frac{2}{c+d} \geq 3,$$

which is equivalent to

$$(a+b-1)(a+b-2) \geq 0.$$

□

**P 3.242.** If  $a, b, c, d$  are nonnegative real numbers such that

$$a \geq b \geq 1 \geq c \geq d, \quad a + b + c + d = 6,$$

then

$$a^2 + b^2 + c^2 + d^2 + 4abcd \leq 26.$$

(Vasile Cîrtoaje, 2015)

**First Solution.** Let

$$E(a, b, c, d) = a^2 + b^2 + c^2 + d^2 + 10abcd.$$

We will show that

$$E(a, b, c, d) \leq E(a, b, x, x) \leq 5,$$

where

$$x = (c+d)/2, \quad a + b + 2x = 3.$$

The left inequality is true since

$$E(a, b, c, d) - E(a, b, x, x) = \frac{1}{2}(c-d)^2(1-2ab) \leq 0.$$

The right inequality can be written as follows:

$$a^2 + b^2 + 2x^2 + 4abx^2 \leq 26,$$

$$(a+b)^2 + 2x^2 + 2ab(2x^2 - 1) \leq 26,$$

$$2s^2 + (6-s)^2 + 2ab[(6-s)^2 - 2] \leq 52,$$

where

$$s = a + b, \quad s \in [4, 6].$$

Case 1:  $(6-s)^2 - 2 \geq 0$ . Since  $ab \leq s^2/4$ , it suffices to show that

$$2s^2 + (6-s)^2 + \frac{1}{2}s^2[(6-s)^2 - 2] \leq 52,$$

which is equivalent to the obvious inequality

$$(s-2)(s-4)[s(s-6)-4] \leq 0.$$

Case 2:  $(6-s)^2 - 2 \leq 0$ . From  $(a-1)(b-1) \geq 0$ , we get  $ab \geq s-1$ . Therefore, it suffices to show that

$$2s^2 + (6-s)^2 + 2(s-1)[(6-s)^2 - 2] \leq 52,$$

which is equivalent to the obvious inequality

$$(s-2)(s-6)(2s-7) \leq 0.$$

The equality holds for  $a = b = 2$ ,  $c = d = 1$ , and for  $a = 5$ ,  $b = 1$ ,  $c = d = 0$ .

**Second Solution.** From

$$(a-1)(b-1)(c-1)(d-1) \geq 0,$$

we have

$$-5 + \sum_{sym} ab - \sum_{sym} abc + abcd \geq 0.$$

Since

$$2 \sum_{sym} ab = 36 - a^2 - b^2 - c^2 - d^2,$$

we get

$$\begin{aligned} -5 + \frac{36 - a^2 - b^2 - c^2 - d^2}{2} - \sum_{sym} abc + abcd &\geq 0, \\ a^2 + b^2 + c^2 + d^2 &\leq 26 - 2 \sum_{sym} abc + 2abcd. \end{aligned}$$

Therefore, it suffices to show that

$$(26 - 2 \sum_{sym} abc + 2abcd) + 4abcd \leq 26,$$

which is equivalent to

$$\sum_{sym} abc \geq 3abcd.$$

For the non-trivial case  $d \neq 0$ , this inequality is equivalent to

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \geq 3.$$

Since

$$\frac{1}{a} + \frac{1}{b} \geq \frac{4}{a+b}$$

and

$$\frac{1}{c} + \frac{1}{d} \geq \frac{4}{c+d},$$

it suffices to show that

$$\frac{4}{a+b} + \frac{4}{c+d} \geq 3,$$

which is equivalent to

$$\begin{aligned}(a+b-2)(a+b-4) &\geq 0, \\ (a+b-2)(2-c-d) &\geq 0.\end{aligned}$$

□

**P 3.243.** If  $a, b, c, d$  are nonnegative real numbers such that

$$a \geq b \geq 1 \geq c \geq d, \quad a + b + c + d = 4,$$

then

$$a^2 + b^2 + c^2 + d^2 + 6abcd \leq 10.$$

(Vasile Cîrtoaje, 2015)

**First Solution.** Let

$$E(a, b, c, d) = a^2 + b^2 + c^2 + d^2 + 6abcd.$$

We will show that

$$E(a, b, c, d) \leq E(a, b, x, x) \leq 10,$$

where

$$x = (c + d)/2.$$

The left inequality is true since

$$E(a, b, c, d) - E(a, b, x, x) = \frac{1}{2}(c-d)^2(1-3ab) \leq 0.$$

The right inequality can be written as follows:

$$a^2 + b^2 + 2x^2 + 6abx^2 \leq 10, \quad a + b + 2x = 4,$$

$$(a+b)^2 + 2x^2 + 2ab(3x^2 - 1) \leq 10,$$

$$2s^2 + (4-s)^2 + ab[3(4-s)^2 - 4] \leq 20,$$

where

$$s = a + b, \quad s \in [2, 4].$$

Case 1:  $3(4-s)^2 - 4 \geq 0$ . Since  $ab \leq s^2/4$ , it suffices to show that

$$2s^2 + (4-s)^2 + \frac{1}{4}s^2[3(4-s)^2 - 4] \leq 20,$$

which is equivalent to the obvious inequality

$$(s-2)^2[3s(s-4)-4] \leq 0.$$

*Case 2:*  $3(4-s)^2-4 \leq 0$ . From  $(a-1)(b-1) \geq 0$ , we get  $ab \geq s-1$ . Therefore, it suffices to show that

$$2s^2 + (4-s)^2 + (s-1)[3(4-s)^2-4] \leq 20,$$

which is equivalent to the obvious inequality

$$(s-2)^2(s-4) \leq 0.$$

**Second Solution** (by *Lingaszayi*). From

$$(a-1)(b-1)(c-1)(d-1) \geq 0,$$

we have

$$-3 + \sum_{sym} ab - \sum_{sym} abc + abcd \geq 0.$$

Since

$$2 \sum_{sym} ab = 16 - a^2 - b^2 - c^2 - d^2,$$

we get

$$10 - a^2 - b^2 - c^2 - d^2 \geq 2 \sum_{sym} abc - 2abcd.$$

Therefore, it suffices to show that

$$2 \sum_{sym} abc - 2abcd \geq 6abcd,$$

which is equivalent to

$$\sum_{sym} abc \geq 4abcd.$$

For the non-trivial case  $d > 0$ , this inequality is equivalent to the Cauchy-Schwarz inequality

$$(a+b+c+d) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \geq 16.$$

**Third Solution** (by *Nguyen Van Quy*). Write the inequality as

$$a^2 + (b+c+d)^2 + 6abcd - 2(bc+cd+db) \leq 10,$$

$$3abcd - (bc+cd+db) \leq (a-1)(3-a).$$

By the AM-GM inequality or the Cauchy-Schwarz inequality, we have

$$bc+cd+db \geq \frac{9bcd}{b+c+d},$$



hence

$$3abcd - (bc + cd + db) \leq 3abcd - \frac{9bcd}{b+c+d} = \frac{3bcd(a-1)(3-a)}{b+c+d}.$$

Since

$$3-a \geq 4-a-b = c+d \geq 0,$$

it suffices to show that

$$\frac{3bcd}{b+c+d} \leq 1.$$

Indeed, using the AM-GM inequality and  $b+c+d = 4-a \leq 3$ , we get

$$\frac{3bcd}{b+c+d} \leq \frac{(b+c+d)^2}{9} \leq 1.$$

□

**P 3.244.** If  $a, b, c, d$  are nonnegative real numbers such that

$$a \geq b \geq 1 \geq c \geq d, \quad a+b+c+d = 4,$$

then

$$a^2 + b^2 + c^2 + d^2 + 6\sqrt{abcd} \leq 10.$$

(Vasile Cîrtoaje, 2015)

**Solution.** Let

$$E(a, b, c, d) = a^2 + b^2 + c^2 + d^2 + 6\sqrt{abcd}.$$

We will show that

$$E(a, b, c, d) \leq E(a, b, x, x) \leq 10,$$

where

$$x = \frac{c+d}{2} = \frac{4-a-b}{2}.$$

The left inequality can be reduced to the obvious form

$$\left(\sqrt{c} - \sqrt{d}\right)^2 \left[6\sqrt{ab} - \left(\sqrt{c} + \sqrt{d}\right)^2\right] \geq 0,$$

while the right inequality is equivalent to

$$a^2 + b^2 + 2x^2 + 6x\sqrt{ab} \leq 10.$$

Since  $2\sqrt{ab} \leq a+b$ , it suffices to show that

$$a^2 + b^2 + 2x^2 + 3x(a+b) \leq 10.$$

which can be rewritten as

$$2(a^2 + b^2) + (4-a-b)^2 + 3(4-a-b)(a+b) \leq 20,$$

$$2(a+b)^2 - 4ab + 16 - 8(a+b) + (a+b)^2 + 12(a+b) - 3(a+b)^2 \leq 20,$$

$$4(a-1)(b-1) \geq 0.$$

□

**P 3.245.** If  $a, b, c, d$  are positive real numbers such that

$$a + b + c + d = 4, \quad a \leq b \leq 1 \leq c \leq d,$$

then

$$9 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \geq 4 + 8(a^2 + b^2 + c^2 + d^2).$$

(Vasile Cîrtoaje, 2021)

**Solution.** For fixed  $b$  and  $d$ , write the required inequality as  $f(c) \geq 0$ , where

$$f(c) = 9 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) - 4 - 8(a^2 + b^2 + c^2 + d^2), \quad a = 4 - b - c - d.$$

We will show that

$$f(c) \geq f(1) \geq 0.$$

Since

$$a + c \leq \frac{a+b}{2} + \frac{c+d}{2} = 2,$$

$$\frac{a+c}{a^2c^2} \geq \frac{16}{(a+c)^3} \geq 2,$$

we have

$$\begin{aligned} f'(c) &= \frac{9}{a^2} - \frac{9}{c^2} + 16(a-c) = 9(c-a) \left( \frac{a+c}{a^2c^2} - \frac{16}{9} \right) \\ &\geq 9(c-a) \left( \frac{a+c}{a^2c^2} - 2 \right) \geq 0, \end{aligned}$$

$f(c)$  is increasing, hence  $f(c) \geq f(1)$ . The inequality  $f(1) \geq 0$  has the form

$$9 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{d} \right) - 3 - 8(a^2 + b^2 + d^2) \geq 0,$$

where

$$a = 3 - b - d.$$

We may write this inequality as  $g(a, b) \geq 0$ , where

$$g(a, b) = 9 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{d} \right) - 3 - 8(a^2 + b^2 + d^2), \quad d = 3 - a - b.$$

We will show that

$$g(a, b) \geq g(x, x) \geq 0,$$

where

$$x = \frac{a+b}{2}, \quad 0 < x \leq 1.$$

We have

$$\begin{aligned} g(a, b) - g(x, x) &= 9 \left( \frac{1}{a} + \frac{1}{b} - \frac{2}{x} \right) - 8(a^2 + b^2 - 2x^2) \\ &= \frac{9(a-b)^2}{2abx} - 4(a-b)^2 = \frac{(a-b)^2(9-8abx)}{2abx} \geq 0 \end{aligned}$$

and

$$\begin{aligned} g(x, x) &= 9 \left( \frac{2}{x} + \frac{1}{d} \right) - 3 - 8(2x^2 + d^2), \quad d = 3 - 2x, \\ g(x, x) &= 9 \left( \frac{2}{x} + \frac{1}{3-2x} \right) - 3 - 16x^2 - 8(3-2x)^2 \\ &= \frac{6(16x^4 - 56x^3 + 73x^2 - 42x + 9)}{x(3-2x)}, \\ &= \frac{6(x-1)^2(4x-3)^2}{x(3-2x)} \geq 0. \end{aligned}$$

The equality holds for  $a = b = c = d = 1$ , and also for  $a = b = \frac{3}{4}$ ,  $c = 1$ ,  $d = \frac{3}{2}$ .

□

**P 3.246.** Let  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) be real numbers such that  $a_1 + a_2 + \dots + a_n = n$ .

(a) If  $a_1 \geq 1 \geq a_2 \geq \dots \geq a_n$ , then

$$a_1^3 + a_2^3 + \dots + a_n^3 + 2n \geq 3(a_1^2 + a_2^2 + \dots + a_n^2).$$

(b) If  $a_1 \leq 1 \leq a_2 \leq \dots \leq a_n$ , then

$$a_1^3 + a_2^3 + \dots + a_n^3 + 2n \leq 3(a_1^2 + a_2^2 + \dots + a_n^2).$$

(Vasile Cîrtoaje, 2007)

**Solution.** (a) Write the inequality as

$$\begin{aligned} \sum (a_1^3 - 3a_1^2 + 3a_1 - 1) &\geq 0, \\ \sum (a_1 - 1)^3 &\geq 0, \\ (a_1 - 1)^3 &\geq (1 - a_2)^3 + \dots + (1 - a_n)^3, \\ [(1 - a_2) + \dots + (1 - a_n)]^3 &\geq (1 - a_2)^3 + \dots + (1 - a_n)^3. \end{aligned}$$

Clearly, the last inequality is true. The equality holds for  $a_1 = a_2 = \dots = a_n = 1$ , and also for  $a_1 = 2$ ,  $a_2 = \dots = a_{n-1} = 1$ ,  $a_n = 0$ .

(b) Similarly, write the inequality as

$$\sum (a_1^3 - 3a_1^2 + 3a_1 - 1) \leq 0,$$

$$\begin{aligned}\sum (1 - a_1)^3 &\geq 0, \\ (1 - a_1)^3 &\geq (a_2 - 1)^3 + \cdots + (a_n - 1)^3, \\ [(a_2 - 1) + \cdots + (a_n - 1)]^3 &\geq (a_2 - 1)^3 + \cdots + (a_n - 1)^3.\end{aligned}$$

The last inequality is obviously true. The equality holds for  $a_1 = a_2 = \cdots = a_n = 1$ , and also for  $a_1 = 0, a_2 = \cdots = a_{n-1} = 1, a_n = 2$ . □

**P 3.247.** Let  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) be nonnegative real numbers such that  $a_1 + a_2 + \cdots + a_n = n$ .

(a) If  $a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n$ , then

$$a_1^4 + a_2^4 + \cdots + a_n^4 + 5n \geq 6(a_1^2 + a_2^2 + \cdots + a_n^2);$$

(b) If  $a_1 \leq 1 \leq a_2 \leq \cdots \leq a_n$ , then

$$a_1^4 + a_2^4 + \cdots + a_n^4 + 6n \leq 7(a_1^2 + a_2^2 + \cdots + a_n^2).$$

(Vasile Cîrtoaje, 2007)

**Solution.** (a) Write the inequality as

$$\begin{aligned}\sum (a_1^4 - 6a_1^2 + 8a_1 - 3) &\geq 0, \\ \sum (a_1 - 1)^3(a_1 + 3) &\geq 0, \\ (a_1 - 1)^3(a_1 + 3) &\geq (1 - a_2)^3(a_2 + 3) + \cdots + (1 - a_n)^3(a_n + 3).\end{aligned}$$

Since

$$(a_1 - 1)^3 = [(1 - a_2) + \cdots + (1 - a_n)]^3 \geq (1 - a_2)^3 + \cdots + (1 - a_n)^3,$$

it suffices to show that

$$[(1 - a_2)^3 + \cdots + (1 - a_n)^3](a_1 + 3) \geq (1 - a_2)^3(a_2 + 3) + \cdots + (1 - a_n)^3(a_n + 3),$$

which is equivalent to the obvious inequality

$$(1 - a_2)^3(a_1 - a_2) + \cdots + (1 - a_n)^3(a_1 - a_n) \geq 0.$$

The equality holds for  $a_1 = a_2 = \cdots = a_n = 1$ .

(b) Write the inequality as

$$\begin{aligned}\sum (a_1^4 - 7a_1^2 + 10a_1 - 4) &\leq 0, \\ \sum (a_1 - 1)^2(a_1^2 + 2a_1 - 4) &\leq 0,\end{aligned}$$

$$(a_2 - 1)^2(a_2^2 + 2a_2 - 4) + \cdots + (a_n - 1)^2(a_n^2 + 2a_n - 4) \leq (1 - a_1)^2(4 - 2a_1 - a_1^2).$$

Since

$$(1 - a_1)^2 = [(a_2 - 1) + \cdots + (a_n - 1)]^2 \geq (a_2 - 1)^2 + \cdots + (a_n - 1)^2,$$

it suffices to show that

$$(a_2 - 1)^2(a_2^2 + 2a_2 - 4) + \cdots + (a_n - 1)^2(a_n^2 + 2a_n - 4) \leq [(a_2 - 1)^2 + \cdots + (a_n - 1)^2](4 - 2a_1 - a_1^2),$$

which is equivalent to

$$(a_2 - 1)^2(a_1^2 + a_2^2 + 2a_1 + 2a_2 - 8) + \cdots + (a_n - 1)^2(a_1^2 + a_n^2 + 2a_1 + 2a_n - 8) \leq 0.$$

This inequality is true if

$$a_1^2 + a_n^2 + 2a_1 + 2a_n - 8 \leq 0.$$

Since

$$a_1 + a_n = n - (a_2 + \cdots + a_{n-1}) = 2 + (1 - a_2) + \cdots + (1 - a_{n-1}) \leq 2,$$

we have

$$a_1^2 + a_n^2 + 2a_1 + 2a_n - 8 = (a_1 + a_n + 1)^2 - 9 - 2a_1a_n \leq (a_1 + a_n + 1)^2 - 9 \leq 0.$$

The equality holds for  $a_1 = a_2 = \cdots = a_n = 1$ , and also for  $a_1 = 0$ ,  $a_2 = \cdots = a_{n-1} = 1$ ,  $a_n = 2$ .

**Remark.** The inequality in (a) remains valid for all real  $a_1, a_2, \dots, a_n$  such that

$$a_1 + a_2 + \cdots + a_n = n, \quad a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n.$$

□

**P 3.248.** If  $a_1, a_2, \dots, a_n$  are positive real numbers such that

$$a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n, \quad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n,$$

then

$$a_1^2 + a_2^2 + \cdots + a_n^2 + 2n \geq 3(a_1 + a_2 + \cdots + a_n).$$

(Vasile Cîrtoaje, 2008)

**Solution.** Write the inequality as follows:

$$(a_1 - 1)(a_1 - 2) + (a_2 - 1)(a_2 - 2) + \cdots + (a_n - 1)(a_n - 2) \geq 0,$$

$$(a_1 - 1)(a_1 - 2) \geq (1 - a_2)(a_2 - 2) + \cdots + (1 - a_n)(a_n - 2),$$

$$\left(1 - \frac{1}{a_1}\right)(a_1^2 - 2a_1) \geq \left(\frac{1}{a_2} - 1\right)(a_2^2 - 2a_2) + \cdots + \left(\frac{1}{a_n} - 1\right)(a_n^2 - 2a_n),$$

$$\begin{aligned}
\left[ \left( \frac{1}{a_2} - 1 \right) + \cdots + \left( \frac{1}{a_n} - 1 \right) \right] (a_1^2 - 2a_1) &\geq \left( \frac{1}{a_2} - 1 \right) (a_2^2 - 2a_2) + \cdots + \left( \frac{1}{a_n} - 1 \right) (a_n^2 - 2a_n), \\
\left( \frac{1}{a_2} - 1 \right) (a_1^2 - 2a_1 - a_2^2 + 2a_2) + \cdots + \left( \frac{1}{a_n} - 1 \right) (a_1^2 - 2a_1 - a_n^2 + 2a_n) &\geq 0, \\
\left( \frac{1}{a_2} - 1 \right) (a_1 - a_2)(a_1 + a_2 - 2) + \cdots + \left( \frac{1}{a_n} - 1 \right) (a_1 - a_n)(a_1 + a_n - 2) &\geq 0.
\end{aligned}$$

Clearly, it suffices to prove that  $a_1 + a_n - 2 \geq 0$ . Indeed,

$$a_1 + a_n - 2 = n - 2 - (a_2 + \cdots + a_{n-1}) = (1 - a_2) + \cdots + (1 - a_{n-1}) \geq 0.$$

The equality holds for  $a_1 = a_2 = \cdots = a_n = 1$ .

□

**P 3.249.** If  $a_1, a_2, \dots, a_n$  are real numbers such that

$$a_1 \leq 1 \leq a_2 \leq \cdots \leq a_n, \quad a_1 + a_2 + \cdots + a_n = n,$$

then

$$(1 + a_1^2)(1 + a_2^2) \cdots (1 + a_n^2) \geq 2^n.$$

(Vasile Cîrtoaje, 2009)

**Solution.** Use the substitutions  $a_1 = 1 - S$  and

$$a_2 = b_2 + 1, \dots, a_n = b_n + 1,$$

where  $S$  and  $b_2, \dots, b_n$  are nonnegative real numbers such that

$$S = b_2 + \cdots + b_n.$$

We have

$$\begin{aligned}
\frac{1}{2}(1 + a_1^2) &= 1 - S + \frac{1}{2}S^2, \\
\frac{1}{2}(1 + a_i^2) &= 1 + b_i + \frac{1}{2}b_i^2, \quad i = 2, \dots, n,
\end{aligned}$$

and, by Lemma below,

$$\frac{1}{2^{n-1}}(1 + a_2^2) \cdots (1 + a_n^2) = \left(1 + b_2 + \frac{1}{2}b_2^2\right) \cdots \left(1 + b_n + \frac{1}{2}b_n^2\right) \geq 1 + S + \frac{1}{2}S^2.$$

Therefore, it suffices to show that

$$\left(1 - S + \frac{1}{2}S^2\right) \left(1 + S + \frac{1}{2}S^2\right) \geq 1,$$

which is equivalent to  $S^4 \geq 0$ . The equality holds for  $a_1 = a_2 = \cdots = a_n = 1$ .

**Lemma.** If  $c_1, c_2, \dots, c_k$  are nonnegative real numbers such that  $c_1 + c_2 + \dots + c_k = S$ , then

$$\left(1 + c_1 + \frac{1}{2}c_1^2\right) \left(1 + c_2 + \frac{1}{2}c_2^2\right) \cdots \left(1 + c_k + \frac{1}{2}c_k^2\right) \geq 1 + S + \frac{1}{2}S^2.$$

*Proof.* We have

$$\begin{aligned} \prod_{1 \leq i \leq k} \left(1 + c_i + \frac{1}{2}c_i^2\right) &\geq 1 + \sum_{1 \leq i \leq k} \left(c_i + \frac{1}{2}c_i^2\right) + \sum_{1 \leq i < j \leq k} \left(c_i + \frac{1}{2}c_i^2\right) \left(c_j + \frac{1}{2}c_j^2\right) \\ &\geq 1 + \sum_{1 \leq i \leq k} \left(c_i + \frac{1}{2}c_i^2\right) + \sum_{1 \leq i < j \leq k} c_i c_j \\ &= 1 + S + \frac{1}{2}S^2. \end{aligned}$$

□

**P 3.250.** If  $a_1, a_2, \dots, a_n$  are positive real numbers such that

$$a_1 \geq 1 \geq a_2 \geq \dots \geq a_n, \quad a_1 a_2 \cdots a_n = 1,$$

then

$$a_1^n + a_2^n + \dots + a_n^n - n \geq n^2 \left( \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} - n \right).$$

(Vasile Cîrtoaje, 2009)

**Solution.** We use the induction method. For  $n = 2$ , the desired inequality is equivalent to

$$(a_1 - 1)^4 \geq 0.$$

Let us denote

$$E_n(a_1, a_2, \dots, a_n) = a_1^n + a_2^n + \dots + a_n^n - n - n^2 \left( \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} - n \right).$$

We will show that

$$E_n(a_1, a_2, a_3, \dots, a_{n-1}, a_n) \geq E_n(a_1, 1, a_3, \dots, a_{n-1}, a_2 a_n) \geq 0.$$

The right inequality can be written as

$$E_{n-1}(a_1, a_3, \dots, a_{n-1}, a_2 a_n) \geq 0.$$

Since

$$a_2 a_n \leq a_n \leq a_{n-1}$$

and

$$a_1 a_3 \cdots a_{n-1} (a_2 a_n) = 1,$$

the inequality follows by the induction hypothesis.

The left inequality is equivalent to

$$a_2^n + a_n^n - 1 - a_2^n a_n^n \geq n^2 \left( \frac{1}{a_2} + \frac{1}{a_n} - 1 - \frac{1}{a_2 a_n} \right),$$

$$n^2 \left( \frac{1}{a_2} - 1 \right) \left( \frac{1}{a_n} - 1 \right) \geq (1 - a_2^n)(1 - a_n^n),$$

which is true if

$$\frac{n^2}{a_2 a_n} \geq (1 + a_2 + \cdots + a_2^{n-1})(1 + a_n + \cdots + a_n^{n-1}).$$

Since  $a_2 \leq 1$  and  $a_n \leq 1$ , this inequality is clearly true. The equality holds for  $a_1 = a_2 = \cdots = a_n = 1$ .

□

**P 3.251.** If  $a_1, a_2, \dots, a_n$  ( $n \geq 3$ ) are real numbers such that

$$a_1 + a_2 + \cdots + a_n = n, \quad a_1 \geq a_2 \geq 1 \geq a_3 \geq \cdots \geq a_n,$$

then

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \geq \frac{14}{3}(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

(Vasile Cîrtoaje, 2009)

**Solution** (by Linqaszayi). Using the substitution

$$a_i = 1 + x_i, \quad i = 1, 2, \dots, n,$$

which implies

$$x_1 \geq x_2 \geq 0 \geq x_3 \geq \cdots \geq x_n, \quad x_1 + x_2 + \cdots + x_n = 0,$$

we need to show that  $E(x_1, x_2, \dots, x_n) \geq 0$ , where

$$E(x_1, x_2, x_3, \dots, x_n) = 3 \sum_{i=1}^n x_i^4 + 12 \sum_{i=1}^n x_i^3 + 4 \sum_{i=1}^n x_i^2.$$

We will prove that

$$E(x_1, x_2, \dots, x_n) \geq E\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, \dots, x_n\right) \geq 0.$$

The left inequality is true because

$$x_1^4 + x_2^4 \geq 2 \left( \frac{x_1 + x_2}{2} \right)^4, \quad x_1^3 + x_2^3 \geq 2 \left( \frac{x_1 + x_2}{2} \right)^3, \quad x_1^2 + x_2^2 \geq 2 \left( \frac{x_1 + x_2}{2} \right)^2.$$



To prove the right inequality, we replace  $x_3, \dots, x_n$  with  $-x_3, \dots, -x_n$ . So, we need to show that

$$x_1 + x_2 = x_3 + \dots + x_n, \quad x_1, x_2, x_3, \dots, x_n \geq 0,$$

involves

$$E + 3(x_3^4 + \dots + x_n^4) - 12(x_3^3 + \dots + x_n^3) + 4(x_3^2 + \dots + x_n^2) \geq 0,$$

where

$$E = 6 \left( \frac{x_1 + x_2}{2} \right)^4 + 24 \left( \frac{x_1 + x_2}{2} \right)^3 + 8 \left( \frac{x_1 + x_2}{2} \right)^2 = 6A^4 + 24A^3 + 8A^2,$$

with

$$A = \frac{x_3 + \dots + x_n}{2}.$$

Since

$$A^4 \geq \frac{x_3^4 + \dots + x_n^4}{16}, \quad A^3 \geq \frac{x_3^3 + \dots + x_n^3}{8}, \quad A^2 \geq \frac{x_3^2 + \dots + x_n^2}{4},$$

we have

$$E \geq \frac{3}{8}(x_3^4 + \dots + x_n^4) + 3(x_3^3 + \dots + x_n^3) + 2(x_3^2 + \dots + x_n^2).$$

Therefore, it suffices to show that

$$\left( \frac{3}{8} + 3 \right) (x_3^4 + \dots + x_n^4) + (3 - 12)(x_3^3 + \dots + x_n^3) + (2 + 4)(x_3^2 + \dots + x_n^2) \geq 0,$$

which is equivalent to the obvious inequality

$$x_3^2(3x_3 - 4)^2 + \dots + x_n^2(3x_n - 4)^2 \geq 0.$$

This completes the proof. The equality holds for  $a_1 = a_2 = \dots = a_n = 1$ , and also for

$$a_1 = a_2 = \frac{5}{3}, \quad a_3 = \dots = a_{n-1} = 1, \quad a_n = \frac{-1}{3}.$$

□

**P 3.252.** If  $a_1, a_2, \dots, a_n$  are nonnegative real numbers such that at most one of them is larger than 1 and  $a_1 a_2 \dots a_n = 1$ , then

$$a_1 + a_2 + \dots + a_n \geq \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}.$$

(Vasile Cîrtoaje, 2009)

**Solution.** For  $n = 2$ , the inequality is an identity. Consider now  $n \geq 3$  and use the induction method. Assume  $a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n \geq 0$  and denote

$$E_n(a_1, a_2, \dots, a_n) = a_1 + a_2 + \cdots + a_n - \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right).$$

We will show that

$$E_n(a_1, a_2, a_3, \dots, a_n) \geq E_n(a_1 a_2, 1, a_3, \dots, a_{n-1}, a_n) \geq 0.$$

The right inequality can be written as

$$E_{n-1}(a_1 a_2, a_3, \dots, a_{n-1}, a_n) \geq 0.$$

Since

$$a_1 a_2 = \frac{1}{a_3 \cdots a_{n-1} a_n} \geq 1$$

and

$$(a_1 a_2) a_3 \cdots a_{n-1} a_n = 1,$$

the inequality follows by the induction hypothesis.

The left inequality is equivalent to

$$\begin{aligned} a_1 + a_2 - \frac{1}{a_1} - \frac{1}{a_2} &\geq a_1 a_2 - \frac{1}{a_1 a_2}, \\ (a_1 + a_2) \left( 1 - \frac{1}{a_1 a_2} \right) &\geq \frac{(a_1 a_2 + 1)(a_1 a_2 - 1)}{a_1 a_2}. \end{aligned}$$

This inequality can be written in the obvious form

$$(a_1 a_2 - 1)(a_1 - 1)(1 - a_2) \geq 0.$$

The proof is finished. The equality occurs when  $n - 2$  numbers  $a_i$  are equal to 1 and the product of the other two is 1.

□

# Appendix A

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